Research Article

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Error term of the mean value theorem for binary Egyptian fractions

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Abstract: In this article, the error term of the mean value theorem for binary Egyptian fractions is studied. An error term of prime number theorem type is obtained unconditionally. Under Riemann hypothesis, a power saving can be obtained. The mean value in short interval is also considered.

Keywords: binary Egyptian fractions, error terms

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1 Introduction

Let $a, n \in \mathbb{N}^+$ and $(a, n) = 1$. Egyptian fractions concern the representation of rational numbers as the finite sum of distinct unit fractions:

$$\frac{a}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k},$$

with positive integers $x_1, x_2, \dots, x_k$. The problem has a long history and has attracted interests of many authors. For $k = 3$, the famous Erdős-Straus conjecture [1] states that the Diophantine equation

$$\frac{4}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$$

is always soluble. This conjecture is still open, although much work has been carried out. See [2–4], for example, for more details about the ternary conjecture. Refer to [5] for more information on Egyptian fractions.

Considering the binary Egyptian fractions when $k = 2$, define

$$R(n; a) = \text{card} \left\{ (x, y) \in \mathbb{N}^2 : \frac{a}{n} = \frac{1}{x} + \frac{1}{y} \right\}.$$

The Diophantine equation is not always necessary to have a solution. For example, for n with all its prime factor p of the form $p \equiv 1 \pmod{a}$, it has no such representation. Thus, it is natural to consider the mean value of $R(n; a)$. For $X > 0$, define

$$S(X; a) = \sum_{\substack{n \leq X \\ (n, a) = 1}} R(n; a).$$

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Huang and Vaughan [6] proved that

$$S(X; a) = D(X; a) + \Delta(X; a), \quad (1)$$

with

$$D(X; a) = \frac{3}{\pi^2 a} \left(\prod_{p|a} \frac{p-1}{p+1} \right) X ((\log X)^2 + C_1(a) \log X + C_0(a)) \quad (2)$$

and

$$\Delta(X; a) \ll \frac{a}{\phi(a)} X^{1/2} (\log X)^5 \prod_{p|a} (1 - p^{-1/2})^{-1}, \quad (3)$$

where

$$C_1(a) = 6\gamma - 4 \frac{\zeta'(2)}{\zeta(2)} - 2 + \sum_{p|a} \frac{6p+2}{p^2-1} \log p$$

and

$$C_0(a) = -2(\log a)^2 - 4(\log a) \sum_{p|a} \frac{\log p}{p-1} + O(a\phi^{-1}(a) \log a).$$

Here $\zeta(s)$ denotes the Riemann zeta function. In [7], Jia got a more explicit expression with better error term for $C_0(a)$. See [8–10] for more results about $R(n; a)$.

In [6], Huang and Vaughan employed for the first time in this area, of complex analytic technique from multiplicative number theory, and gave an innovative counting function with a different criterion with Croot et al. in [11]. Their estimate (3) holds uniformly for X and $a \in \mathbb{N}$ and the error term is almost optimal. The bound for the error here is as strong as can be established on generalized Riemann hypothesis (GRH) with their method. The aim of this paper is to improve the estimate (3) concerning X .

Theorem 1. Let $X > 0$ and $\Delta(X; a)$ be defined in (1). We have

$$\Delta(X; a) \ll a^{2\alpha+\varepsilon} X^{1/2} \exp\{-c(\log X)^{3/5} (\log \log X)^{-1/5}\},$$

with

$$\alpha = 1864/5073 = 0.36744 \dots$$

and some absolute constant $c > 0$. Here the implied constant depends only on ε .

Remark 1. Theorem 1 is better than (3) when a is small. Obviously, Theorem 1 is a PNT (prime number theorem) type result, which depends on the zero-free region of the Riemann zeta-function. So in some sense, it is the best possible result under the present methods in the analytic number theory. It is impossible to improve the exponent $1/2$ in Theorem 1 without better zero-free region of the Riemann zeta-function.

Under Riemann hypothesis (RH), we can prove the following power saving result.

Theorem 2. Let $X > 0$ and $\Delta(X; a)$ be defined in (1). Assuming RH, we have

$$\Delta(X; a) \ll a^{2\alpha+\varepsilon} X^{(1-\beta)/(3-4\beta)},$$

with α as above and

$$\beta = 2498/5073 = 0.4924108 \dots.$$

Here the implied constant depends only on ε .

Remark 2. One main tool in [6] is the well-known Perron formula. Our proof relies on the convolution method, the results concerning divisor problems with Dirichlet characters (see [12,13]) and moment results of the Dirichlet L -functions.

In the authors' another work [14], the mean square of the error term under RH was studied. An asymptotic formula can be obtained, which suggests that the average size of the error term is $O_a(X^{1/3+\varepsilon})$. Both Theorems 1 and 2 depend on zeros of $\zeta(s)$. To avoid this point, it is interesting to consider the average value in short intervals. For $X, Y > 0$, we define

$$S(X, Y; a) = \sum_{\substack{X \leq n \leq X+Y \\ (n, a)=1}} R(n; a).$$

Then we can get the following estimate.

Theorem 3. Let $\varepsilon > 0$, $X > a^{90}$ and $X^{\beta+\varepsilon} \leq Y \leq X$. We have

$$S(X, Y; a) = D(X + Y; a) - D(X; a) + O\left(a^\varepsilon \frac{Y}{X^\varepsilon} + a^{2\alpha+\varepsilon} X^{\beta+\varepsilon}\right),$$

with same α and β as above, where $D(\cdot; a)$ is defined by (2).

Remark 3. For the proof of Theorem 3, we use the technique of Zhai [15] who considered the short interval distribution of a class of integers, and his lemma (see Lemma 8) can help us to get some saving in short interval. The constraint for $X > a^{90}$ is needed for the character sum (see Lemma 2). We can also use Lemma 1 to get a slightly worse result if it is removed from the theorem.

In what follows, $c_1, c_2, \dots$ and $C_1, C_2, C_3, \dots$ denote absolute positive constants. Denote by ε small positive constant which may take different values at each occurrence. For $s \in \mathbb{C}$, we denote $s = \sigma + it$.

2 Preliminary

In this section, we list some lemmas which will be needed in the proof.

We cite a theorem of Friedlander and Iwaniec [12].

Lemma 1. Let $\chi_j(\bmod q_j)$, $j = 1, 2, 3$ be primitive characters and denote $Q = q_1 q_2 q_3$. For any $u \geq 1$, we have

$$\sum_{n_1 n_2 n_3 \leq u} \chi_1(n_1) \chi_2(n_2) \chi_3(n_3) = M_1(u) + O(Q^{2\alpha_0} u^{\beta_0+\varepsilon}),$$

with $\alpha_0 = 38/75$ and $\beta_0 = 37/75$, where the main term is given by

$$M_1(u) = \operatorname{Res}_{s=1} L(s, \chi_1) L(s, \chi_2) L(s, \chi_3) u^s / s,$$

and the implied constant depends only on ε .

