Research Article

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Dynamics of a diffusive delayed competition and cooperation system

<https://doi.org/10.1515/math-2020-0064>

received December 15, 2019; accepted July 24, 2020

Abstract: In this manuscript, we first consider the diffusive competition and cooperation system subject to Neumann boundary conditions without delay terms and get the conclusion that the unique positive constant equilibrium is locally asymptotically stable. Then, we study the diffusive delayed competition and cooperation system subject to Neumann boundary conditions, and the existence of Hopf bifurcation at the positive equilibrium is obtained by regarding delay term as the parameter. By the theory of center manifold and normal form, an algorithm for determining the direction and stability of Hopf bifurcation is derived. Finally, some numerical simulations and summarizations are carried out for illustrating the theoretical analytic results.

Keywords: competition and cooperation system, diffusion, stability, local Hopf bifurcation

MSC 2020: 34K10, 34K20, 37L10

1 Introduction

In the past few decades, delay differential equations that change in time and involve delays have become a hot research topic. Many complicated and large-scale systems in nature and society can be modeled as delay differential systems, due to their flexibility and generality for representing virtually any natural and man-made structure. They have received much attention in interdisciplinary subjects including natural science [1–3], engineering [4], life sciences, and others [5,6]. In particular, many scientists paid their attention to the stability and bifurcation phenomena of the predator-prey system with multiple delays (see, for example, [7–13]). When the delay is continuous and modeled by a convolution, the problem on the periodic phenomenon can be restricted on the critical manifold, and the limit cycle can be detected by the zeros of Melnikov function, see [14–16]. In fact, much commonness is reflected between species that co-evolve in nature and different enterprises that co-exist in economic society, so numerous researchers have widely presented the competition and cooperation model of the enterprises [17,18], which are governed by the following ordinary differential equation:

$$\begin{cases} \dot{x}_1(t) = r_1 x_1(t) \left(1 - \frac{x_1(t)}{K_1} - \frac{\alpha(x_2(t) - c_2)^2}{K_2} \right), \\ \dot{x}_2(t) = r_2 x_2(t) \left(1 - \frac{x_2(t)}{K_2} + \frac{\beta(x_1(t) - c_1)^2}{K_1} \right), \end{cases} \quad (1.1)$$

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where $x_1(t), x_2(t)$ denote the output of enterprise x_1 and enterprise x_2 at time t , respectively; $(x_1(t), x_2(t)) \in \mathbb{R}^1 \times \mathbb{R}^1$. The parameters r_i ($i = 1, 2$) represent the intrinsic growth for output of two enterprises; K_i ($i = 1, 2$) measure the load capacity of two enterprises in an unrestricted natural market; α, β stand for the coefficient of competition of enterprise x_1 and x_2 ; c_i ($i = 1, 2$) denote the initial production of them. All the parameters in the above model are positive.

Let $a_1 = \frac{r_1}{K_1}$, $a_2 = \frac{r_2}{K_2}$, $b_1 = \frac{\alpha r_1}{K_2}$, $b_2 = \frac{\beta r_2}{K_1}$, $d_i = r_i - a_i c_i$, $u(t) = x_1(t) - c_1$, $v(t) = x_2(t) - c_2$. System (1.1) becomes

$$\begin{cases} \frac{du(t)}{dt} = (u(t) + c_1)(d_1 - a_1 u(t) - b_1 v^2(t)), \\ \frac{dv(t)}{dt} = (v(t) + c_2)(d_2 - a_2 v(t) + b_2 u^2(t)), \\ u(0) > 0, v(0) > 0. \end{cases} \quad (1.2)$$

Taking into account the influence of history, the researchers introduce the time delay τ to the feedback in model (1.2), which is a more realistic approach to the understanding of competition and cooperation dynamics. Delays can induce various oscillations and periodic solutions through bifurcations as the delay is increasing. Therefore, it is interesting to investigate the following delayed model:

$$\begin{cases} \frac{du(t)}{dt} = (u(t) + c_1)(d_1 - a_1 u(t - \tau_1) - b_1 v^2(t - \tau_2)), \\ \frac{dv(t)}{dt} = (v(t) + c_2)(d_2 - a_2 v(t - \tau_3) + b_2 u^2(t - \tau_4)), \\ u(0) > 0, v(0) > 0, \end{cases} \quad (1.3)$$

where τ_i ($i = 1, 2, 3, 4$) ≥ 0 represent the time delay, system (1.3) had been studied extensively by many researchers and some interesting conclusions have also been obtained in [17–19].

In the natural economical environment, due to limited customer resources, enterprises are not evenly distributed in the space, and in order to survive, enterprises will look for customers everywhere, which will lead to migration and diffusion. Therefore, considering the heterogeneity of enterprise spatial distribution, motivated by the present situation stated above, we take the inhomogeneity of the spatial distribution into account and obtain the following competition and cooperation system incorporating diffusion and delay subject to Neumann boundary conditions

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = e_1 \Delta u + (u(t) + c_1)(d_1 - a_1 u(t) - b_1 v^2(t - \tau)), & x \in \Omega, \quad t > 0, \\ \frac{\partial v(x, t)}{\partial t} = e_2 \Delta v + (v(t) + c_2)(d_2 - a_2 v(t) + b_2 u^2(t)), & x \in \Omega, \quad t > 0, \\ u_x(0, t) = v_x(0, t) = 0, u_x(l\pi, t) = v_x(l\pi, t) = 0, & t \geq 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \bar{\Omega}, \end{cases} \quad (1.4)$$

where e_1 and e_2 are the diffusion coefficients of competition and cooperation enterprises, $l \in \mathbb{R}^+$, and $\Omega = (0, l\pi)$ is a bounded domain with a smooth boundary $\partial\Omega$. Note that the homogeneous Neumann boundary condition means that neither enterprise can cross the boundary. The appearance of the spatial dispersal makes the dynamics and behaviors of the organisms even more complicated [20–22]. The investigation in connection with the dynamics of the diffusive competition and cooperation system (1.4) will make significant economic implications. It is worth mentioning that in the study of systems with diffusion terms, different boundary conditions represent different practical meanings. For instance, for the predator-prey system, homogeneous Dirichlet boundary conditions are imposed so that both species die out on the boundary [23,24], and the homogeneous Robin boundary condition means that the prey or predator can cross the boundary [25].

This paper aims to investigate the stability of equilibria and the properties of Hopf bifurcation at the unique positive constant equilibrium of system (1.4). The rest of this paper is organized as follows. In Section 2, the stability properties of the equilibria are studied for system (1.4) with $\tau = 0$. In Section 3, as for system (1.4), the existence of Hopf bifurcation at the positive equilibrium is obtained by regarding delay term as the parameter, and the direction and stability of spatial Hopf bifurcating periodic solutions are determined. Finally, some numerical simulations and summarizations are given in Sections 4 and 5.

2 Stability analysis of equilibria for the diffusive system

In this section, we only consider the diffusive competition and cooperation system without delay terms subject to Neumann boundary conditions

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = e_1 \Delta u + (u(t) + c_1)(d_1 - a_1 u(t) - b_1 v^2(t)), & x \in \Omega, \quad t > 0, \\ \frac{\partial v(x, t)}{\partial t} = e_2 \Delta v + (v(t) + c_2)(d_2 - a_2 v(t) + b_2 u^2(t)), & x \in \Omega, \quad t > 0, \\ u_x(0, t) = v_x(0, t) = 0, \quad u_x(l\pi, t) = v_x(l\pi, t) = 0, & t \geq 0, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, & x \in \bar{\Omega}. \end{cases} \quad (2.5)$$

Lemma 1. Assume that

$$(H_1) \quad a_2^2 d_1 > b_1 d_2^2$$

holds, then system (2.5) has a unique positive equilibrium $E^* = (u^*, v^*)$.

Proof. The proof is similar to that in [17], we omit it here.

Suppose (H_1) holds, then $E^* = (u^*, v^*)$ is the unique positive constant equilibrium of system (2.5). We discuss the stability of the unique positive constant equilibrium. First, we transform $E^* = (u^*, v^*)$ of system (2.5) to the origin via the translation $\hat{u} = u - u^*$, $\hat{v} = v - v^*$ and drop the hats for simplicity of notation, then system (2.5) is transformed into

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = e_1 \Delta u + (u(t) + u^* + c_1)(d_1 - a_1(u(t) + u^*) - b_1(v(t) + v^*)^2), \\ \frac{\partial v(x, t)}{\partial t} = e_2 \Delta v + (v(t) + v^* + c_2)(d_2 - a_2(v(t) + v^*) + b_2(u(t) + u^*)^2). \end{cases} \quad (2.6)$$

Define the real-valued Sobolev space

$$X := \{(u, v)^T : u, v \in H^2(0, l\pi), (u_x, v_x)|_{x=0, l\pi} = (0, 0)\}$$

and the complexification of X to be

$$X_{\mathbb{C}} := X \oplus iX = \{x_1 + ix_2 \mid x_1, x_2 \in X\}.$$

System (2.6) is equivalent to the following abstract operator equation:

$$\dot{U}(t) = e\Delta U(t) + LU(t) + F(U(t)), \quad (2.7)$$

where $U := (u, v) \in H^2(0, l\pi)$, $e = \text{diag}(e_1, e_2)$, $\text{dom}(e\Delta) = X$,

$$L = \begin{pmatrix} -a_1(u^* + c_1) & -2b_1 v^*(u^* + c_1) \\ 2b_2 u^*(v^* + c_2) & -a_2(v^* + c_2) \end{pmatrix},$$

$$F(U) = \begin{pmatrix} (u(t) + u^* + c_1)(d_1 - a_1(u(t) + u^*) - b_1(v(t) + v^*)^2) + a_1(u^* + c_1)u + 2b_1 v^*(u^* + c_1)v \\ (v(t) + v^* + c_2)(d_2 - a_2(v(t) + v^*) + b_2(u(t) + u^*)^2) - 2b_2 u^*(v^* + c_2)u + a_2(v^* + c_2)v \end{pmatrix}.$$

