Research Article

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Instantaneous blow-up of solutions to the Cauchy problem for the fractional Khokhlov-Zabolotskaya equation

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Abstract: In this paper, we consider the Cauchy problem for a second-order nonlinear equation with mixed fractional derivatives related to the fractional Khokhlov-Zabolotskaya equation. We prove the nonexistence of a classical local in time solution. The obtained instantaneous blow-up result is proved via the nonlinear capacity method.

Keywords: nonlinear capacity, blow-up, fractional differential equations

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1 Introduction

The study of the blow-up phenomena for many initial-boundary value problems that arise in particular in the theory of ion sound waves in plasma [1], waves in semiconductors [2] and electric oscillations [3] was detailed in many studies with the aim to obtain global or local unsolvability results by using the method of test functions.

The Khokhlov-Zabolotskaya equation arose in the theory of nonlinear underwater acoustics. This equation has the following form (see [4]):

$$\partial_t \partial_x u(t, w) - \frac{c_0}{2} \Delta_{(y,z)} u(t, w) = \frac{\varepsilon}{2c_0^2} \partial_t^2 u^2(t, w),$$

for $t > 0$, $w \in \mathbb{R}^3$, c_0 is the equilibrium speed of sound and ε is a nonlinear acoustic parameter. Here

$$\Delta_{(y,z)} u(t, w) := (\partial_y^2 + \partial_z^2) u(t, w), \quad \text{for } w = (x, y, z) \in \mathbb{R}^3.$$

Recently, in [5], the Cauchy problem for the aforementioned second-order nonlinear equation with mixed derivatives with the initial data

$$u(0, w) = u_0(w) \quad \text{and} \quad u_t(0, w) = u_1(w), \quad \text{for } w \in \mathbb{R}^3,$$

was studied. In particular, the blow-up of its classical local-in-time solution is proved.

Based on the nonlinear capacity method developed in [6], in this paper, we obtain an instantaneous blow-up result for the fractional version of the Khokhlov-Zabolotskaya equation. More precisely, let $T > 0$ be fixed, $\alpha \in (0, 1)$ and $Q_T = [0, T] \times \mathbb{R}^3$. First, we consider the problem

$$\partial_t^\alpha \partial_x u(t, w) - \Delta_{(y,z)} u(t, w) = \frac{1}{2} \partial_t^{\alpha+1} u^2(t, w), \quad (t, w) \in Q_T, \quad (1)$$

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under the initial data

$$u(0, w) = u_0(w) \quad \text{and} \quad u_t(0, w) = u_1(w), \quad \text{for } w \in \mathbb{R}^3, \quad (2)$$

where $\partial_t^\alpha u$ is the left-sided Liouville-Caputo derivative of order α of the function u .

Then, for $\alpha, \beta \in (0, 1)$ we will be interested in the system

$$\begin{cases} \partial_t^\alpha \partial_x u(t, w) - \Delta_{(y,z)} u(t, w) = \frac{1}{2} \partial_t^{\alpha+1} v^2(t, w) & \text{in } Q_T, \\ \partial_t^\beta \partial_x v(t, w) - \Delta_{(y,z)} v(t, w) = \frac{1}{2} \partial_t^{\beta+1} u^2(t, w) & \text{in } Q_T, \end{cases} \quad (3)$$

under the initial data

$$u(0, w) = u_0(w), \quad u_t(0, w) = u_1(w), \quad v(0, w) = v_0(w) \quad \text{and} \quad v_t(0, w) = v_1(w) \quad \text{for } w \in \mathbb{R}^3. \quad (4)$$