For primitive character $\chi(\bmod q)$, define by

$$\sum_{n=1}^{\infty} \frac{d_3(n; \chi)}{n^s} = \zeta(s) L(s, \chi) L(s, \bar{\chi}), \quad (4)$$

whence we get

$$d_3(n; \chi) = \sum_{n=n_1 n_2 n_3} \chi(n_2) \bar{\chi}(n_3)$$

and

$$d_3(n; \chi) \ll d_3(n).$$

Also for $\chi = \chi_0$,

$$d_3(n; \chi_0) = d_3(n).$$

The following lemma is due to Nowak [13], which has more precise estimation than the theorem given by Friedlander and Iwaniec for the special case given below.

Lemma 2. Let $q > 1$. Let $\chi \pmod{q}$ be a primitive character. For any $u \geq q^2$ and $\lambda := \frac{2 \log q}{\log u}$, we have

$$\sum_{n \leq u} d_3(n; \chi) = M_2(u) + \Delta_0(u; \chi),$$

with

$$\Delta_0(u; \chi) \ll q^{2\alpha} u^{\beta+\varepsilon}$$

and

$$(\alpha, \beta) := \begin{cases} \left(\frac{1864}{5073}, \frac{2498}{5073} \right), & 0 < \lambda \leq \lambda_0 = \frac{2852}{25487} = 0.1119 \dots, \\ \left(\frac{4672}{12387}, \frac{6086}{12387} \right), & \frac{2852}{25487} < \lambda \leq 1. \end{cases}$$

Here the main term is given by

$$M_2(u) = |L(1, \chi)|^2 u,$$

and the implied constant depends only on ε .

Lemma 3. For any $u \geq 1$, we have

$$\sum_{n \leq u} d_3(n) = M_3(u) + O(u^{43/96+\varepsilon}),$$

with the implied constant depending only on ε . Here the main term is given by

$$M_3(u) = \operatorname{Res}_{s=1} \zeta^3(s) u^s / s = u P_2(\log u),$$

where $P_2(\cdot)$ is a polynomial of degree 2.

For $y > 0$, define

$$f_y(s) = \sum_{n > y} \frac{\mu(n)}{n^s}. \quad (5)$$

Then we have

Lemma 4. Assuming RH, then we have

$$f_y(s) \ll y^{1/2-\sigma+\varepsilon} (1 + |t|)^\varepsilon, \quad 1/2 + \varepsilon < \operatorname{Re} s \leq 1.$$

Proof. Well-known. □

The estimation for the mean value of the Möbius function depends on the zero-free region of Riemann zeta function (see for example Theorem 12.7 [16]).

Lemma 5. Let $u > 0$. There is an absolute constant $c_1 > 0$ such that

$$\sum_{n \leq u} \mu(n) \ll u \exp\{-c_1(\log u)^{3/5}(\log \log u)^{-1/5}\}. \quad (6)$$

The following results about moments of zeta function and Dirichlet L -functions are well-known (see [17] and Theorem 10.1 of [18]).

Lemma 6. *Let $T \geq 2$, then we have*

$$\int_{-T}^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \ll T \log T$$

and

$$\sum_{\chi(\bmod a)}^* \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \ll \phi(a) T (\log(aT))^4,$$

where $\sum^*$ indicates that the sum is over the primitive characters modulo a .

Lemma 7. *Let $T \geq 2$, then*

$$\sum_{\chi(\bmod a)} \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi^*\right) \right|^4 dt \ll a T (\log(aT))^4,$$

where χ^* is the primitive character inducing χ .

Proof. Let χ^* modulo q be a primitive character. Then we have

$$\sum_{\chi(\bmod a)} \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi^*\right) \right|^4 dt = \sum_{q|a} \sum_{\chi^*(\bmod q)}^* \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi^*\right) \right|^4 dt.$$

Then according to Lemma 6, we get

$$\sum_{\chi(\bmod a)} \int_{-T}^T \left| \left(\frac{1}{2} + it, \chi^*\right) \right|^4 dt \ll T (\log(aT))^4 \sum_{q|a} \phi(q) = a T (\log(aT))^4.$$

□

The following lemma is due to [15], which gives an upper bound for some summation in short intervals.

Lemma 8. *Let $x > 0$ and $x^{1/5+\varepsilon} \leq y \leq x$. Then*

$$\sum_{\substack{x < m\ell^2 < x+y \\ \ell > x^\varepsilon}} 1 \ll yx^{-\varepsilon} + x^{1/5} \log x.$$

3 Proof of Theorem 1

First of all, we assume that

$$(\log X)^{3/5} (\log \log X)^{-1/5} \gg \log a. \quad (7)$$

Otherwise, the theorem follows from the upper bound (3).

We start from the identity (see (3.1) in [6])

$$S(X; a) = \frac{1}{\phi(a)} \sum_{\chi \pmod{a}} \bar{\chi}(-1) \sum_{n \leq X} D_{\chi}(n), \quad (8)$$

where

$$D_{\chi}(n) = \bar{\chi}(n) \sum_{u|n^2} \chi(u).$$

For $\text{Re } s > 1$, define

$$F_{\chi}(s) := \sum_{n=1}^{\infty} \frac{D_{\chi}(n)}{n^s}.$$

Then $F_{\chi}(s)$ can be written as the form (see (3.2) in [6])

$$F_{\chi}(s) = \frac{L(s, \chi_0)}{L(2s, \chi_0)} L(s, \chi) L(s, \bar{\chi}), \quad \text{Re } s > 1,$$

where χ_0 is the principal character modulo a and $F_{\chi}(s)$ can be analytically continued to $\mathbb{C}$.

If χ is a non-principal character modulo q (χ^* is the primitive character induced from χ), we have

$$\begin{aligned} F_{\chi}(s) &= \frac{L(s, \chi_0)}{L(2s, \chi_0)} L(s, \chi) L(s, \bar{\chi}) \\ &= \frac{\zeta(s)}{\zeta(2s)} L(s, \chi^*) L(s, \bar{\chi}^*) \frac{\prod_{p|a} (1 - p^{-s})(1 - \chi^*(p)p^{-s})(1 - \bar{\chi}^*(p)p^{-s})}{\prod_{p|a} (1 - p^{-2s})} \\ &=: \frac{\zeta(s)}{\zeta(2s)} L(s, \chi^*) L(s, \bar{\chi}^*) \sum_{n=1}^{\infty} \frac{h_1(n; \chi)}{n^s}. \end{aligned}$$

If $\chi = \chi_0$, we have

$$F_{\chi}(s) = \frac{L^3(s, \chi_0)}{L(2s, \chi_0)} = \frac{\zeta(s)^3}{\zeta(2s)} \frac{\prod_{p|a} (1 - p^{-s})^3}{\prod_{p|a} (1 - p^{-2s})} =: \frac{\zeta(s)^3}{\zeta(2s)} \sum_{n=1}^{\infty} \frac{h_2(n; \chi_0)}{n^s}.$$

Noting that $h_i(n, \chi)$, $i = 1, 2$, is supported on $n|a^{\infty}$ and the series

$$\sum_{n=1}^{\infty} \frac{h_i(n, \chi)}{n^s}$$

converges absolutely for $\sigma > 0$, we have

$$\sum_{n \leq u} |h_i(n, \chi)| \ll (au)^{\varepsilon}, \quad u > 0. \quad (9)$$

Hence, we can write

$$\sum_{n \leq X} D_{\chi}(n) = \sum_{n_3 \leq X} h(n_3; \chi) I\left(\frac{X}{n_3}; \chi^*\right), \quad (10)$$

where

$$h(\cdot; \chi) := \begin{cases} h_1(\cdot; \chi), & \chi \neq \chi_0, \\ h_2(\cdot; \chi), & \chi = \chi_0, \end{cases}$$

and for $T > 2$ and primitive character χ^* modulo $q|a$

$$I(T; \chi^*) := \sum_{n_1 n_2^2 \leq T} d_3(n_1; \chi^*) \mu(n_2) \quad (11)$$

with $d_3(\cdot; \chi^*)$ being defined by (4).