Then the linearization of system (2.7) near $(0, 0)$ has the form:

$$\dot{U}(t) = e\Delta U(t) + LU(t), \quad (2.8)$$

the associated characteristic equation of (2.8) is given by

$$\lambda y - e\Delta y - Ly = 0, \quad y \in \text{dom}(e\Delta), \quad y \neq 0. \quad (2.9)$$

For convenience, we denote $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It is well known that the eigenvalue problem

$$-\varphi'' = \mu\varphi, \quad x \in (0, l\pi), \quad \varphi'(0) = \varphi'(l\pi) = 0$$

has eigenvalues $\mu_n = n^2/l^2 (n \in \mathbb{N}_0)$ with corresponding eigenfunctions $\varphi_n(x) = \cos\left(\frac{n}{l}x\right)$. Substituting

$$y = \sum_{n=0}^{\infty} \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix} \cos \frac{n}{l}x$$

into the characteristic Eq. (2.9), it follows that

$$\begin{pmatrix} -e_1 \frac{n^2}{l^2} - a_1(u^* + c_1) & -2b_1 v^*(u^* + c_1) \\ 2b_2 u^*(v^* + c_2) & -e_2 \frac{n^2}{l^2} - a_2(v^* + c_2) \end{pmatrix} \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix} = \lambda \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix}, \quad n \in \mathbb{N}_0.$$

Therefore, the characteristic Eq. (2.9) is equivalent to

$$\lambda^2 - \lambda T_n + D_n = 0, \quad n \in \mathbb{N}_0, \quad (2.10)$$

where

$$\begin{aligned} T_n &= -n^2 \frac{e_1 + e_2}{l^2} - a_1(u^* + c_1) - a_2(v^* + c_2), \\ D_n &= n^4 \frac{e_1 e_2}{l^4} + \frac{n^2}{l^2} [a_1 e_2 (u^* + c_1) + a_2 e_1 (v^* + c_2)] + a_1 a_2 (u^* + c_1)(v^* + c_2) + 4b_1 b_2 u^* v^* (u^* + c_1)(v^* + c_2). \end{aligned} \quad (2.11)$$

Clearly, the roots of (2.10) are given by

$$\lambda_{1,2}^n = \frac{1}{2} \left[T_n \pm \sqrt{T_n^2 - 4D_n} \right], \quad n \in \mathbb{N}_0. \quad (2.12)$$

Based on the aforementioned statements, we have the following conclusions. $\square$

Theorem 1. Suppose (H_1) holds, then $T_n < 0$ and $D_n > 0$, for $n \in \mathbb{N}_0$, that is, all the roots of Eq. (2.12) have negative real parts. More precisely, the unique positive constant equilibrium is locally asymptotically stable.

Proof. Let $z = n^2$, by Eq. (2.11) and denote

$$\begin{aligned} T_z &= -z \frac{e_1 + e_2}{l^2} - a_1(u^* + c_1) - a_2(v^* + c_2), \\ D_z &= z^2 \frac{e_1 e_2}{l^4} + \frac{z}{l^2} [a_1 e_1 (u^* + c_1) + a_2 e_2 (v^* + c_2)] + a_1 a_2 (u^* + c_1)(v^* + c_2) + 4b_1 b_2 u^* v^* (u^* + c_1)(v^* + c_2). \end{aligned}$$

Obviously, $T'_z = -\frac{e_1 + e_2}{l^2} < 0$, that is T_z is decreasing with respect to $z \geq 0$. Hence,

$$T_n = T(n^2) \leq T(0) = -a_1(u^* + c_1) - a_2(v^* + c_2) < 0, \quad n \in \mathbb{N}_0.$$

$D'_z = 2z \frac{e_1 e_2}{l^4} + \frac{1}{l^2} [a_1 e_1 (u^* + c_1) + a_2 e_2 (v^* + c_2)] > 0$, that is, D_z is increasing with respect to $z \geq 0$. Hence,

$$D_n = D(n^2) \geq D(0) = a_1 a_2 (u^* + c_1)(v^* + c_2) + 4b_1 b_2 u^* v^* (u^* + c_1)(v^* + c_2) > 0, \quad n \in \mathbb{N}_0. \quad \square$$

Remark 1. By Theorem 2.2 in [17], assume that (H_1) holds, and the positive equilibrium E^* of system (1.2) is asymptotically stable. In this paper, we show that the positive equilibrium of system (2.5) is also asymptotically stable. That is, the diffusive term has no influence on the dynamical behavior of system (1.2).

3 Dynamical analysis of the diffusive delayed system

In this section, we consider the diffusive delayed competition and cooperation system subject to Neumann boundary conditions

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = e_1 \Delta u + (u(t) + c_1)(d_1 - a_1 u(t) - b_1 v^2(t - \tau)), & x \in \Omega, \quad t > 0, \\ \frac{\partial v(x, t)}{\partial t} = e_2 \Delta v + (v(t) + c_2)(d_2 - a_2 v(t) + b_2 u^2(t)), & x \in \Omega, \quad t > 0, \\ u_x(0, t) = v_x(0, t) = 0, \quad u_x(l\pi, t) = v_x(l\pi, t) = 0, & t \geq 0, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, & x \in \bar{\Omega}, \end{cases} \quad (3.13)$$

where $\tau > 0$ stands for the feedback delay of the one enterprise to the other one.

Suppose (H_1) holds and $\tau \neq 0$, and we discuss the existence of local Hopf bifurcations occurring at the unique constant positive equilibrium $E^* = (u^*, v^*)$ by regarding delay term τ as the parameter. We first transform $E^* = (u^*, v^*)$ of (3.13) to the origin via the translation $\hat{u} = u - u^*$, $\hat{v} = v - v^*$ and drop the hats for simplicity of notation, then system (3.13) is transformed into

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = e_1 \Delta u + (u + u^* + c_1)(d_1 - a_1(u + u^*) - b_1(v(t - \tau) + v^*)^2), \\ \frac{\partial v(x, t)}{\partial t} = e_2 \Delta v + (v + v^* + c_2)(d_2 - a_2(v + v^*) + b_2(u + u^*)^2). \end{cases} \quad (3.14)$$

In the phase space $C_\tau := C([- \tau, 0], X)$, system (3.14) can be regarded as the following abstract functional differential equation:

$$\dot{U}(t) = e \Delta U(t) + L(U_t) + F(U_t), \quad (3.15)$$

where $U := (u, v) \in H^2(0, l\pi)$, $e = \text{diag}(e_1, e_2)$, $\text{dom}(e \Delta) = X$, $L : C_\tau \mapsto X$, and $F : C_\tau \mapsto X$ are defined by

$$L(\phi) := \begin{pmatrix} -a_1(u^* + c_1) & 0 \\ 2b_2 u^*(v^* + c_2) & -a_2(v^* + c_2) \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \begin{pmatrix} 0 & -2b_1 v^*(u^* + c_1) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-\tau) \\ \phi_2(-\tau) \end{pmatrix},$$

$$F(\phi) = \begin{pmatrix} (u(t) + u^* + c_1)(d_1 - a_1(u(t) + u^*) - b_1(v(t) + v^*)^2) + a_1(u^* + c_1)\phi_1(0) + 2b_1 v^*(u^* + c_1)\phi_2(-\tau) \\ (v(t) + v^* + c_2)(d_2 - a_2(v(t) + v^*) + b_2(u(t) + u^*)^2) - 2b_2 u^*(v^* + c_2)\phi_1(0) + a_2(v^* + c_2)\phi_2(0) \end{pmatrix}$$

for $\phi = (\phi_1, \phi_2)^T \in C_\tau$.

Then the linearization of system (3.15) at the origin is given by

$$\dot{U}(t) = e \Delta U(t) + L(U_t). \quad (3.16)$$

According to [26], we obtain that the characteristic equation for linear system (3.16) is given by

$$\lambda y - e \Delta y - L(e^\lambda y) = 0, \quad y \in \text{dom}(d \Delta), \quad y \neq 0. \quad (3.17)$$

It is well known that the eigenvalue problem

$$-\varphi'' = \mu \varphi, \quad x \in (0, l\pi), \quad \varphi'(0) = \varphi'(l\pi) = 0$$

has eigenvalues $\mu_n = \frac{n^2}{l^2}$ ($n \in \mathbb{N}_0$) with corresponding eigenfunctions $\varphi_n(x) = \cos \frac{n}{l}x$.