2 Preliminaries

Recall that, for $\alpha \in (0, 1)$ and $\Psi \in C^2[0, T]$, the left-sided and right-sided fractional derivatives of orders α and $\alpha + 1$ of the function Ψ are given by

$$\partial_t^\alpha \Psi(t) = \mathcal{I}_{0|t}^{1-\alpha}(\Psi_t)(t) \quad \text{and} \quad \partial_t^{1+\alpha} \Psi(t) = \mathcal{I}_{0|t}^{1-\alpha}(\Psi_{tt})(t), \quad (5)$$

where

$$\mathcal{I}_{0|t}^\alpha(\Psi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Psi(s) ds \quad \text{and} \quad \mathcal{I}_{t|T}^\alpha(\Psi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} \Psi(s) ds \quad (6)$$

are the left-sided and right-sided fractional integrals of order α . It is easy (see [7]) to see that for all $\alpha \in (0, 1)$ and $\Psi, \Phi \in C[0, T]$, we have

$$\int_0^T (\mathcal{I}_{0|t}^\alpha \Psi)(t) \Phi(t) dt = \int_0^T \Psi(t) (\mathcal{I}_{t|T}^\alpha \Phi)(t) dt. \quad (7)$$

For $T > 0$, $\alpha \in (0, 1)$ and $R \gg 1$, we define

$$[0, T] \ni t \mapsto v(t) := \left(1 - \frac{t}{T}\right)^{1+\alpha}, \quad \text{and} \quad \mathbb{R}^3 \ni w \mapsto \mu(w) := \xi^4 \left(\frac{x^2 + R^2(y^2 + z^2)}{R^4} \right), \quad (8)$$

with $\xi \in C^\infty([0, \infty))$ be defined by

$$\xi(r) = \begin{cases} 1, & 0 \leq r \leq 1, \\ \searrow, & 1 \leq r \leq 2, \\ 0, & r \geq 2. \end{cases}$$

Also, we denote by $Q_{T,R} = [0, T] \times B_R$, where

$$B_R = \left\{ w \in \mathbb{R}^3; \quad 1 \leq \frac{x^2 + R^2(y^2 + z^2)}{R^4} \leq 2 \right\}.$$

Using (6) and (8), it is easy to show that

Lemma 2.1. *Let $\alpha \in (0, 1)$ and $k \in \{0, 1, 2\}$. Then*

$$(\mathcal{I}_{t|T}^{1-\alpha} v)^{(k)}(t) = (-1)^k \frac{\Gamma(2+\alpha)}{\Gamma(3-k)} T^{-(1+\alpha)} (T-t)^{2-k}, \quad \text{for all } t \in [0, T].$$

Lemma 2.2. For all $R > 0$, there exists $c_\xi > 0$ depending only on ξ such that

$$\int_{B_R} \frac{|\Delta_{(y,z)}\mu(w)|^2 + |\partial_x\mu(w)|^2}{\mu(w)} dw \leq c_\xi.$$

3 Results

Let $T > 0$, be fixed, $\alpha \in (0, 1)$ and $R \gg 1$. We define

$$Q_T \ni (t, w) \mapsto \eta(t, w) := v(t)\mu(w),$$

where the function v and μ are given in (8).

We have the following result:

Theorem 3.1. Let u_0 and u_1 be two functions defined on $\mathbb{R}^3$ such that $u_0 \in L^2(\mathbb{R}^3)$ and $u_0 u_1 \in L^1(\mathbb{R}^3)$. Then

- If $u_0 \equiv 0$ a.e. and $u_1 \not\equiv 0$, then the instantaneous blow-up occurs for any solution to (1)–(2).
- If $u_0 \not\equiv 0$ and $\int_{\mathbb{R}^3} u_0(w)u_1(w)dw < 0$, then the time period existence $T_{\max}$ for any local solution to (1)–(2) satisfies

$$T_{\max} \leq -\frac{\int_{\mathbb{R}^3} u_0(w)u_1(w)dw}{\int_{\mathbb{R}^3} u_0^2(w)dw},$$

and for all $0 < T < T_{\max}$, we have

$$\int_{Q_T} u^2(t, w) dt dw \leq \int_{\mathbb{R}^3} (Tu_0^2(w) + T^2 u_0(w)u_1(w)) dw.$$