Proposition 1. Let $T \gg a^4$ and $\chi^* \neq \chi_0$. We have

$$I(T; \chi^*) = \frac{6}{\pi^2} T |L(1, \chi^*)|^2 + O(a^{2\alpha} T^{1/2} \exp\{-c_2(\log T)^{3/5}(\log \log T)^{-1/5}\}). \quad (12)$$

Proof. When $\chi^* \neq \chi_0$, we write the sum (11) as

$$\begin{aligned} I(T; \chi^*) &= \sum_{n_1 \leq z} d_3(n_1; \chi^*) \sum_{n_2 \leq (T/n_1)^{1/2}} \mu(n_2) + \sum_{n_2 \leq (T/z)^{1/2}} \mu(n_2) \sum_{n_1 \leq T/n_2^2} d_3(n_1; \chi^*) - \sum_{n_2 \leq (T/z)^{1/2}} \mu(n_2) \sum_{n_1 \leq z} d_3(n_1; \chi^*) \\ &=: \sum_1 + \sum_2 - \sum_3, \end{aligned}$$

where $a^2 \ll z < \sqrt{T}$ is a parameter to be determined.

For the first sum, by using (6) and Lemma 3, we get

$$\begin{aligned} \sum_1 &\ll \sum_{n_1 \leq z} d_3(n_1) \sqrt{\frac{T}{n_1}} \exp\left\{-c_1 \left(\log \frac{T}{n_1}\right)^{3/5} \left(\log \log \frac{T}{n_1}\right)^{-1/5}\right\} \\ &\ll T^{1/2} \exp\{-c_3(\log T)^{3/5}(\log \log T)^{-1/5}\} \sum_{n_1 \leq z} \frac{d_3(n_1)}{n_1^{1/2}} \\ &\ll T^{1/2} \exp\{-c_3(\log T)^{3/5}(\log \log T)^{-1/5}\} z^{1/2} \log^2 z. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \sum_3 &\ll \sqrt{\frac{T}{z}} \exp\left\{-c_1 \left(\log \sqrt{\frac{T}{z}}\right)^{3/5} \left(\log \log \sqrt{\frac{T}{z}}\right)^{-1/5}\right\} \cdot z \log^2 z \\ &\ll T^{1/2} \exp\{-c_4(\log T)^{3/5}(\log \log T)^{-1/5}\} z^{1/2} \log^2 z. \end{aligned}$$

Since $T \gg a^4$, as a consequence of Lemma 2, we have

$$\sum_2 = \sum_{n_2 \leq (T/z)^{1/2}} \mu(n_2) (M(T/n_2^2) + O(a^{2\alpha} (T/n_2^2)^{\beta+\varepsilon})) = \sum_{21} + O(a^{2\alpha} T^{1/2} z^{-1/2+\beta+\varepsilon}), \quad (13)$$

with

$$\sum_{21} = \sum_{n_2 \leq (T/z)^{1/2}} \mu(n_2) \operatorname{Res}_{s=1} \zeta(s) L(s, \chi^*) L(s, \bar{\chi}^*) (T/n_2^2)^s / s.$$

Then, for $\chi^* \neq \chi_0$, we have by using (6)

$$\begin{aligned} \sum_{21} &= \sum_{n_2 \leq (T/z)^{1/2}} \mu(n_2) |L(1, \chi^*)|^2 (T/n_2^2) = T |L(1, \chi^*)|^2 \sum_{n_2 \leq (T/z)^{1/2}} \frac{\mu(n_2)}{n_2^2} \\ &= T |L(1, \chi^*)|^2 \zeta(2)^{-1} + O(T^{1/2} |L(1, \chi^*)|^2 z^{1/2} \exp\{-c_5(\log T)^{3/5}(\log \log T)^{-1/5}\}). \end{aligned}$$

Taking

$$z = \exp\{c_0(\log T)^{3/5}(\log \log T)^{-1/5}\}$$

for some sufficiently small $c_0 > 0$, we can get the result. $\square$

Remark 4. If we remove the constraint $T > a^4$, we can employ Lemma 1 instead of Lemma 2 in (13). Then the power 2α of a will be replaced by $2\alpha_0$.

Proposition 2. Let $T > 0$. We have

$$I(T; \chi_0) = T \sum_{n_2} \frac{\mu(n_2)}{n_2^2} \left(\frac{1}{2} \left(\log \frac{T}{n_2^2} \right)^2 + C_1 \log \frac{T}{n_2^2} + C_2 \right) + O(T^{1/2} \exp\{-c_6(\log T)^{3/5}(\log \log T)^{-1/5}\}). \quad (14)$$

Proof. We can apply similar argument to the proof above and get

$$I(T; \chi_0) = \mathcal{D}(T, z) + O(T^{1/2} z^{1/2+\varepsilon} \exp\{-c_7(\log T)^{3/5}(\log \log T)^{-1/5}\}),$$

with

$$\mathcal{D}(T, z) = \sum_{n_2 \leq (T/z)^{1/2}} \mu(n_2) \operatorname{Res}_{s=1} \zeta^3(s) \frac{(T/n_2^2)^s}{s} = T \sum_{n_2 \leq (T/z)^{1/2}} \frac{\mu(n_2)}{n_2^2} \left(\frac{1}{2} \left(\log \frac{T}{n_2^2} \right)^2 + C_1 \log \frac{T}{n_2^2} + C_2 \right),$$

where C_1 and C_2 are absolute constants. By using (6) again, we have

$$\mathcal{D}(T, z) = T \sum_{n_2} \frac{\mu(n_2)}{n_2^2} \left(\frac{1}{2} \left(\log \frac{T}{n_2^2} \right)^2 + C_1 \log \frac{T}{n_2^2} + C_2 \right) + O(T^{1/2} z^{1/2} \exp\{-c_8(\log T)^{3/5}(\log \log T)^{-1/5}\}),$$

which implies (14) by taking z as the proof in Proposition 1. $\square$

By an appeal to (8), (10) and (12), we have

$$S_1(X; a) := \frac{1}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) \sum_{n_3 \leq X} h(n_3; \chi) I\left(\frac{X}{n_3}; \chi^*\right) = \mathcal{M}_1(X, a) + \mathcal{E}_1(X, a), \quad (15)$$

with

$$\mathcal{M}_1(X, a) = \frac{6}{\pi^2} \frac{X}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) |L(1, \chi^*)|^2 \sum_{n_3 \leq X} \frac{h(n_3; \chi)}{n_3}$$

and

$$\begin{aligned} \mathcal{E}_1 &\ll \frac{a^{2\alpha}}{\phi(a)} X^{1/2} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} |\bar{\chi}(-1)| \sum_{n_3 \leq X/a^{20}} \frac{|h(n_3; \chi)|}{n_3^{1/2}} \exp\left\{-c_2 \left(\log \frac{X}{n_3}\right)^{3/5} \left(\log \log \frac{X}{n_3}\right)^{-1/5}\right\} \\ &+ \frac{a^{2\alpha_0}}{\phi(a)} X^{1/2} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} |\bar{\chi}(-1)| \sum_{n_3 > X/a^{20}} \frac{|h(n_3; \chi)|}{n_3^{1/2}} \exp\left\{-c_2 \left(\log \frac{X}{n_3}\right)^{3/5} \left(\log \log \frac{X}{n_3}\right)^{-1/5}\right\}. \end{aligned}$$