Substituting

$$y = \sum_{n=0}^{\infty} \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix} \cos \frac{n}{l}x$$

into the characteristic Eq. (3.17), it follows that

$$\begin{pmatrix} -e_1 \frac{n^2}{l^2} - a_1(u^* + c_1) & -2b_1 v^*(u^* + c_1)e^{-\lambda \tau} \\ 2b_2 u^*(v^* + c_2) & -e_2 \frac{n^2}{l^2} - a_2(v^* + c_2) \end{pmatrix} \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix} = \lambda \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix}, \quad n \in \mathbb{N}_0.$$

Therefore, the characteristic Eq. (3.17) is equivalent

$$\Delta_n(\lambda, \tau) = \lambda^2 + \lambda A_n + B_n - C_n e^{-\lambda \tau} = 0, \quad n \in \mathbb{N}_0, \quad (3.18)$$

where

$$\begin{aligned}
A_n &= (e_1 + e_2) \frac{n^2}{l^2} + a_1(u^* + c_1) + a_2(v^* + c_2), \\
B_n &= \left[e_1 \frac{n^2}{l^2} + a_1(u^* + c_1) \right] \left[e_2 \frac{n^2}{l^2} + a_2(v^* + c_2) \right], \\
C_n &= -4b_1b_2u^*v^*(u^* + c_1)(v^* + c_2).
\end{aligned}$$

By Eq. (3.18), we can get $\Delta_n(0, \tau) = B_n - C_n > 0$ and obtain the following lemma.

Lemma 2. Suppose (H_1) is satisfied. Then $\lambda = 0$ is not a root of Eq. (3.18) for any $n \in \mathbb{N}_0$.

We make the following hypotheses

$$(H_2) \quad a_1a_2 > 4b_1b_2u^*v^*,$$

$$(H_3) \quad a_1a_2 < 4b_1b_2u^*v^*.$$

From the result of [27], the sum of the multiplicities of the roots of (3.18) in the open right-half plane changes only if a root appears on or crosses the imaginary axis. In the following, we will derive the conditions under which the aforementioned cases occur. Denote

$$\bar{N} = l \sqrt{\frac{1}{2e_1e_2} \{-a_1e_2(u^* + c_1) - a_2e_1(v^* + c_2) + \sqrt{[a_1e_2(u^* + c_1) + a_2e_1(v^* + c_2)]^2 - \tilde{P}}\}},$$

where $\tilde{P} = 4e_1e_2(u^* + c_1)(v^* + c_2)(a_1a_2 - 4b_1b_2u^*v^*)$, and

$$N_1 = \begin{cases} [\bar{N}], & \bar{N} \notin \mathbb{N}, \\ \bar{N} - 1, & \bar{N} \in \mathbb{N}. \end{cases}$$

Then we have the following lemma.

Lemma 3. Suppose (H_1) holds, then the following statements are true.

- (i) If (H_2) holds, (3.18) has no purely imaginary roots for $n \geq 0$.
- (ii) If (H_3) holds, (3.18) has a pair of purely imaginary roots

$$i\omega_n^\pm, \quad 0 \leq n \leq N_1 \quad \text{at} \quad \tau_n^{\pm,j}.$$

Here,

$$\begin{aligned}
\tau_n^{\pm,j} &= \tau_n^{\pm,0} + \frac{2j\pi}{\omega_n}, \quad j \in \mathbb{N}_0, \\
\tau_n^{\pm,0} &= \frac{1}{\omega_n^\pm} \arccos \frac{B_n - (\omega_n^\pm)^2}{C_n}, \\
\omega_n^\pm &= \sqrt{\frac{1}{2} [-(A_n^2 - 2B_n) \pm \sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - C_n^2)}]}.
\end{aligned}$$

Applying the same analytical steps as those in Ruan and Wei [27], when $\tau > 0$, $\lambda = i\omega$ ($\omega > 0$) is a root of (3.18) if and only if ω satisfies

$$-\omega^2 + i\omega A_n + B_n - C_n(\cos \omega\tau - i \sin \omega\tau) = 0.$$

Then we have

$$\begin{cases} -\omega^2 + B_n - C_n \cos \omega\tau = 0, \\ \omega A_n + C_n \sin \omega\tau = 0, \end{cases}$$

which leads to

$$\omega^4 + \omega^2(A_n^2 - 2B_n) + B_n^2 - C_n^2 = 0. \quad (3.19)$$

Let $z = \omega^2$, then (3.19) can be rewritten into the following form

$$z^2 + z(A_n^2 - 2B_n) + B_n^2 - C_n^2 = 0,$$

and its roots are given by

$$z = \frac{1}{2} \left[-(A_n^2 - 2B_n) \pm \sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - C_n^2)} \right].$$

We obtain

$$\begin{aligned} B_n - C_n &= e_1 e_2 \frac{n^4}{l^4} + [a_1 e_2 (u^* + c_1) + a_2 e_1 (v^* + c_2)] \frac{n^2}{l^2} + a_1 a_2 (u^* + c_1)(v^* + c_2) \\ &\quad + 4b_1 b_2 u^* v^* (u^* + c_1)(v^* + c_2) = D_n > 0, \end{aligned}$$

and

$$B_n + C_n = e_1 e_2 \frac{n^4}{l^4} + [a_1 e_2 (u^* + c_1) + a_2 e_1 (v^* + c_2)] \frac{n^2}{l^2} + a_1 a_2 (u^* + c_1)(v^* + c_2) - 4b_1 b_2 u^* v^* (u^* + c_1)(v^* + c_2)$$

is increasing with respect $n \geq 0$, then

$$B_n + C_n > 0 \quad \text{for } n \geq 0.$$

Under (H_3) , we have

$$B_n + C_n < 0 \quad \text{for } 0 \leq n \leq N_1 \quad \text{and} \quad B_n + C_n > 0 \quad \text{for } n \geq N_1.$$

In addition,

$$A_n^2 - 2B_n = (e_1^2 + e_2^2) \frac{n^4}{l^4} + 2[a_1 e_1 (u^* + c_1) + a_2 e_2 (v^* + c_2)] \frac{n^2}{l^2} + a_1^2 (u^* + c_1)^2 + a_2^2 (v^* + c_2)^2$$

is increasing with respect $n \geq 0$. Hence,

$$A_n^2 - 2B_n \geq A_0^2 - 2B_0 = a_1^2 (u^* + c_1)^2 + a_2^2 (v^* + c_2)^2 > 0, \quad n \in \mathbb{N}_0.$$

Based on the discussion above, the two statements hold and $\omega_n^\pm = \sqrt{z_\pm}$.

For simplicity, we only consider case (ii) in Lemma 3 and use τ_n^j to denote $\tau_n^{+,j}$. Note that $\tau_m^j = \tau_n^k$, for some $m \neq n$ may occur. In this paper, we do not consider this case. In other words, we consider

$$\tau \in \mathcal{D} := \{\tau_n^j : \tau_m^j \neq \tau_n^k, m \neq n, 0 \leq m, n \leq N_1, j, k \in \mathbb{N}_0\}.$$

Let $\lambda_n = \alpha_n(\tau) + i\omega_n(\tau)$ be the root of (3.18) satisfying $\alpha_n(\tau_n^j) = 0$ and $\omega_n(\tau_n^j) = \omega_n$ when τ is close to τ_n^j . Then we have the following transversality condition.

Lemma 4. Suppose (H_1) is satisfied. Then

$$\alpha_n'(\tau_n^j) < 0 \quad \text{for } \tau \in \mathcal{D} \text{ and } j \in \mathbb{N}_0.$$

Differentiating two sides of (3.18) with respect τ , we have

$$\begin{aligned} 2\lambda \frac{d\lambda}{d\tau} + \frac{d\lambda}{d\tau} A_n - C_n e^{-\lambda\tau} \left(-\frac{d\lambda}{d\tau} \tau - \lambda \right) &= 0, \\ \left(\frac{d\lambda}{d\tau} \right)^{-1} &= \frac{2\lambda + A_n}{\lambda C_n e^{-\lambda\tau}} + \frac{\tau}{\lambda}. \end{aligned}$$

Then

$$\left[\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\tau=\tau_n^j} = \left[\operatorname{Re} \left(\frac{2\lambda + A_n}{\lambda C_n e^{-\lambda\tau}} + \frac{\tau}{\lambda} \right) \right]_{\tau=\tau_n^j} = \left[\operatorname{Re} \left(\frac{(2i\omega + A_n)(\cos \omega\tau + i \sin \omega\tau)}{i\omega C_n} \right) \right]_{\tau=\tau_n^j}$$

$$= \left[\frac{2\omega \cos \omega\tau + A_n \sin \omega\tau}{\omega C_n} \right]_{\tau=\tau_n^j} = \frac{-2\omega^2}{C_n^2} < 0.$$

Therefore, $\alpha'_n(\tau_n^j) < 0$.

Denote

$$\tau_*^0 = \min_{0 \leq i \leq N_1} \{\tau_i^0\}.$$

By Lemma 3, we know that if assumptions (H_1) and (H_3) hold, there are two sequence values of τ , $\tau_n^{+,j}$ and $\tau_n^{-,j}$, such that characteristic Eq. (3.18) has a pair of purely imaginary roots when $\tau = \tau_n^{\pm,j}$, respectively. By Lemma 4, the transversality condition is also satisfied. According to Ruan and Wei [27], we have the following theorem.

Theorem 2. For system (1), suppose (H_1) holds, if (H_2) or (H_3) also holds, then the following statements are true.

- (i) If $\tau \in [0, \tau_*^0)$, then the equilibrium E^* is locally asymptotically stable;
- (ii) If $\tau > \tau_*^0$, then the equilibrium E^* is unstable;
- (iii) $\tau = \tau_0^j$ ($j \in \mathbb{N}_0$) are Hopf bifurcation values of system (1.4), and the bifurcating periodic solutions are spatially homogeneous, which coincide with the periodic solutions of the corresponding functional differential equation (FDE) system; when $\tau \in \mathcal{D} \setminus \{\tau_0^k : k \in \mathbb{N}_0\}$, system (1.4) also undergoes a Hopf bifurcation and the bifurcating periodic solutions are spatially non-homogeneous.