Theorem 3.2. Let u_0, v_0, u_1 and v_1 be defined on $\mathbb{R}^3$ such that $u_0, v_0 \in L^2(\mathbb{R}^3)$, and $u_0 u_1, v_0 v_1 \in L^1(\mathbb{R}^3)$. Then

- If $u_0 \equiv 0$ a.e. and $u_1 \not\equiv 0$ or $v_0 \equiv 0$ a.e. and $v_1 \not\equiv 0$, then the instantaneous blow-up occurs for the solution of systems (3) and (4).
- If $u_0 \not\equiv 0, v_0 \not\equiv 0, \int_{\mathbb{R}^3} u_0(w)u_1(w)dw < 0$ and $\int_{\mathbb{R}^3} v_0(w)v_1(w)dw < 0$, then the period existence $T_{\max}$ for any local pair solution to (3) and (4) satisfies

$$T_{\max} \leq -\max \left\{ \frac{\int_{\mathbb{R}^3} u_0 u_1 dw}{\int_{\mathbb{R}^3} u_0^2 dw}, \frac{\int_{\mathbb{R}^3} v_0 v_1 dw}{\int_{\mathbb{R}^3} v_0^2 dw} \right\},$$

and for all $0 < T < T_{\max}$, we have

$$\int_{Q_T} u^2 dt dw \leq \int_{\mathbb{R}^3} (Tu_0^2 + T^2 u_0 u_1) dw \quad \text{and} \quad \int_{Q_T} v^2 dt dw \leq \int_{\mathbb{R}^3} (Tv_0^2 + T^2 v_0 v_1) dw.$$

Proof. Assume that there exists a solution u to (1) and (2), defined on Q_T for some $T > 0$. Then multiplying (1) by η , we get

$$\frac{1}{2} \int_{Q_T} \eta(\partial_t^{\alpha+1} u^2) dt dw = \int_{Q_T} \eta(\partial_t^\alpha \partial_x u) dt dw - \int_{Q_T} \eta(\Delta_{(y,z)} u) dt dw. \quad (9)$$

Thanks to (7), have

$$\begin{aligned}
\int_{Q_T} \eta(t, w) (\partial_t^\alpha \partial_x u)(t, w) dt dw &= \int_0^T v(t) \left(\int_{\mathbb{R}^3} \partial_x (\partial_t^\alpha u)(t, w) \mu(w) dw \right) dt \\
&= - \int_0^T v(t) \left(\int_{\mathbb{R}^3} (\partial_t^\alpha u)(t, w) \partial_x \mu(w) dw \right) dt \\
&= - \int_{B_R} \partial_x \mu(w) \left(\int_0^T (\mathcal{I}_{0|t}^{1-\alpha} u_t)(t, w) v(t) dt \right) dw \\
&= - \int_{B_R} \partial_x \mu(w) \left(\int_0^T u_t(t, w) (\mathcal{I}_{t|T}^{1-\alpha} v)(t) dt \right) dw \\
&= \int_{Q_{T,R}} \partial_x \mu(w) (\mathcal{I}_{t|T}^{1-\alpha} v)_t(t) dt dw - \int_{B_R} \partial_x \mu(w) [u(\mathcal{I}_{t|T}^{1-\alpha} v)_t(t)]_{t=0}^{t=T} dw.
\end{aligned}$$

Therefore, from Lemma 2.1, we deduce that

$$\int_{Q_T} \eta(\partial_t^\alpha \partial_x u) dt dw = \frac{\Gamma(2 + \alpha) T^{1-\alpha}}{2} \left(\int_{B_R} u_0 \partial_x \mu dw - 2T^{-2} \int_{Q_{T,R}} u \partial_x \mu (T - t) dt dw \right). \quad (10)$$