Thanks to (7) and (9), the first term on the right is

$$\ll a^{2\alpha+\varepsilon} X^{1/2} \exp\{-c_9(\log X)^{3/5}(\log \log X)^{-1/5}\}$$

and the second term on the right is

$$\ll a^{10+2\alpha_0+\varepsilon} X^\varepsilon.$$

Therefore, we have

$$\mathcal{E}_1(X, a) \ll a^{2\alpha+\varepsilon} X^{1/2} \exp\{-c_9(\log X)^{3/5}(\log \log X)^{-1/5}\}. \quad (16)$$

For the main term,

$$\mathcal{M}_1(X, a) = \frac{6}{\pi^2} \prod_{p|a} \left(\frac{p}{p+1} \right) \frac{X}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) |L(1, \chi)|^2 + O((aX)^\varepsilon). \quad (17)$$

Therefore, we obtain

$$S_1(X; a) = \frac{6}{\pi^2} \prod_{p|a} \left(\frac{p}{p+1} \right) \frac{X}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) |L(1, \chi)|^2 + O(a^{2\alpha+\varepsilon} X^{1/2} \exp\{-c_9(\log X)^{3/5}(\log \log X)^{-1/5}\}). \quad (18)$$

By an appeal to (8), (10) and (14), we have

$$S_2(X; a) := \frac{1}{\phi(a)} \sum_{n_3 \leq X} h(n_3; \chi_0) I\left(\frac{X}{n_3}; \chi_0\right) = \mathcal{M}_2(X, a) + \mathcal{E}_2(X, a), \quad (19)$$

where

$$\mathcal{M}_2(X, a) = \frac{X}{\phi(a)} \sum_{n_3} \frac{h(n_3; \chi)}{n_3} \sum_{n_2} \frac{\mu(n_2)}{n_2^2} \left[\frac{1}{2} \left(\log \frac{X}{n_2^2 n_3} \right)^2 + C_1 \log \frac{X}{n_2^2 n_3} + C_2 \right] + R(X) \quad (20)$$

with

$$R(X) \ll \frac{a^\varepsilon}{\phi(a)} X^\varepsilon$$

and

$$\mathcal{E}_2(X, a) \ll \frac{a^\varepsilon}{\phi(a)} X^{1/2} \exp\{-c_{10}(\log X)^{3/5}(\log \log X)^{-1/5}\}.$$

It is easy to check that the main term in (20) equals to

$$\frac{1}{\phi(a)} \operatorname{Res}_{s=1} \frac{\zeta^3(s)}{\zeta(2s)} \sum_n \frac{h(n, \chi)}{n^s} \frac{X^s}{s}.$$

Therefore, we obtain

$$S_2(X; a) = \frac{1}{\phi(a)} \operatorname{Res}_{s=1} \frac{\zeta^3(s)}{\zeta(2s)} \prod_{p|a} \frac{(p^s - 1)^2}{p^s(p^s + 1)} \frac{X^s}{s} + O\left(\frac{a^\varepsilon X^{1/2}}{\phi(a)} \exp\{-c_{10}(\log X)^{3/5}(\log \log X)^{-1/5}\}\right). \quad (21)$$

Combining (18) and (21), and according to the computation of Huang and Vaughan in [6] about the residue, we get

$$S(X; a) = C_a X ((\log X)^2 + C_1(a) \log X + C_0(a)) + O\left(a^{2a+\varepsilon} X^{1/2} \exp\left\{-c \frac{(\log X)^{3/5}}{(\log \log X)^{1/5}}\right\}\right),$$

where C_a , $C_1(a)$ and $C_0(a)$ are defined in (1).

4 Proof of Theorem 2

It is easy to get the upper bound from (3) when $X \leq a^{99}$. So we assume

$$X \geq a^{99}.$$

We start from (11). Let $1 < y \ll X^{1/2}$ be a parameter to be chosen. We write the sum as

$$I(T; \chi^*) = \sum_{n_2 \leq y} \mu(n_2) \sum_{n_1 \leq \frac{T}{n_2^2}} d_3(n_1; \chi^*) + \sum_{n_2 > y} \mu(n_2) \sum_{n_1 \leq \frac{T}{n_2^2}} d_3(n_1; \chi^*) =: \mathfrak{S}_1 + \mathfrak{S}_2. \quad (22)$$

We write according to Lemmas 2 and 3

$$\sum_{n_1 \leq \frac{T}{n_2^2}} d_3(n_1; \chi^*) = M\left(\frac{T}{n_2^2}; \chi^*\right) + \Delta_0\left(\frac{T}{n_2^2}; \chi^*\right),$$

where

$$M(u; \chi^*) = \operatorname{Res}_{s=1} \zeta(s) L(s, \chi^*) L(s, \overline{\chi^*}) u^s / s,$$

and for $u \geq a^2$

$$\Delta_0(u) = \Delta_0(u; \chi^*) \ll \begin{cases} a^{2\alpha} u^{\beta+\varepsilon}, & \chi^* \neq \chi_0, \\ u^{43/96+\varepsilon}, & \chi^* = \chi_0. \end{cases} \quad (23)$$

Then

$$\begin{aligned} \mathfrak{S}_1 &= \sum_{n_2 \leq y} \mu(n_2) M\left(\frac{T}{n_2^2}; \chi^*\right) + \sum_{n_2 \leq y} \mu(n_2) \Delta_0\left(\frac{T}{n_2^2}; \chi^*\right) \\ &=: \sum_{n_2 \leq y} \mu(n_2) M\left(\frac{T}{n_2^2}; \chi^*\right) + \mathfrak{S}_1^*(y, T, \chi^*). \end{aligned} \quad (24)$$

By (23) and Lemma 2 (here the exponent 18 in the following is chosen to make sure that $\lambda < \lambda_0$), for $\chi^* \neq \chi_0$,

$$\mathfrak{S}_1^*(y, T, \chi^*) \ll \begin{cases} a^{2\alpha} T^{\beta+\varepsilon} y^{1-2\beta+\varepsilon}, & y \leq T^{1/2}/a^{18}, \\ a^{2\alpha_0} T^{\beta_0+\varepsilon} y^{1-2\beta_0+\varepsilon}, & y \geq T^{1/2}/a^{18}, \end{cases} \quad (25)$$

and for $\chi^* = \chi_0$

$$\mathfrak{S}_1^*(y, T, \chi_0) \ll T^{43/96+\varepsilon} y^{5/48+\varepsilon}. \quad (26)$$

It remains to compute $\mathfrak{S}_2$. We first consider the case $\chi^* \neq \chi_0$. Noting that

$$\sum_n \sum_{\substack{n=n_1 n_2^2 \\ n_2 > y}} \frac{d_3(n_1; \chi^*) \mu(n_2)}{n^s} = \zeta(s) L(s, \chi^*) L(s, \overline{\chi^*}) f_y(2s),$$

with $f_y(s)$ defined by (5), as a consequence of Perron's formula

$$\mathfrak{S}_2 = \frac{1}{2\pi i} \int_{1+\frac{1}{\log T}-i\mathfrak{T}}^{1+\frac{1}{\log T}+i\mathfrak{T}} \zeta(s) L(s, \chi^*) L(s, \overline{\chi^*}) f_y(2s) \frac{T^s}{s} ds + O\left(\frac{T^{1+\varepsilon}}{\mathfrak{T}}\right), \quad (27)$$

for $\mathfrak{T} = T^{10}$.