In this part, we shall study the direction of Hopf bifurcation and stability of the bifurcating periodic solution of system (3.13) by applying center manifold theorem and normal form theorem of partial functional differential equations [26–28]. For fixed $j \in \mathbb{N}_0$ and $0 \leq n \leq N_1$, we denote $\tilde{\tau} = \tau_n^j$. Let $\tilde{u}(x, t)$ and $\tilde{v}(x, t)$ denote $u(x, \tau t)$ and $v(x, \tau t)$, respectively. For convenience, we drop the tilde. Then system (3.13) can be transformed into

$$\begin{cases} \frac{\partial u}{\partial t} = \tau \left[e_1 \frac{\partial^2 u}{\partial x^2} + (u(t) + u^* + c_1)(d_1 - a_1(u(t) + u^*) - b_1(v(t - \tau) + v^*)^2) \right], \\ \frac{\partial v}{\partial t} = \tau \left[e_2 \frac{\partial^2 v}{\partial x^2} + (v(t) + v^* + c_2)(d_2 - a_2(v(t) + v^*) + b_2(u(t) + u^*)^2) \right]. \end{cases} \quad (3.20)$$

Let $\tau = \tilde{\tau} + \mu$, $u_1(t) = u(\cdot, t)$, $u_2(t) = v(\cdot, t)$, and $U = (u_1, u_2)^T$.

Then system (3.20) can be rewritten in an abstract form in the phase space $C_1 := C([-1, 0], X)$

$$\frac{dU(t)}{dt} = \tilde{\tau} e \Delta U(t) + L_{\tilde{\tau}}(U_t) + F(U_t), \quad (3.21)$$

where

$$L_{\mu}(\phi) = \mu \begin{pmatrix} -a_1(u^* + c_1)\phi_1(0) & -2b_1v^*(u^* + c_1)\phi_2(-1) \\ 2b_2u^*(v^* + c_2)\phi_1(0) & -a_2(v^* + c_2)\phi_2(0) \end{pmatrix},$$

and

$$F(\phi, \mu) = \mu e \Delta \phi + L_{\mu}(\phi) + f(\phi, \mu),$$

$$f(\phi) = (\tilde{\tau} + \mu) \times \begin{pmatrix} (\phi_1(0) + c_1)(d_1 - a_1\phi_1(0) - b_1\phi_2^2(-1)) + a_1(u^* + c_1)\phi_1(0) + 2b_1v^*(u^* + c_1)\phi_2(-1) \\ (\phi_2(0) + c_2)(d_2 - a_2\phi_2(0) + b_2\phi_1^2(0)) - 2b_2u^*(v^* + c_2)\phi_1(0) + a_2(v^* + c_2)\phi_2(0) \end{pmatrix},$$

respectively, for $\phi = (\phi_1, \phi_2)^T \in C_{\tau}$.

Consider the linear equation

$$\frac{dU(t)}{dt} = \tilde{\tau} e \Delta U(t) + L_{\tilde{\tau}}(U_t). \quad (3.22)$$

Obviously, $(0, 0)$ is an equilibrium of system (3.20), and $\Lambda_n := \{i\omega_n \tilde{\tau}, -i\omega_n \tilde{\tau}\}$ are characteristic values of system (3.20) and the linear functional differential equation:

$$\frac{dz(t)}{dt} = -\tilde{\tau} e^{\frac{n^2}{l^2} z(t)} + L_{\tilde{\tau}}(z_t). \quad (3.23)$$

By the Riesz representation theorem, there exists a 2×2 matrix function $\eta^n(\theta, \tilde{\tau})$ ($\theta \in [-1, 0]$), whose elements are of bounded variation functions such that

$$-\tilde{\tau} e^{\frac{n^2}{l^2} \phi(0)} + L_{\tilde{\tau}}(\phi) = \int_{-1}^0 d\eta^n(\theta, \tau) \phi(\theta) \quad (3.24)$$

for $\phi \in C([-1, 0], \mathbb{R}^2)$.

In fact, we can choose

$$\eta^n(\theta, \tau) = \begin{cases} \tau \begin{pmatrix} 0 & -2b_1 v^*(u^* + c_1) \\ 0 & 0 \end{pmatrix}, & \theta = -1, \\ 0, & \theta \in (-1, 0), \\ \tau \begin{pmatrix} -a_1(u^* + c_1) - e_1 \frac{n^2}{l^2} & 0 \\ 2b_2 u^*(v^* + c_2) & -a_2(v^* + c_2) - e_2 \frac{n^2}{l^2} \end{pmatrix}, & \theta = 0. \end{cases} \quad (3.25)$$

Let $A(\tilde{\tau})$ denote the infinitesimal generators of semigroup included by the solutions of Eq. (3.23) and A^* be the formal adjoint of $A(\tilde{\tau})$ under the bilinear pairing

$$\begin{aligned} \langle \psi, \phi \rangle &= \psi(0) \phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \psi(\xi - \theta) d\eta^n(\theta, \tilde{\tau}) \phi(\xi) d\xi \\ &= \psi(0) \phi(0) + \tilde{\tau} \int_{-1}^0 \psi(\xi + 1) \begin{pmatrix} 0 & -2b_1 v^*(u^* + c_1) \\ 0 & 0 \end{pmatrix} \phi(\xi) d\xi \end{aligned} \quad (3.26)$$

for $\phi \in C([-1, 0], \mathbb{R}^2)$, $\psi \in C([-1, 0], \mathbb{R}^2)$. $A(\tilde{\tau})$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_n \tilde{\tau}$, and they are also eigenvalues of A^* . Let P and P^* be the center subspace, that is, the generalized eigenspace of $A(\tilde{\tau})$ and A^* associated with Λ_n , respectively. Then P^* is the adjoint space of P and $\dim P = \dim P^* = 2$.

It can be verified that

$$p_1(\theta) = (1, \xi)^T e^{i\omega_n \tilde{\tau} \theta} \quad (\theta \in [-1, 0]), \quad p_2(\theta) = \overline{p_1(\theta)}$$

is a basis of A^* with Λ_n and

$$q_1(r) = (1, \eta) e^{-i\omega_n \tilde{\tau} r} \quad (r \in [0, 1]), \quad q_2(r) = \overline{q_1(r)}$$

is a basis of A^* with Λ_n , where

$$\xi = \frac{i\omega_n + a_1(u^* + c_1) + e_1 \frac{n^2}{l^2}}{-2b_1 v^*(u^* + c_1)}, \quad \eta = \frac{-i\omega_n + a_1(u^* + c_1) + e_1 \frac{n^2}{l^2}}{2b_2 u^*(v^* + c_2)} e^{-i\omega_n \tilde{\tau}}.$$

Let $\Phi = (\Phi_1, \Phi_2)$ and $\Phi^* = (\Phi_1^*, \Phi_2^*)$ with

$$\begin{aligned} \Phi_1(\theta) &= \frac{p_1(\theta) + p_2(\theta)}{2} = \begin{pmatrix} \operatorname{Re}(e^{i\omega_n \tilde{\tau} \theta}) \\ \operatorname{Re}(\xi e^{i\omega_n \tilde{\tau} \theta}) \end{pmatrix} = \begin{pmatrix} \cos(\omega_n \tilde{\tau} \theta) \\ \frac{\omega_n}{2b_1 v^*(u^* + c_1)} \sin(\omega_n \tilde{\tau} \theta) - \frac{e_1 \frac{n^2}{l^2} + a_1(u^* + c_1)}{2b_1 v^*(u^* + c_1)} \cos(\omega_n \tilde{\tau} \theta) \end{pmatrix}, \\ \Phi_2(\theta) &= \frac{p_1(\theta) - p_2(\theta)}{2i} = \begin{pmatrix} \operatorname{Im}(e^{i\omega_n \tilde{\tau} \theta}) \\ \operatorname{Im}(\xi e^{i\omega_n \tilde{\tau} \theta}) \end{pmatrix} = \begin{pmatrix} \sin(\omega_n \tilde{\tau} \theta) \\ -\frac{\omega_n}{2b_1 v^*(u^* + c_1)} \cos(\omega_n \tilde{\tau} \theta) - \frac{e_1 \frac{n^2}{l^2} + a_1(u^* + c_1)}{2b_1 v^*(u^* + c_1)} \sin(\omega_n \tilde{\tau} \theta) \end{pmatrix} \end{aligned}$$

for $\theta \in [-1, 0)$, and

$$\Psi_1^*(r) = \frac{q_1(r) + q_2(r)}{2} = \begin{pmatrix} \operatorname{Re}(e^{i\omega_n \tilde{r}}) \\ \operatorname{Re}(\eta e^{i\omega_n \tilde{r}}) \end{pmatrix} = \begin{pmatrix} \cos(\omega_n \tilde{r}) \\ -\frac{\omega_n}{2b_2 u^*(v^* + c_2)} \sin(\omega_n \tilde{r}(r+1)) + \frac{e_1 \frac{n^2}{l^2} + a_1(u^* + c_1)}{2b_2 u^*(v^* + c_2)} \cos(\omega_n \tilde{r}(r+1)) \end{pmatrix},$$

$$\Psi_2^*(r) = \frac{q_1(r) - q_2(r)}{2i} = \begin{pmatrix} \operatorname{Im}(e^{i\omega_n \tilde{r}}) \\ \operatorname{Im}(\eta e^{i\omega_n \tilde{r}}) \end{pmatrix} = \begin{pmatrix} -\sin(\omega_n \tilde{r}) \\ -\frac{\omega_n}{2b_2 u^*(v^* + c_2)} \cos(\omega_n \tilde{r}(r+1)) + \frac{e_1 \frac{n^2}{l^2} - a_1(u^* + c_1)}{2b_2 u^*(v^* + c_2)} \sin(\omega_n \tilde{r}(r+1)) \end{pmatrix}$$

for $r \in [0, 1]$. Then we can compute by (3.26)

$$D_1^* := (\Psi_1^*, \Phi_1), \quad D_2^* := (\Psi_1^*, \Phi_2), \quad D_3^* := (\Psi_2^*, \Phi_1), \quad D_4^* := (\Psi_2^*, \Phi_2).$$