Similarly, using (7) and Lemma 2.1, we obtain

$$\begin{aligned}
\int_{Q_T} \eta(\partial_t^{\alpha+1} u^2) dt dw &= \int_{\mathbb{R}^3} \mu(w) \left(\int_0^T v(t) (\partial_t^{1+\alpha} u^2)(t) dt \right) dw \\
&= \int_{\mathbb{R}^3} \mu(w) \left(\int_0^T (\mathcal{I}_{0|t}^{1-\alpha} u^2)_{tt}(t, w) v(t) dt \right) dw \\
&= \int_{\mathbb{R}^3} \mu(w) \left(\int_0^T (u^2)_{tt}(t, w) (\mathcal{I}_{t|T}^{1-\alpha} v)(t) dt \right) dw \\
&= \int_{Q_T} u^2 \mu(w) (\mathcal{I}_{t|T}^{1-\alpha} v)_{tt}(t) dt dw + \int_{\mathbb{R}^3} \mu(w) [(u^2)_t (\mathcal{I}_{t|T}^{1-\alpha} v) - u^2 (\mathcal{I}_{t|T}^{1-\alpha} v)_t]_{t=0}^{t=T} dw \\
&= \Gamma(2 + \alpha) T^{-(1+\alpha)} \left(\int_{Q_T} u^2 \mu dt dw - \int_{\mathbb{R}^3} (Tu_0^2 + T^2 u_0 u_1) \mu dw \right).
\end{aligned} \quad (11)$$

On the other hand,

$$\int_{Q_T} \eta(t, w) (\Delta_{(y,z)} u)(t, w) dt dw = \int_{Q_{R,T}} u(t, w) v(t) (\Delta_{(y,z)} \mu)(w) dt dw. \quad (12)$$

Collecting (10)–(12), we deduce from (9) that

$$\int_{Q_T} u^2 \mu dt dw \leq c_T \left(\int_{Q_{R,T}} |u(\Delta_{(y,z)} \mu)| dt dw + \int_{Q_{R,T}} |u \partial_x \mu| dt dw + \int_{B_T} |u_0 \partial_x \mu| dw \right) + \int_{\mathbb{R}^3} (Tu_0^2 + T^2 u_0 u_1) \mu dw. \quad (13)$$

Applying ε -Young inequality, we get

$$\int_{Q_{R,T}} |u|(|\Delta_{(y,z)}\mu| + |\partial_x\mu|) dt dw \leq \varepsilon \int_{Q_{R,T}} u^2 \mu dt dw + c_\varepsilon \int_{B_R} \frac{|\Delta_{(y,z)}\mu|^2 + |\partial_x\mu|^2}{\mu} dw.$$

Using this and Lemma 2.2, (13) becomes

$$\int_{Q_T} u^2 \mu dt dw \leq c_T \left(1 + \int_{B_T} |u_0 \partial_x \mu| dw \right) + \int_{\mathbb{R}^3} (Tu_0^2 + T^2 u_0 u_1) \mu dw.$$

Since $u_0^2, u_0 u_1 \in L^1(\mathbb{R}^3)$, then $\int_{B_T} |u_0 \partial_x \mu| dw$ tends to zero as R tends to $+\infty$. This yields

$$\int_{Q_T} u^2 \mu dt dw \leq c_T + \int_{\mathbb{R}^3} (Tu_0^2 + T^2 u_0 u_1) dw.$$

Hence u belongs to $L^2(\mathbb{R}^3)$.

Now, applying Hölder to (13), we obtain

$$\int_{Q_T} u^2 \mu dt dw \leq c_T \left(\int_{Q_{R,T}} u^2 \mu dt dw \right)^{\frac{1}{2}} \left(\int_{B_R} \frac{|\Delta_{(y,z)}\mu|^2 + |\partial_x\mu|^2}{\mu} dw \right)^{\frac{1}{2}} + \int_{B_T} |u_0 \partial_x \mu| dw + \int_{\mathbb{R}^3} (Tu_0^2 + T^2 u_0 u_1) \mu dw.$$

Finally, since $u \in L^2(Q_T)$, then $\int_{Q_{R,T}} u^2 \mu dt dw$ tends to zero as R tends to $+\infty$. This yields

$$\int_{Q_T} u^2(t, w) dt dw \leq \int_{\mathbb{R}^3} (Tu_0^2(w) + T^2 u_0(w) u_1(w)) dw.$$

Then the result of Theorem 3.1 holds.