Move the line of integral in (27) to $\operatorname{Re} s = 1/2$. The residue theorem gives

$$\frac{1}{2\pi i} \int_{1+\frac{1}{\log T}-i\mathfrak{T}}^{1+\frac{1}{\log T}+i\mathfrak{T}} \zeta(s) L(s, \chi^*) L(s, \overline{\chi^*}) f_y(2s) \frac{T^s}{s} ds = \operatorname{Res}_{s=1} \zeta(s) L(s, \chi^*) L(s, \overline{\chi^*}) f_y(2s) \frac{T^s}{s} + I_{\chi^*}(T) + I_h(T),$$

with

$$I_{\chi^*}(T) = \frac{1}{2\pi i} \int_{1/2-i\mathfrak{T}}^{1/2+i\mathfrak{T}} \zeta(s) L(s, \chi^*) L(s, \overline{\chi^*}) f_y(2s) \frac{T^s}{s} ds \quad (28)$$

and

$$I_h(T) = \frac{1}{2\pi i} \left(\int_{1/2+i\mathfrak{T}}^{1+\frac{1}{\log T}+i\mathfrak{T}} + \int_{1+\frac{1}{\log T}-i\mathfrak{T}}^{1/2-i\mathfrak{T}} \right) \zeta(s) L(s, \chi^*) L(s, \overline{\chi^*}) f_y(2s) \frac{T^s}{s} ds.$$

Assuming RH, using convexity bound for Dirichlet L -function and Lemma 4, the contribution from the horizontal path is

$$I_h(T) \ll a^{1/2+\varepsilon} \mathfrak{T}^{-1/2+\varepsilon} y^{-1/2+\varepsilon} T. \quad (29)$$

Thus, we have

$$\mathfrak{S}_2 = \operatorname{Res}_{s=1} \zeta(s) L(s, \chi^*) L(s, \overline{\chi^*}) f_y(2s) \frac{T^s}{s} + I_{\chi^*}(T) + I_h(T) + O(T^{-9+\varepsilon}). \quad (30)$$

Combining (22), (24), (25), (29) and (30), we can get for $\chi^* \neq \chi_0$

$$\begin{aligned} I(T; \chi^*) &= \sum_{n_1 n_2^2 \leq T} d_3(n_1; \chi^*) \mu(n_2) = \operatorname{Res}_{s=1} \zeta(s) L(s, \chi^*) L(s, \overline{\chi^*}) \sum_{n_2 \leq y} \mu(n_2) \frac{T^s}{n^{2s}} \frac{1}{s} \\ &\quad + \operatorname{Res}_{s=1} \zeta(s) L(s, \chi^*) L(s, \overline{\chi^*}) \sum_{n_2 > y} \mu(n_2) \frac{T^s}{n^{2s}} \frac{1}{s} + I_{\chi^*}(T) + \mathfrak{R}_1(y, T), \end{aligned}$$

with $I_{\chi^*}(T)$ defined by (28) and

$$\mathfrak{R}_1(y, T) \ll \begin{cases} a^{2\alpha} T^{\beta+\varepsilon} y^{1-2\beta+\varepsilon}, & y \leq T^{1/2}/a^{18}, \\ a^{2\alpha_0} T^{\beta_0+\varepsilon} y^{1-2\beta_0+\varepsilon}, & y > T^{1/2}/a^{18}. \end{cases} \quad (31)$$

It implies

$$\sum_{n_1 n_2^2 \leq T} d_3(n_1; \chi^*) \mu(n_2) = \operatorname{Res}_{s=1} \frac{\zeta(s) L(s, \chi^*) L(s, \overline{\chi^*})}{\zeta(2s)} \frac{T^s}{s} + I_{\chi^*}(T) + \mathfrak{R}_1(y, T). \quad (32)$$

Similarly, we can get for $\chi^* = \chi_0$

$$\sum_{n_1 n_2^2 \leq T} d_3(n_1) \mu(n_2) = \operatorname{Res}_{s=1} \frac{\zeta^3(s)}{\zeta(2s)} \frac{T^s}{s} + I_{\chi_0}(T) + \mathfrak{R}_2(y, T), \quad (33)$$

with

$$I_{\chi_0}(T) = \frac{1}{2\pi i} \int_{1/2-i\mathfrak{T}}^{1/2+i\mathfrak{T}} \zeta^3(s) f_y(2s) \frac{T^s}{s} ds \quad (34)$$

and

$$\mathfrak{R}_2(y, T) \ll T^{43/96+\varepsilon} y^{5/48+\varepsilon},$$

with the help of (26).

Insert (32) and (33) into (10) and take $T = X/n_3$. We get for $\chi \neq \chi_0$

$$\sum_{n \leq X} D_{\chi}(n) = \sum_{n_3 \leq X} h(n_3; \chi) \left(\operatorname{Res}_{s=1} \frac{\zeta(s) L(s, \chi^*) L(s, \overline{\chi^*})}{\zeta(2s)} \frac{(X/n_3)^s}{s} + I_{\chi^*} \left(\frac{X}{n_3} \right) + \mathfrak{R}_1 \left(y, \frac{X}{n_3} \right) \right),$$

and for $\chi^* = \chi_0$

$$\sum_{n \leq X} D_{\chi_0}(n) = \sum_{n_3 \leq X} h(n_3; \chi_0) \left(\operatorname{Res}_{s=1} \frac{\zeta^3(s)}{\zeta(2s)} \frac{(X/n_3)^s}{s} + I_{\chi_0} \left(\frac{X}{n_3} \right) + \mathfrak{R}_2 \left(y, \frac{X}{n_3} \right) \right).$$

Let $S_1(X; a)$ be defined by (15). Then

$$\begin{aligned} S_1(X; a) &= \frac{1}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) \sum_{n_3 \leq X} h(n_3; \chi) \times \left(\operatorname{Res}_{s=1} \frac{\zeta(s) L(s, \chi^*) L(s, \overline{\chi^*})}{\zeta(2s)} \frac{(X/n_3)^s}{s} + I_{\chi^*} \left(\frac{X}{n_3} \right) + \mathfrak{R}_1 \left(y, \frac{X}{n_3} \right) \right) \\ &= \mathcal{M}_1(X; a) + \mathcal{E}_1(X, a) + \mathcal{E}'_1(X, a), \end{aligned}$$

where $\mathcal{M}_1(X; a)$ is the same with (17), say,

$$\mathcal{M}_1(X; a) = \frac{6}{\pi^2} \prod_{p|a} \left(\frac{p}{p+1} \right) \frac{X}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) |L(1, \chi)|^2 + O(a^\varepsilon X^\varepsilon),$$

$$\mathcal{E}'_1(X; a) = \frac{1}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) \sum_{n_3 \leq X} h(n_3; \chi) \Re_1 \left(y, \frac{X}{n_3} \right)$$

and

$$\mathcal{E}_1(X; a) = \frac{1}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) \sum_{n_3 \leq X} h(n_3; \chi) I_{\chi^*} \left(\frac{X}{n_3} \right).$$