Define $(\Psi^*, \Phi) = (\Psi_j^*, \Phi_k) = \begin{pmatrix} D_1^* & D_2^* \\ D_3^* & D_4^* \end{pmatrix}$ and construct a new basis Ψ for P^* by

$$\Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)^{-1} \Psi^*.$$

Then $(\Psi, \Phi) = I_2$. In addition, define $f_n := (\beta_n^1, \beta_n^2)$, where

$$\beta_n^1 = \begin{pmatrix} \cos \frac{n}{l} X \\ 0 \end{pmatrix}, \quad \beta_n^2 = \begin{pmatrix} 0 \\ \cos \frac{n}{l} X \end{pmatrix}.$$

We also define

$$c \cdot f_n = c_1 \beta_n^1 + c_2 \beta_n^2, \quad \text{for } c = (c_1, c_2)^T \in C_1.$$

Thus, the center subspace of linear Eq. (3.6) is given by $P_{CN}C_1 \oplus P_SC_1$ and P_SC_1 manifests the complement subspace of $P_{CN}C_1$ in C_1 ,

$$\langle u, v \rangle := \frac{1}{l\pi} \int_0^{l\pi} u_1 \bar{v}_1 dx + \frac{1}{l\pi} \int_0^{l\pi} u_1 \bar{v}_2 dx$$

for $u = (u_1, u_2)$, $v = (v_1, v_2)$, $u, v \in X$, and $\langle \phi, f_0 \rangle = (\langle \phi, f_0^1 \rangle, \langle \phi, f_0^2 \rangle)^T$.

Let $A_{\tilde{r}}$ denote the infinitesimal generator of an analytic semigroup induced by the linear system (3.23), and Eq. (3.20) can be rewritten as the following abstract form:

$$\frac{dU(t)}{dt} = A_{\tilde{r}} U_t + R(U_t, \mu),$$

where

$$R(U_t, \mu) = \begin{cases} 0, & \theta \in [-1, 0), \\ F(U_t, \mu), & \theta = 0. \end{cases}$$

By the decomposition of $C_1 = P_{CN}C_1 \oplus P_SC_1$, the solution above can be written as

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} f_n + h(x_1, x_2, \mu),$$

where

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (\Psi, \langle U_t, f_n \rangle),$$

and

$$h(x_1, x_2, \mu) \in P_SC_1, \quad h(0, 0, 0) = 0, \quad Dh(0, 0, 0) = 0.$$

In particular, the solution of (3.21) on the center manifold is given by

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} f_n + h(x_1, x_2, 0). \quad (3.27)$$

Let $z = x_1 - ix_2$, and note that $p_1 = \Phi_1 + i\Phi_2$. Then we have

$$\Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} f_n = (\Phi_1, \Phi_2) \begin{pmatrix} \frac{z+\bar{z}}{2} \\ \frac{i(z-\bar{z})}{2} \end{pmatrix} f_n = \frac{1}{2}(p_1 z + \overline{p_1 \bar{z}}) f_n,$$

and

$$h(x_1, x_2, 0) = h\left(\frac{z+\bar{z}}{2}, \frac{i(z-\bar{z})}{2}, 0\right).$$

Hence, Eq. (3.27) can be transformed into

$$U_t = \frac{1}{2}(p_1 z + \overline{p_1 \bar{z}}) f_n + h\left(\frac{z+\bar{z}}{2}, \frac{i(z-\bar{z})}{2}, 0\right) = \frac{1}{2}(p_1 z + \overline{p_1 \bar{z}}) f_n + W(z, \bar{z}), \quad (3.28)$$

where

$$W(z, \bar{z}) = h\left(\frac{z+\bar{z}}{2}, \frac{i(z-\bar{z})}{2}, 0\right).$$

By [29,30], z satisfies

$$\dot{z} = i\omega_n \bar{z} z + g(z, \bar{z}), \quad (3.29)$$

where

$$g(z, \bar{z}) = (\Psi_1(0) - i\Psi_2(0)) \langle F(U_t, 0), f_n \rangle.$$

Let

$$W(z, \bar{z}) = W_{20} \frac{z^2}{2} + W_{11} z\bar{z} + W_{02} \frac{\bar{z}^2}{2} + \dots, \quad (3.30)$$

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + \dots, \quad (3.31)$$

from Eqs. (3.28) and (3.30), we have

$$\begin{aligned} u_t(0) &= \frac{1}{2}(z + \bar{z}) \cos\left(\frac{nx}{l}\right) + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \dots, \\ v_t(0) &= \frac{1}{2}(\xi z + \bar{\xi} \bar{z}) \cos\left(\frac{nx}{l}\right) + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots, \\ u_t(-1) &= \frac{1}{2}(ze^{-i\omega_n \bar{\tau}} + \bar{z}e^{i\omega_n \bar{\tau}}) \cos\left(\frac{nx}{l}\right) + W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + \dots, \\ v_t(-1) &= \frac{1}{2}(\xi ze^{-i\omega_n \bar{\tau}} + \bar{\xi} \bar{z}e^{i\omega_n \bar{\tau}}) \cos\left(\frac{nx}{l}\right) + W_{20}^{(2)}(-1) \frac{z^2}{2} + W_{11}^{(2)}(-1) z\bar{z} + W_{02}^{(2)}(-1) \frac{\bar{z}^2}{2} + \dots, \end{aligned} \quad (3.32)$$

and

$$\begin{aligned} \bar{F}_1(U_t, 0) &= \frac{1}{\bar{\tau}} F_1 = -a_1 u_t^2(0) - b_1(u^* + c_1) v_t^2(-1) - 2b_1 v^* u_t(0) v_t(-1) - b_1 u_t(0) v_t^2(-1), \\ \bar{F}_2(U_t, 0) &= \frac{1}{\bar{\tau}} F_2 = -a_2 v_t^2(0) + b_2(v^* + c_2) u_t^2(0) + 2b_2 u^* u_t(0) v_t(0) + b_2 u_t^2(0) v_t(0). \end{aligned} \quad (3.33)$$

Substitute Eq. (3.32) into Eq. (3.33), hence,

$$\begin{aligned}
 \bar{F}_1(U_t, 0) = & \frac{z^2}{2} \cos^2\left(\frac{nx}{l}\right) \left[-\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi^2 e^{-2i\omega_n \tilde{\tau}} - b_1 v^* \xi e^{-i\omega_n \tilde{\tau}} \right] \\
 & + z\bar{z} \cos^2\left(\frac{nx}{l}\right) \left[-\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi\bar{\xi} - \frac{1}{2}b_1 v^* (\xi e^{-i\omega_n \tilde{\tau}} + \bar{\xi} e^{i\omega_n \tilde{\tau}}) \right] \\
 & + \frac{\bar{z}^2}{2} \cos^2\left(\frac{nx}{l}\right) \left[-\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\bar{\xi}^2 e^{2i\omega_n \tilde{\tau}} - b_1 v^* \bar{\xi} e^{i\omega_n \tilde{\tau}} \right] \\
 & + \frac{z^2 \bar{z}}{2} \left\{ -a_1 \cos\left(\frac{nx}{l}\right) [W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)] - \frac{1}{2}b_1 \xi \cos\left(\frac{nx}{l}\right) \left[\bar{\xi} \cos^2\left(\frac{nx}{l}\right) + \frac{1}{2}\xi e^{-2i\omega_n \tilde{\tau}} \right] \right. \\
 & - b_1(u^* + c_1) \cos\left(\frac{nx}{l}\right) [2\xi e^{-i\omega_n \tilde{\tau}} W_{11}^{(2)}(-1) + \bar{\xi} e^{i\omega_n \tilde{\tau}} W_{20}^{(2)}(-1)] \\
 & \left. - 2b_1 v^* \cos\left(\frac{nx}{l}\right) \left[W_{11}^{(2)}(-1) + W_{20}^{(2)}(-1) + \frac{1}{2}W_{20}^{(1)}(0)\bar{\xi} e^{i\omega_n \tilde{\tau}} + W_{11}^{(1)}(0)\xi e^{-i\omega_n \tilde{\tau}} \right] \right\} + \dots,
 \end{aligned} \tag{3.34}$$