Assume that there exists a pair (u, v) solution to system (3)–(4), defined on Q_T for some $T > 0$. Then

$$\frac{1}{2} \int_{Q_T} \eta(\partial_t^{\alpha+1} v^2) dt dw = \int_{Q_T} \eta(\partial_t^\alpha \partial_x u) dt dw - \int_{Q_T} \eta(\Delta_{(y,z)} u) dt dw$$

and

$$\frac{1}{2} \int_{Q_T} \eta(\partial_t^{\beta+1} u^2) dt dw = \int_{Q_T} \eta(\partial_t^\beta \partial_x v) dt dw - \int_{Q_T} \eta(\Delta_{(y,z)} v) dt dw.$$

The analogue of the estimation (13), for system (3)–(4), is the following

$$\int_{Q_T} v^2 \mu dt dw \leq c_T \left(\int_{Q_{R,T}} |u(\Delta_{(y,z)}\mu)| dt dw + \int_{Q_{R,T}} |u \partial_x \mu| dt dw + \int_{B_T} |u_0 \partial_x \mu| dw \right) + \int_{\mathbb{R}^3} (Tv_0^2 + T^2 v_0 v_1) \mu dw, \quad (14)$$

together with

$$\int_{Q_T} u^2 \mu dt dw \leq c_T \left(\int_{Q_{R,T}} |v(\Delta_{(y,z)}\mu)| dt dw + \int_{Q_{R,T}} |v \partial_x \mu| dt dw + \int_{B_T} |v_0 \partial_x \mu| dw \right) + \int_{\mathbb{R}^3} (Tu_0^2 + T^2 u_0 u_1) \mu dw. \quad (15)$$

As in the case of single equation, we use ε -Young and Hölder inequalities to deduce that

$$\int_{Q_T} u^2 dt dw \leq \int_{\mathbb{R}^3} (Tu_0^2 + T^2 u_0 u_1) dw \quad \text{and} \quad \int_{Q_T} v^2 dt dw \leq \int_{\mathbb{R}^3} (Tv_0^2 + T^2 v_0 v_1) dw.$$

Therefore, the result in Theorem 3.2 holds. $\square$

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References

- [1] Mohamed Jleli, *Instantaneous blow-up for a fractional-in-time evolution equation arising in plasma theory*, Math. Methods Appl. Sci. (2020), DOI: 10.1002/mma.6309.
- [2] Maxim O. Korpusov, Aleksandra K. Matveeva, and Dmitry V. Lukyanenko, *Diagnostics of instant decomposition of solution in the nonlinear equation of theory of waves in semiconductors*, Vestnik YuUrGU. Ser. Mat. Model. Progr. **12** (2019), no. 4, 104–113, DOI: 10.14529/mmp190408.
- [3] Maxim O. Korpusov and Evgenii A. Ovsyannikov, *Blow-up instability in non-linear wave models with distributed parameters*, Izv. Math. **84** (2020), no. 3, 449–501, DOI: 10.1070/IM8820.
- [4] Sergey N. Gurbatov, Oleg V. Rudenko, and Aleksandr I. Saichev, *Waves and Structures in Nonlinear Nondispersive Media*, Springer-Verlag Berlin Heidelberg, 2011.
- [5] Maxim O. Korpusov and S. Mikhailenko, *Instantaneous blow-up of classical solutions to the Cauchy problem for the Khokhlov-Zabolotskaya equation*, Comput. Math. Math. Phys. **57** (2017), no. 7, 1167–1172, DOI: 10.1134/S0965542517030095.
- [6] Enzo Mitidieri and Stanislav I. Pohozaev, *Apriori estimates and blow-up of solutions to partial differential equations and inequalities*, Proc. Steklov Inst. Math. **234** (2001), 3–383.
- [7] Stefan G. Samko, Anatoly A. Kilbas, and Oleg I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, Switzerland, 1993.