According to (31), we have

$$\mathcal{E}'_1(X; a) \ll \frac{1}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) \sum_{n_3 \leq X/a^{36}y^2} h(n_3; \chi) \Re_1 \left(y, \frac{X}{n_3} \right) + \frac{1}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) \sum_{n_3 > X/a^{36}y^2} h(n_3; \chi) \Re_1 \left(y, \frac{X}{n_3} \right).$$

Thus, we have

$$\mathcal{E}'_1(X; a) \ll a^{2\alpha} X^{\beta+\varepsilon} y^{1-2\beta+\varepsilon} + a^{2\alpha_0+36\beta_0+\varepsilon} X^\varepsilon y^{1+\varepsilon}.$$

Thus, under RH, we can improve (18) by

$$S_1(X; a) = \frac{6}{\pi^2} \prod_{p|a} \left(\frac{p}{p+1} \right) \frac{X}{\phi(a)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) |L(1, \chi)|^2 + \mathcal{E}_1(X; a) + \mathcal{E}'_1(X; a). \quad (35)$$

Let $S_2(X; a)$ be defined by (19). Then

$$\begin{aligned} S_2(X; a) &= \frac{1}{\phi(a)} \sum_{n_3 \leq X} h(n_3; \chi_0) \left(\operatorname{Res}_{s=1} \frac{\zeta^3(s)}{\zeta(2s)} \frac{(X/n_3)^s}{s} + I_{\chi_0} \left(\frac{X}{n_3} \right) + \Re_2 \left(y, \frac{X}{n_3} \right) \right) \\ &= \mathcal{M}_2(X; a) + \mathcal{E}_2(X; a) + \mathcal{E}'_2(X; a), \end{aligned}$$

where $\mathcal{M}_2(X; a)$ is the same with (20), say,

$$\mathcal{M}_2(X; a) = \frac{X}{\phi(a)} \sum_{n_3} \frac{h_1(n_3; \chi_0)}{n_3} \sum_{n_2} \left[\frac{1}{2} \left(\log \frac{X}{n_2^2 n_3} \right)^2 + C_1 \log \frac{X}{n_2^2 n_3} + C_2 \right] + O(a^\varepsilon X^\varepsilon)$$

with

$$\mathcal{E}'_2(X; a) \ll \frac{a^\varepsilon}{\phi(a)} X^{43/96+\varepsilon} y^{5/48+\varepsilon}$$

and

$$\mathcal{E}_2(X; a) = \frac{1}{\phi(a)} \sum_{n_3 \leq X} h(n_3; \chi_0) I_{\chi_0} \left(\frac{X}{n_3} \right).$$

Then, under RH, we can improve (21) by

$$S_2(X; a) = \frac{1}{\phi(a)} \operatorname{Res}_{s=1} \frac{\zeta^3(s)}{\zeta(2s)} \prod_{p|a} \frac{(p^s - 1)^2}{p^s(p^s + 1)} \frac{X^s}{s} + \mathcal{E}_2(X; a) + \mathcal{E}'_2(X; a). \quad (36)$$

In order to estimate $\mathcal{E}_1(X; a)$ and $\mathcal{E}_2(X; a)$, we have

$$\begin{aligned}\mathcal{E}_1(X; a) + \mathcal{E}_2(X; a) &= \frac{1}{\phi(a)} \sum_{\chi \pmod{a}} \bar{\chi}(-1) \sum_{n_3 \leq X} h(n_3; \chi) I_{\chi^*} \left(\frac{X}{n_3} \right) \\ &= \frac{1}{\phi(a)} \sum_{\chi \pmod{a}} \bar{\chi}(-1) \sum_{n_3 \leq X} h(n_3; \chi) \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \zeta(s) L(s, \chi^*) L(s, \bar{\chi}^*) f_Y(2s) \frac{(X/n_3)^s}{s} ds.\end{aligned}$$

Noting that $\mathfrak{T} = (X/n_3)^{10}$, Lemma 4 implies that

$$\mathcal{E}_1(X; a) + \mathcal{E}_2(X; a) \ll y^{-\frac{1}{2}+\varepsilon} \frac{a^\varepsilon}{\phi(a)} X^{1/2+\varepsilon} \sum_{\chi \pmod{a}} \sum_{2^k \leq \mathfrak{T}} \int_{2^{k-1}}^{2^k} \left| \zeta \left(\frac{1}{2} + it \right) \right| \left| L \left(\frac{1}{2} + it, \chi^* \right) \right|^2 \frac{1}{|t|} dt.$$

Then Hölder's inequality indicates

$$\begin{aligned}\mathcal{E}_1(X; a) + \mathcal{E}_2(X; a) &\ll y^{-\frac{1}{2}+\varepsilon} \frac{a^\varepsilon}{\phi(a)} X^{1/2+\varepsilon} \sum_{2^k \leq \mathfrak{T}} 2^{-k} \sum_{\chi \pmod{a}} \int_{2^{k-1}}^{2^k} \left| \zeta \left(\frac{1}{2} + it \right) \right| \left| L \left(\frac{1}{2} + it, \chi^* \right) \right|^2 dt \\ &\ll y^{-\frac{1}{2}+\varepsilon} \frac{a^\varepsilon}{\phi(a)} X^{1/2+\varepsilon} \sum_{2^k \leq \mathfrak{T}} 2^{-k} \left(\sum_{\chi} \int_{2^{k-1}}^{2^k} \left| \left(\frac{1}{2} + it \right) \right|^2 dt \right)^{1/2} \left(\sum_{\chi} \int_{2^{k-1}}^{2^k} \left| L \left(\frac{1}{2} + it, \chi^* \right) \right|^4 dt \right)^{1/2}.\end{aligned}$$

Therefore, we obtain by Lemmas 6 and 7

$$\mathcal{E}_1(X; a) + \mathcal{E}_2(X; a) \ll y^{-\frac{1}{2}+\varepsilon} X^{1/2+\varepsilon} \frac{a^{1+\varepsilon}}{\phi(a)} (\log aX)^4. \quad (37)$$

Taking (α, β) in Lemma 2 and $y = X^{(1-2\beta)/(3-4\beta)}$, combining (35), (36) and (37), we can complete the proof.