$$\begin{aligned}
 \bar{F}_2(U_t, 0) = & \frac{z^2}{2} a_2 \cos^2\left(\frac{nx}{l}\right) \left[\frac{1}{2}a_2 \xi^2 - \frac{1}{2}b_2(v^* + c_2) - b_2 u^* \xi \right] \\
 & + z\bar{z} \cdot \frac{1}{2} \cos^2\left(\frac{nx}{l}\right) [-a_2 \xi \bar{\xi} + b_2(v^* + c_2) + b_2 u^* \xi] \\
 & + \frac{\bar{z}^2}{2} \cdot \frac{1}{2} \cos^2\left(\frac{nx}{l}\right) [-a_2 \bar{\xi}^2 + b_2(v^* + c_2) + 2b_2 u^* \bar{\xi}] \\
 & + \frac{z^2 \bar{z}}{2} \left\{ -a_2 \cos\left(\frac{nx}{l}\right) [2\xi W_{11}^{(2)}(0) + \bar{\xi} W_{20}^{(2)}(0)] + b_2(v^* + c_2) \cos\left(\frac{nx}{l}\right) \left[\frac{1}{2}W_{20}^{(1)}(0) + W_{11}^{(1)}(0) \right] \right. \\
 & + b_2 u^* \cos\left(\frac{nx}{l}\right) \left[W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \frac{1}{2}\bar{\xi} \cos\left(\frac{nx}{l}\right) W_{20}^{(1)}(0) + \xi \cos\left(\frac{nx}{l}\right) W_{11}^{(1)}(0) \right] \\
 & \left. + \frac{1}{4}b_2 \left(\xi + \frac{1}{2}\bar{\xi} \right) \cos^3\left(\frac{nx}{l}\right) \right\} + \dots,
 \end{aligned} \tag{3.35}$$

$$\begin{aligned}
 \langle F(U_t, 0), f_n \rangle = & \tilde{\tau} (\bar{F}_1(U_t, 0) f_n^1 + \bar{F}_2(U_t, 0) f_n^2) \\
 = & \frac{z^2}{2} \tilde{\tau} \left(\begin{aligned} & -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi^2 e^{-2i\omega_n \tilde{\tau}} - b_1 v^* \xi e^{-i\omega_n \tilde{\tau}} \\ & a_2 \left[\frac{1}{2}a_2 \xi^2 - \frac{1}{2}b_2(v^* + c_2) - b_2 u^* \xi \right] \end{aligned} \right) \frac{1}{l\pi} \int_0^{l\pi} \cos^3\left(\frac{nx}{l}\right) dx \\
 & + z\bar{z} \tilde{\tau} \left(\begin{aligned} & -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi\bar{\xi} - \frac{1}{2}b_1 v^* (\xi e^{-i\omega_n \tilde{\tau}} + \bar{\xi} e^{i\omega_n \tilde{\tau}}) \\ & \frac{1}{2} [-a_2 \xi \bar{\xi} + b_2(v^* + c_2) + b_2 u^* \xi] \end{aligned} \right) \frac{1}{l\pi} \int_0^{l\pi} \cos^3\left(\frac{nx}{l}\right) dx \\
 & + \frac{\bar{z}^2}{2} \tilde{\tau} \left(\begin{aligned} & -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\bar{\xi}^2 e^{2i\omega_n \tilde{\tau}} - b_1 v^* \bar{\xi} e^{i\omega_n \tilde{\tau}} \\ & \frac{1}{2} [-a_2 \bar{\xi}^2 + b_2(v^* + c_2) + 2b_2 u^* \bar{\xi}] \end{aligned} \right) \frac{1}{l\pi} \int_0^{l\pi} \cos^3\left(\frac{nx}{l}\right) dx \\
 & + \frac{z^2 \bar{z}}{2} \tilde{\tau} \left(\begin{aligned} & -2(a_1 + b_1 v^* \xi e^{-i\omega_n \tilde{\tau}}) \left\langle W_{11}^{(1)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \\ & + a_1 b_1 v^* \bar{\xi} e^{i\omega_n \tilde{\tau}} \left\langle W_{20}^{(1)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \\ & - 2b_1 [v^* + (u^* + c_1)\xi e^{-i\omega_n \tilde{\tau}}] \left\langle W_{11}^{(2)}(-1) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \\ & - b_1 [(u^* + c_1)\bar{\xi} e^{i\omega_n \tilde{\tau}} + v^*] \left\langle W_{20}^{(2)}(-1) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \\ & + b_2(v^* + c_2) \left\langle W_{11}^{(1)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \\ & + (b_2 u^* - 2a_1 \xi) \left\langle W_{11}^{(2)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \\ & + \frac{1}{2}b_2(v^* + c_2) \left\langle W_{20}^{(1)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \\ & + \left(\frac{1}{2}b_2 u^* - a_1 \bar{\xi} \right) \left\langle W_{20}^{(2)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \end{aligned} \right) + \dots.
 \end{aligned} \tag{3.36}$$

Denote

$$\Gamma = \frac{1}{l\pi} \int_0^{l\pi} \cos^3\left(\frac{nx}{l}\right) dx,$$

$$\Psi_1(0) - i\Psi_2(0) := (\gamma_1 \gamma_2).$$

Note that

$$\frac{1}{l\pi} \int_0^{l\pi} \cos^3\left(\frac{nx}{l}\right) dx = 0, \quad n \in \mathbb{N},$$

and we have

$$\begin{aligned} & (\Psi_1(0) - i\Psi_2(0)) \langle F(U_t, 0), f_n \rangle \\ &= \frac{z^2}{2} \left\{ \gamma_1 \left[-\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi^2 e^{-2i\omega_n \tilde{t}} - b_1 v^* \xi e^{-i\omega_n \tilde{t}} \right] + \gamma_2 a_2 \left[\frac{1}{2}a_2 \xi^2 - \frac{1}{2}b_2(v^* + c_2) - b_2 u^* \xi \right] \right\} \Gamma \tilde{t} \\ &+ z\bar{z} \left\{ \gamma_1 \left[-\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi\bar{\xi} - \frac{1}{2}b_1 v^* (\xi e^{-i\omega_n \tilde{t}} + \bar{\xi} e^{i\omega_n \tilde{t}}) \right] + \frac{1}{2}\gamma_2 [-a_2 \xi\bar{\xi} + b_2(v^* + c_2) + b_2 u^* \xi] \right\} \Gamma \tilde{t} \\ &+ \frac{\bar{z}^2}{2} \left\{ \gamma_1 \left[-\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\bar{\xi}^2 e^{2i\omega_n \tilde{t}} - b_1 v^* \bar{\xi} e^{i\omega_n \tilde{t}} \right] + \frac{1}{2}\gamma_2 [-a_2 \bar{\xi}^2 + b_2(v^* + c_2) + 2b_2 u^* \bar{\xi}] \right\} \Gamma \tilde{t} \\ &+ \frac{z^2 \bar{z}}{2} \tilde{t} k + \dots, \end{aligned} \quad (3.37)$$

where

$$\begin{aligned} k &= \gamma_1 \left\{ -2(a_1 + b_1 v^* \xi e^{-i\omega_n \tilde{t}}) \left\langle W_{11}^{(1)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle + a_1 b_1 v^* \bar{\xi} e^{i\omega_n \tilde{t}} \left\langle W_{20}^{(1)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \right. \\ &\quad - 2b_1[v^* + (u^* + c_1)\xi e^{-i\omega_n \tilde{t}}] \left\langle W_{11}^{(2)}(-1) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \\ &\quad \left. - b_1[(u^* + c_1)\bar{\xi} e^{i\omega_n \tilde{t}} + v^*] \left\langle W_{20}^{(2)}(-1) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \right\} \\ &+ \gamma_2 \left\{ b_2(v^* + c_2) \left\langle W_{11}^{(1)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle + (b_2 u^* - 2a_1 \xi) \left\langle W_{11}^{(2)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \right. \\ &\quad \left. + \frac{1}{2}b_2(v^* + c_2) \left\langle W_{20}^{(1)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle + \left(\frac{1}{2}b_2 u^* - a_1 \bar{\xi}\right) \left\langle W_{20}^{(2)}(0) \cos\left(\frac{nx}{l}\right), \cos\left(\frac{nx}{l}\right) \right\rangle \right\}. \end{aligned}$$

Then by (3.30), (3.25), and (3.37), we have $g_{20} = g_{11} = g_{02} = 0$ for $n \in \mathbb{N}$. If $n = 0$, we have the following quantities:

$$\begin{aligned} g_{20} &= \tilde{t} \gamma_1 \left[-\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi^2 e^{-2i\omega_n \tilde{t}} - b_1 v^* \xi e^{-i\omega_n \tilde{t}} \right] + \tilde{t} \gamma_2 a_2 \left[\frac{1}{2}a_2 \xi^2 - \frac{1}{2}b_2(v^* + c_2) - b_2 u^* \xi \right], \\ g_{11} &= -\frac{\tilde{t}}{2} \gamma_1 [a_1 + b_1(u^* + c_1)\xi\bar{\xi} + b_1 v^* (\xi e^{-i\omega_n \tilde{t}} + \bar{\xi} e^{i\omega_n \tilde{t}})] + \frac{\tilde{t}}{2} \gamma_2 [-a_2 \xi\bar{\xi} + b_2(v^* + c_2) + b_2 u^* \xi], \\ g_{02} &= \tilde{t} \gamma_1 \left[-\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\bar{\xi}^2 e^{2i\omega_n \tilde{t}} - b_1 v^* \bar{\xi} e^{i\omega_n \tilde{t}} \right] + \tilde{t} \gamma_2 \left[-\frac{1}{2}a_2 \bar{\xi}^2 + \frac{1}{2}b_2(v^* + c_2) + b_2 u^* \bar{\xi} \right]. \end{aligned}$$

And for $n \in \mathbb{N}_0$, $g_{21} = \tilde{t}k$.

Now, a complete expression for g_{21} depends on the algorithm for $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta \in [-1, 0)$ which we shall compute.