5 Mean value in short interval

Appeal to (8), (10) and (11), we can write

$$S(X, Y; a) = \frac{1}{\phi(a)} \sum_{\chi \pmod{a}} \bar{\chi}(-1) \sum_{n_1 \leq X+Y} h(n_1; \chi) \sum_{\substack{\frac{X}{n_1} < m\ell^2 < \frac{X+Y}{n_1}}} \mu(\ell) d_3(m; \chi^*).$$

Observing that

$$\frac{1}{\phi(a)} \sum_{\chi \pmod{a}} \bar{\chi}(-1) \sum_{n_1 > X^{3/4}} h(n_1; \chi) \sum_{\substack{\frac{X}{n_1} < m\ell^2 < \frac{X+Y}{n_1}}} \mu(\ell) d_3(m; \chi^*) \ll \frac{1}{\phi(a)} \sum_{\chi \pmod{a}} \sum_{n_1 > X^{3/4}} |h(n_1; \chi)| \sum_{\substack{\frac{X}{n_1} < m\ell^2 < \frac{X+Y}{n_1}}} d_3(m) \ll a^\varepsilon X^{1/4+\varepsilon},$$

we have

$$S(X, Y; a) = \frac{1}{\phi(a)} \sum_{\chi \pmod{a}} \bar{\chi}(-1) \sum_{n_1 \leq X^{3/4}} h(n_1; \chi) \sum_{\substack{\frac{X}{n_1} < m\ell^2 < \frac{X+Y}{n_1}}} \mu(\ell) d_3(m; \chi^*) + O(a^\varepsilon X^{1/4+\varepsilon}).$$

For the cases where characters are non-principal, denote by

$$\mathfrak{P}_{11} := \frac{1}{\phi(a)} \sum_{\chi \neq \chi_0 \pmod{a}} \bar{\chi}(-1) \sum_{n_1 \leq X^{3/4}} h(n_1; \chi) \sum_{\substack{\frac{X}{n_1} < m\ell^2 < \frac{X+Y}{n_1} \\ \ell \leq (X/n_1)^\varepsilon}} \mu(\ell) d_3(m; \chi^*)$$

and

$$\mathfrak{P}_{12} := \frac{1}{\phi(a)} \sum_{\chi \neq \chi_0 \pmod{a}} \bar{\chi}(-1) \sum_{n_1 \leq X^{3/4}} h(n_1; \chi) \sum_{\substack{\frac{X}{n_1} < m\ell^2 < \frac{X+Y}{n_1} \\ \ell > (X/n_1)^\varepsilon}} \mu(\ell) d_3(m; \chi^*).$$

For principal character case, denote by

$$\mathfrak{P}_{21} := \frac{1}{\phi(a)} \sum_{n_1 \leq X^{3/4}} h(n_1; \chi_0) \sum_{\substack{\frac{X}{n_1} < m\ell^2 < \frac{X+Y}{n_1} \\ \ell \leq (X/n_1)^\varepsilon}} \mu(\ell) d_3(m)$$

and

$$\mathfrak{P}_{22} := \frac{1}{\phi(a)} \sum_{n_1 \leq X^{3/4}} h(n_1; \chi_0) \sum_{\substack{\frac{X}{n_1} < m\ell^2 < \frac{X+Y}{n_1} \\ \ell > (X/n_1)^\varepsilon}} \mu(\ell) d_3(m).$$

Thus, we have

$$S(X, Y; a) = \mathfrak{P}_{11} + \mathfrak{P}_{12} + \mathfrak{P}_{21} + \mathfrak{P}_{22} + O(a^\varepsilon X^{1/4+\varepsilon}). \quad (38)$$

Proposition 3. *We have*

$$\mathfrak{P}_{11} = \operatorname{Res}_{s=1} \frac{\zeta(s)}{\zeta(2s)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) L(s, \chi^*) L(s, \bar{\chi}^*) \sum_{n=1}^{\infty} \frac{h_1(n; \chi)}{n^s} \frac{(X+Y)^s - X^s}{s} + O(X^{-\varepsilon} Y a^\varepsilon + a^{2\alpha+\varepsilon} X^{\beta+\varepsilon}).$$

Proof. Rewrite $\mathfrak{P}_{11}$ as the form

$$\mathfrak{P}_{11} = \frac{1}{\phi(a)} \sum_{\chi \neq \chi_0 \pmod{a}} \bar{\chi}(-1) \sum_{n_1 \leq X^{3/4}} h(n_1; \chi) \sum_{\ell \leq (X/n_1)^\varepsilon} \mu(\ell) \sum_{\substack{\frac{X}{n_1 \ell^2} < m < \frac{X+Y}{n_1 \ell^2}}} d_3(m; \chi^*).$$

Then according to Lemma 2, for $X \geq a^{90}$, we have

$$\mathfrak{P}_{11} = \frac{1}{\phi(a)} \sum_{\chi \neq \chi_0 \pmod{a}} \bar{\chi}(-1) \sum_{n_1 \leq X^{3/4}} h(n_1; \chi) \sum_{\ell \leq (X/n_1)^\varepsilon} \mu(\ell) \times \left(\frac{Y}{n_1 \ell^2} |L(1, \chi^*)|^2 + O\left(a^{2\alpha} \left(\frac{X}{n_1 \ell^2}\right)^{\beta+\varepsilon}\right) \right). \quad (39)$$

The main term above is

$$\begin{aligned} M_{11} &:= \frac{Y}{\phi(a)} \sum_{\chi \neq \chi_0 \pmod{a}} \bar{\chi}(-1) |L(1, \chi^*)|^2 \sum_{n_1 \leq X^{3/4}} \frac{h(n_1; \chi)}{n_1} \sum_{\ell \leq (X/n_1)^\varepsilon} \frac{\mu(\ell)}{\ell^2} \\ &= \frac{Y}{\phi(a)} \sum_{\chi \neq \chi_0 \pmod{a}} \bar{\chi}(-1) |L(1, \chi^*)|^2 \sum_{n_1 \leq X^{3/4}} \frac{h(n_1; \chi)}{n_1} \left(\frac{6}{\pi^2} + O\left(\left(\frac{X}{n_1}\right)^{-\varepsilon}\right) \right). \end{aligned}$$

Then we can get easily

$$M_{11} = \operatorname{Res}_{s=1} \frac{\zeta(s)}{\zeta(2s)} \sum_{\substack{\chi \pmod{a} \\ \chi \neq \chi_0}} \bar{\chi}(-1) L(s, \chi^*) L(s, \bar{\chi}^*) \sum_{n=1}^{\infty} \frac{h_1(n; \chi)}{n^s} \frac{(X+Y)^s - X^s}{s} + O(a^\varepsilon X^{-\varepsilon} Y).$$

The error term in (39) is

$$E_{11} \ll a^{2\alpha+\varepsilon} X^{\beta+\varepsilon}.$$

□

Proposition 4. *We have*

$$\mathfrak{P}_{21} = \frac{1}{\phi(a)} \operatorname{Res}_{s=1} \frac{\zeta^3(s)}{\zeta(2s)} \sum_{n=1}^{\infty} \frac{h_2(n; \chi_0)}{n^s} \frac{(X+Y)^s - X^s}{s} + O\left(\frac{a^\varepsilon(YX^{-\varepsilon} + X^{43/96+\varepsilon})}{\phi(a)}\right).$$

Proof. We can write

$$\mathfrak{P}_{21} = \frac{1}{\phi(a)} \sum_{n_1 \leq X^{3/4}} h(n_1; \chi_0) \sum_{\ell \leq (X/n_1)^\varepsilon} \mu(\ell) \sum_{\substack{\frac{X}{n_1 \ell^2} < m < \frac{X+Y}{n_1 \ell^2}}} d_3(m).$$