By [26], we have

$$\dot{W}(z, \bar{z}) = W_{20} z \dot{z} + W_{11} \dot{z} \bar{z} + W_{02} \bar{z} \dot{z} + \dots,$$

$$A_{\bar{t}} W(z, \bar{z}) = A_{\bar{t}} W_{20} \frac{z^2}{2} + A_{\bar{t}} W_{11} z \bar{z} + A_{\bar{t}} W_{02} \frac{\bar{z}^2}{2} + \cdots,$$

and $W(z, \bar{z})$ satisfies

$$\dot{W}(z, \bar{z}) = A_{\bar{t}} W + H(z, \bar{z}),$$

where

$$\begin{aligned} H(z, \bar{z}) &= H_{20} \frac{z^2}{2} + H_{11} z \bar{z} + H_{02} \frac{\bar{z}^2}{2} + \cdots \\ &= X_0 F(U_t, 0) - \Phi(\Psi, \langle X_0 F(U_t, 0), f_n \rangle \cdot f_n). \end{aligned} \quad (3.38)$$

Hence, we have

$$(2i\omega_n \tilde{t} - A_{\bar{t}}) W_{20} = H_{20}, \quad -A_{\bar{t}} W_{11} = H_{11}, \quad (-2i\omega_n \tilde{t} - A_{\bar{t}}) W_{02} = H_{02}, \quad (3.39)$$

that is,

$$W_{20} = (2i\omega_n \tilde{t} - A_{\bar{t}})^{-1} H_{20}, \quad W_{11} = -A_{\bar{t}}^{-1} H_{11}, \quad W_{02} = (-2i\omega_n \tilde{t} - A_{\bar{t}})^{-1} H_{02}. \quad (3.40)$$

By (3.37), we have that for $\theta \in [-1, 0)$,

$$\begin{aligned} H(z, \bar{z}) &= -\Phi(0) \Psi(0) \langle F(U_t, 0), f_n \rangle \cdot f_n \\ &= -\left(\frac{p_1(\theta) + p_2(\theta)}{2}, \frac{p_1(\theta) - p_2(\theta)}{2i} \right) \begin{pmatrix} \Phi_1(0) \\ \Phi_2(0) \end{pmatrix} \left\langle F(U_t, 0), f_n \right\rangle \cdot f_n \\ &= -\frac{1}{2} [p_1(\theta)(\Phi_1(0) - i\Phi_2(0)) + p_2(\theta)(\Phi_1(0) + i\Phi_2(0))] \langle F(U_t, 0), f_n \rangle \cdot f_n \\ &= -\frac{1}{2} \left[(p_1(\theta)g_{20} + p_2(\theta)\bar{g}_{02}) \frac{z^2}{2} + (p_1(\theta)g_{11} + p_2(\theta)\bar{g}_{11}) z \bar{z} + (p_1(\theta)g_{02} + p_2(\theta)\bar{g}_{20}) \frac{\bar{z}^2}{2} \right] + \cdots. \end{aligned}$$

Therefore, by (3.38), for $\theta \in [-1, 0)$,

$$\begin{aligned} H_{20}(\theta) &= \begin{cases} 0, & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\bar{g}_{02}) \cdot f_0, & n = 0, \end{cases} \\ H_{11}(\theta) &= \begin{cases} 0, & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{11} + p_2(\theta)\bar{g}_{11}) \cdot f_0, & n = 0, \end{cases} \\ H_{02}(\theta) &= \begin{cases} 0, & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{02} + p_2(\theta)\bar{g}_{02}) \cdot f_0, & n = 0, \end{cases} \end{aligned}$$

and

$$H(z, \bar{z})(0) = F(U_t, 0) - \Phi(\Psi, \langle F(U_t, 0), f_n \rangle \cdot f_n),$$

where

$$H_{20}(0) = \begin{cases} \tilde{t} \begin{pmatrix} -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi^2 e^{-2i\omega_n \tilde{t}} - b_1 v^* \xi e^{-i\omega_n \tilde{t}} \\ a_2 \left[\frac{1}{2}a_2 \xi^2 - \frac{1}{2}b_2(v^* + c_2) - b_2 u^* \xi \right] \end{pmatrix} \cos^2 \left(\frac{n\pi}{l} \right), & n \in \mathbb{N}, \\ \tilde{t} \begin{pmatrix} -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi^2 e^{-2i\omega_n \tilde{t}} - b_1 v^* \xi e^{-i\omega_n \tilde{t}} \\ a_2 \left[\frac{1}{2}a_2 \xi^2 - \frac{1}{2}b_2(v^* + c_2) - b_2 u^* \xi \right] \end{pmatrix} - \frac{1}{2}(p_1(0)g_{20} + p_2(0)\bar{g}_{02}) \cdot f_0, & n = 0, \end{cases}$$

$$\begin{aligned}
H_{11}(0) &= \begin{cases} \tilde{\tau} \begin{pmatrix} -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi\bar{\xi} - \frac{1}{2}b_1v^*(\xi e^{-i\omega_n\tilde{\tau}} + \bar{\xi}e^{i\omega_n\tilde{\tau}}) \\ \frac{1}{2}[-a_2\xi\bar{\xi} + b_2(v^* + c_2) + b_2u^*\xi] \end{pmatrix} \cos^2\left(\frac{n\pi}{l}\right), & n \in \mathbb{N}, \\ \tilde{\tau} \begin{pmatrix} -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi\bar{\xi} - \frac{1}{2}b_1v^*(\xi e^{-i\omega_n\tilde{\tau}} + \bar{\xi}e^{i\omega_n\tilde{\tau}}) \\ \frac{1}{2}[-a_2\xi\bar{\xi} + b_2(v^* + c_2) + b_2u^*\xi] \end{pmatrix} - \frac{1}{2}(p_1(0)g_{11} + p_2(0)\bar{g}_{11}) \cdot f_0, & n = 0, \end{cases} \\
H_{02}(0) &= \begin{cases} \tilde{\tau} \begin{pmatrix} -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\bar{\xi}^2e^{2i\omega_n\tilde{\tau}} - b_1v^*\bar{\xi}e^{i\omega_n\tilde{\tau}} \\ \frac{1}{2}[-a_2\bar{\xi}^2 + b_2(v^* + c_2) + 2b_2u^*\bar{\xi}] \end{pmatrix} \cos^2\left(\frac{n\pi}{l}\right), & n \in \mathbb{N}, \\ \tilde{\tau} \begin{pmatrix} -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\bar{\xi}^2e^{2i\omega_n\tilde{\tau}} - b_1v^*\bar{\xi}e^{i\omega_n\tilde{\tau}} \\ \frac{1}{2}[-a_2\bar{\xi}^2 + b_2(v^* + c_2) + 2b_2u^*\bar{\xi}] \end{pmatrix} - \frac{1}{2}(p_1(0)g_{02} + p_2(0)\bar{g}_{02}) \cdot f_0, & n = 0. \end{cases}
\end{aligned}$$

By the definition of $A_{\tilde{\tau}}$ and (3.39), we have

$$\dot{W}_{20} = A_{\tilde{\tau}} W_{20} = 2i\omega_n \tilde{\tau} W_{20} + \frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\bar{g}_{02}) \cdot f_n, \quad \theta \in [-1, 0).$$

Note that $p_1(\theta) = p_1(0)e^{i\omega_n\tilde{\tau}\theta}$, $-1 \leq \theta \leq 0$. That is,

$$W_{20}(\theta) = \frac{i}{2i\omega_n \tilde{\tau}} \left(g_{20}p_1(\theta) + \frac{\bar{g}_{02}}{3}p_2(\theta) \right) \cdot f_n + E_1 e^{2i\omega_n \tilde{\tau}\theta},$$

where

$$E_1 = \begin{cases} W_{20}(0), & n \in \mathbb{N}, \\ W_{20}(0) - \frac{i}{2i\omega_n \tilde{\tau}} \left(g_{20}p_1(0) + \frac{\bar{g}_{02}}{3}p_2(0) \right) \cdot f_n, & n = 0. \end{cases}$$

Using the definition of $A_{\tilde{\tau}}$ and (3.39), we have that for $\theta \in [-1, 0)$

$$\begin{aligned}
& - \left(g_{20}p_1(0) + \frac{\bar{g}_{02}}{3}p_2(0) \right) \cdot f_0 + 2i\omega_n \tilde{\tau} E_1 - A_{\tilde{\tau}} \left(\frac{i}{2\omega_n \tilde{\tau}} \left(g_{20}p_1(0) + \frac{\bar{g}_{02}}{3}p_2(0) \right) \cdot f_0 \right) \\
& - A_{\tilde{\tau}} E_1 - L_{\tilde{\tau}} \left(\frac{i}{2\omega_n \tilde{\tau}} \left(g_{20}p_1(0) + \frac{\bar{g}_{02}}{3}p_2(0) \right) \cdot f_n + E_1 e^{2i\omega_n \tilde{\tau}\theta} \right) \\
& = \tilde{\tau} \begin{pmatrix} -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi\bar{\xi} - \frac{1}{2}b_1v^*(\xi e^{-i\omega_n\tilde{\tau}} + \bar{\xi}e^{i\omega_n\tilde{\tau}}) \\ \frac{1}{2}[-a_2\xi\bar{\xi} + b_2(v^* + c_2) + b_2u^*\xi] \end{pmatrix} - \frac{1}{2}(g_{20}p_1(0) + \bar{g}_{02}p_2(0)) \cdot f_n.
\end{aligned}$$