Lemma 3 implies

$$\mathfrak{P}_{21} = \frac{1}{\phi(a)} \sum_{n_1 \leq X^{3/4}} h(n_1; \chi_0) \sum_{\ell \leq (X/n_1)^\varepsilon} \mu(\ell) \times \left(\frac{X+Y}{n_1 \ell^2} P_2\left(\log \frac{X+Y}{n_1 \ell^2}\right) - \frac{X}{n_1 \ell^2} P_2\left(\log \frac{X}{n_1 \ell^2}\right) + O\left(\left(\frac{X}{n_1 \ell^2}\right)^{43/96+\varepsilon}\right) \right).$$

Observing that

$$(X+Y)P_2\left(\log \frac{X+Y}{n_1 \ell^2}\right) - XP_2\left(\log \frac{X}{n_1 \ell^2}\right) \ll Y(\log^2 X + \log^2 n_1 \ell^2),$$

it is easy to check that

$$\mathfrak{P}_{21} = \frac{1}{\phi(a)} \sum_{n_1 \leq X^{3/4}} h(n_1; \chi_0) \sum_{\ell} \mu(\ell) \left(\frac{X+Y}{n_1 \ell^2} P_2\left(\log \frac{X+Y}{n_1 \ell^2}\right) - \frac{X}{n_1 \ell^2} P_2\left(\log \frac{X}{n_1 \ell^2}\right) \right) + O(a^\varepsilon \phi(a)^{-1} (YX^{-\varepsilon} + X^{43/96+\varepsilon})).$$

Removing the constraint for n_1 , we can get a smaller error term. Thus,

$$\mathfrak{P}_{21} = \frac{1}{\phi(a)} \sum_{n_1} h(n_1; \chi_0) \sum_{\ell} \mu(\ell) \left(\frac{X+Y}{n_1 \ell^2} P_2\left(\log \frac{X+Y}{n_1 \ell^2}\right) - \frac{X}{n_1 \ell^2} P_2\left(\log \frac{X}{n_1 \ell^2}\right) \right) + O(a^\varepsilon \phi(a)^{-1} (YX^{-\varepsilon} + X^{43/96+\varepsilon})),$$

which infers

$$\mathfrak{P}_{21} = \frac{1}{\phi(a)} \operatorname{Res}_{s=1} \frac{\zeta^3(s)}{\zeta(2s)} \sum_{n=1}^{\infty} \frac{h_2(n; \chi_0)}{n^s} \frac{(X+Y)^s - X^s}{s} + O\left(a^\varepsilon \frac{YX^{-\varepsilon} + X^{43/96+\varepsilon}}{\phi(a)}\right).$$

□

Proposition 5. *We have*

$$\mathfrak{P}_{12} \ll a^\varepsilon X^{\varepsilon^2} (YX^{-\varepsilon} + X^{1/5} \log X)$$

and

$$\mathfrak{P}_{22} \ll a^\varepsilon \frac{X^{\varepsilon^2}}{\phi(a)} (YX^{-\varepsilon} + X^{1/5} \log X).$$

Proof. In order to use Lemma 8, we shall cut the length of the summation over n_1 and get

$$\mathfrak{P}_{12} \ll \frac{X^{\varepsilon^2}}{\phi(a)} \sum_{\chi \neq \chi_0 \pmod{a}} \sum_{n_1 \leq X^{3/4}} |h(n_1; \chi)| \sum_{\substack{\frac{X}{n_1} < m \ell^2 < \frac{X+Y}{n_1} \\ \ell > (X/n_1)^\varepsilon}} 1 + O(a^\varepsilon YX^{-1/3+\varepsilon}).$$

According to Lemma 8, we get

$$\mathfrak{P}_{12} \ll \frac{X^{\varepsilon^2}}{\phi(a)} \sum_{\chi \neq \chi_0 \pmod{a}} \sum_{n_1 \leq X^{3/4}} |h(n_1; \chi)| \left(\frac{Y}{n_1} \left(\frac{X}{n_1} \right)^{-\varepsilon} + \left(\frac{X}{n_1} \right)^{1/5} \log \frac{X}{n_1} \right) + O(a^\varepsilon YX^{-1/3+\varepsilon}).$$

Then it is easy to check

$$\mathfrak{P}_{12} \ll a^\varepsilon X^{\varepsilon^2} (YX^{-\varepsilon} + X^{1/5} \log X).$$

Similar argument can get the second assertion. $\square$

Combining (38) and Propositions 3–5, we can deduce Theorem 3.

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References

- [1] P. Erdős, *AZ $\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = \frac{a}{b}$ egyenlet egész számú megoldásairól (On a Diophantine equation)*, Mat. Lapok **1** (1950), 192–210.
- [2] C. Elsholtz and T. Tao, *Counting the number of solutions to the Erdős-Straus equation on unit fractions*, J. Aust. Math. Soc. **94** (2013), no. 1, 50–105.
- [3] C. Jia, *The estimate for mean values on prime numbers relative to $\frac{4}{p} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$* , Sci. China Math. **55** (2012), 465–474.
- [4] R. C. Vaughan, *On a problem of Erdős, Straus and Schinzel*, Mathematika **17** (1970), 193–198.
- [5] R. K. Guy, *Unsolved Problems in Number Theory*, 2nd edition, Springer-Verlag, 1994.
- [6] J. Huang and R. C. Vaughan, *Mean value theorems for binary Egyptian fractions*, Acta Arith. **155** (2012), 287–296.
- [7] C. Jia, *Mean value from representation of rational number as sum of two Egyptian fractions*, J. Number Theory **132** (2012), 701–713.
- [8] T. D. Browning and C. Elsholtz, *The number of representations of rationals as a sum of unit fractions*, Illinois J. Math. **55** (2011) no. 2, 685–696.
- [9] J. Huang and R. C. Vaughan, *Mean value theorems for binary Egyptian fractions II*, J. Number Theory **131** (2011), 1641–1656.
- [10] J. Huang and R. C. Vaughan, *On the exceptional set for binary Egyptian fractions*, Bull. Lond. Math. Soc. **45** (2013), 861–874.
- [11] C. Croot, D. Dobbs, J. Friedlander, A. Hetzel, and F. Pappalardi, *Binary Egyptian fractions*, J. Number Theory **84** (2000), 63–79.
- [12] J. Friedlander and H. Iwaniec, *Summation formulae for coefficients of L-functions*, Canad. J. Math. **57** (2005), no. 3, 494–505.
- [13] W. G. Nowak, *Refined estimates for exponential sums and a problem concerning the product of three l-series*, in: C. Elsholtz, P. Grabner (eds.), Number Theory-Diophantine Problems, Uniform Distribution and Applications, Springer, Cham, 2017, pp. 333–345.
- [14] X. X. Xiao and W. G. Zhai, *Mean square of the error term of the mean value of binary Egyptian fractions*, Front. Math. China **15** (2020), no. 1, 183–204.
- [15] W. G. Zhai, *Short interval results for a class of integers*, Monatsh. Math. **140** (2003), 233–257.
- [16] A. Ivic, *The Riemann Zeta-Functions: Theory and Applications*, reprint ed., Dover Publications, 2003.
- [17] G. H. Hardy and J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math. **41** (1916), no. 1, 119–196.
- [18] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Springer-Verlag, 1971.