As

$$A_{\tilde{\tau}} p_1(0) + L_{\tilde{\tau}}(p_1 \cdot f_0) = i\omega_0 p_1(0) \cdot f_0,$$

$$A_{\tilde{\tau}} p_2(0) + L_{\tilde{\tau}}(p_2 \cdot f_0) = i\omega_0 p_2(0) \cdot f_0,$$

we have

$$2i\omega_n E_1 - A_{\tilde{\tau}} E_1 - L_{\tilde{\tau}} E_1 e^{2i\omega_n \tilde{\tau}} = \tilde{\tau} \begin{pmatrix} -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi^2 e^{-2i\omega_n \tilde{\tau}} - b_1v^*\xi e^{-i\omega_n \tilde{\tau}} \\ a_2 \left[\frac{1}{2}a_2\xi^2 - \frac{1}{2}b_2(v^* + c_2) - b_2u^*\xi \right] \end{pmatrix} \cos^2\left(\frac{n\pi}{l}\right), \quad n \in \mathbb{N}_0.$$

That is,

$$E_1 = \tilde{\tau} E \begin{pmatrix} -\frac{1}{2}a_1 - \frac{1}{2}b_1(u^* + c_1)\xi^2 e^{-2i\omega_n \tilde{\tau}} - b_1v^*\xi e^{-i\omega_n \tilde{\tau}} \\ a_2 \left[\frac{1}{2}a_2\xi^2 - \frac{1}{2}b_2(v^* + c_2) - b_2u^*\xi \right] \end{pmatrix} \cos^2\left(\frac{n\pi}{l}\right),$$

where

$$E = \begin{pmatrix} 2i\omega_n \tilde{\tau} + e_1 \frac{n^2}{l^2} + a_1(u^* + c_1) & 2b_1 v^*(u^* + c_1) e^{-2i\omega_n \tilde{\tau}} \\ -2b_2 u^*(v^* + c_2) & 2i\omega_n \tilde{\tau} + e_2 \frac{n^2}{l^2} + a_2(v^* + c_2) \end{pmatrix}^{-1}.$$

Similarly, from (3.40), we have

$$-\dot{W}_{11} = \frac{i}{2\omega_n \tilde{\tau}} (p_1(\theta) g_{11} + p_2(\theta) \bar{g}_{11}) \cdot f_n, \quad \theta \in [-1, 0).$$

That is,

$$W_{11}(\theta) = \frac{i}{2\omega_n \tilde{\tau}} (p_1(\theta) \bar{g}_{11} - p_2(\theta) g_{11}) + E_2.$$

Similar to the procedure of computing W_{20} , we have

$$E_2 = \tilde{\tau} E^* \begin{pmatrix} -\frac{1}{2} a_1 - \frac{1}{2} b_1(u^* + c_1) \xi \bar{\xi} - \frac{1}{2} b_1 v^* (\xi e^{-i\omega_n \tilde{\tau}} + \bar{\xi} e^{i\omega_n \tilde{\tau}}) \\ \frac{1}{2} [-a_2 \xi \bar{\xi} + b_2(v^* + c_2) + b_2 u^* \xi] \end{pmatrix} \cos^2 \left(\frac{n\pi}{l} \right),$$

where

$$E^* = \begin{pmatrix} e_1 \frac{n^2}{l^2} + a_1(u^* + c_1) & 2b_1(u^* + c_1) \\ -2b_2 u^*(v^* + c_2) & e_2 \frac{n^2}{l^2} + a_2(v^* + c_2) \end{pmatrix}^{-1}.$$

So far, $W_{20}(\theta)$ and $W_{11}(\theta)$ have been expressed by the parameters of system (4), g_{21} can also be given. Thus, we can compute the following quantities which determine the direction and stability of bifurcating periodic orbits:

$$\begin{aligned} c_1(0) &= i/2\omega_n \tilde{\tau} (g_{11} g_{20} - 2|g_{11}|^2 - |g_{02}|^2/3) + g_{21}/2, \\ \mu_2 &= -\operatorname{Re}\{c_1(0)\} / \operatorname{Re}(\lambda'(\tau_n^j)), \\ \beta_2 &= 2 \operatorname{Re}\{c_1(0)\}, \\ T_2 &= -[\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}(\lambda'(\tau_n^j))] / \omega_n \tau. \end{aligned} \quad (3.41)$$

Then we have the following theorem.

Theorem 3. For any critical value τ_n^j , assume that (H_1) and (H_2) hold. Then

- (i) μ_2 determines the direction of the Hopf bifurcation. If $\mu_2 > 0$ (resp. $\mu_2 < 0$), then the Hopf bifurcation is supercritical (resp. subcritical), that is, the bifurcating periodic solutions exists for $\tau > \tau_n^j$ (resp. $\tau < \tau_n^j$);
- (ii) β_2 determines the stability of the bifurcating periodic solutions. If $\beta_2 < 0$ (resp. $\beta_2 > 0$), then bifurcating periodic solution is stable (resp. unstable);
- (iii) T_2 determines the period of the bifurcating periodic solutions. If $T_2 > 0$ (resp. $T_2 < 0$), then periods of periodic solutions increase (resp. decrease).

4 Numerical simulations

Through the previous discussion, we conclude that the delay term plays an important role in the diffusive delayed system, which can let the stable equilibrium unstable. In this section, we shall give some numerical simulations and actual conclusions to support the theoretical analysis discussed in the previous section.

First, we consider the following diffusive model contains no delay term

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = \Delta u + (u(t) + 1)(0.6 - 0.2u(t) - 0.4v^2(t)), & x \in \Omega, \quad t > 0, \\ \frac{\partial v(x, t)}{\partial t} = \Delta v + (v(t) + 3)(0.4 - 0.6v(t) + 0.2u^2(t)), & x \in \Omega, \quad t > 0, \\ u_x(0, t) = v_x(0, t) = 0, \quad u_x(3\pi, t) = v_x(3\pi, t) = 0, & t \geq 0, \\ u(x, 0) = 0.3 + 0.1 \cos x, \quad v(x, 0) = 0.5 + 0.1 \cos x, & x \in \overline{\Omega}, \end{cases} \quad (4.42)$$

which satisfies (H_1) , by computing, the positive equilibrium $E^* = (1, 1)$. By Theorem 1, we get that E^* is asymptotically stable as demonstrated in Figure 1(a) and (b) in the $u - x - t$ space and $v - x - t$ space, respectively. That is as time increases, the numerical solution tends to the positive equilibrium E^* .

Furthermore, we study the following specific diffusive model with delay term

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = \Delta u + (u(t) + 1)(0.6 - 0.2u(t) - 0.4v^2(t - \tau)), & x \in \Omega, \quad t > 0, \\ \frac{\partial v(x, t)}{\partial t} = \Delta v + (v(t) + 3)(0.4 - 0.6v(t) + 0.2u^2(t)), & x \in \Omega, \quad t > 0, \\ u_x(0, t) = v_x(0, t) = 0, \quad u_x(3\pi, t) = v_x(3\pi, t) = 0, & t \geq 0, \\ u(x, 0) = 0.3 + 0.1 \cos x, \quad v(x, 0) = 0.5 + 0.1 \cos x, & x \in \overline{\Omega}, \end{cases} \quad (4.43)$$

which satisfies (H_1) (H_3) . By computing, $E^* = (1, 1)$, $A_0 = 2.8$, $B_0 = 0.96$, $C = -2.56$, $\omega_0 \approx 0.8339$, $\tau_0^0 \approx 1.7594$. By Theorem 2, we get that E^* is asymptotically stable for $\tau \in [0, 1.7594)$ as displayed in Figures 2(a) and (b), where we choose $\tau = 1.5$. However, when τ crosses the critical value τ_0^0 , a family of inhomogeneous periodic solutions is bifurcated from E^* . We choose $\tau = 2$, E^* loses its stability and Hopf bifurcation occurs when τ crosses τ_0^0 as illustrated in Figure 3(a) and (b), respectively.

5 Conclusions

In this paper, a diffusive competition and cooperation system subject to local delayed feedback control under Neumann boundary value conditions has been studied in detail to show its rich spatial-temporal patterns. From the economical aspect, the most interesting results are the following: under certain hypotheses, the patterns caused by the Turing instability can be expected for the competition and cooperation model. In particular, it is interesting that delayed feedback control can break the stability of the system and stabilize the unstable oscillation in an originally spatially stable domain. With the increase of delay, the constant equilibrium may switch finite times from stability to instability to stability and become unstable, and a sequence of inhomogeneous periodic solutions bifurcates from the equilibrium eventually. That is, delayed feedback control plays an essential role in destabilizing the spatially extended system. Moreover, the short-term data observed in nature may be misleading to make predictions due to complex dynamical behaviors. The analysis and interesting observations in this paper may be useful both in the mathematical and economical research areas.

Acknowledgement: The authors would like to thank the referees for their valuable suggestions which greatly improved the presentation of this paper. This work was supported by Provincial Natural Science Research Project of Anhui Colleges (No. KJ2019A0672 and KJ2019A0666), Program for Excellent Young Talents in University of Anhui Province (No. gxyq2017092 and gxyq2018102), Teaching Research Project of Anhui Province (No. 2019jyxm0468) and Key Teaching Research Project of Suzhou University (No. szxy2018jy).

Figure 1: The positive equilibrium E^* of system (4.42) is locally asymptotically stable in the $u - x - t$ space and $v - x - t$ space, respectively.

Figure 2: The positive equilibrium E^* of system (4.43) is locally asymptotically stable in the $u - x - t$ space and $v - x - t$ space, respectively.

Figure 3: The positive equilibrium E^* of system (4.43) is unstable in the $u - x - t$ space and $v - x - t$ space, respectively.

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