Research Article

İlkay Arslan Güven*

Some integral curves with a new frame

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Abstract: In this paper, some new integral curves are defined in three-dimensional Euclidean space by using a new frame of a polynomial spatial curve. The Frenet vectors, curvature and torsion of these curves are obtained by means of new frame and curvatures. We give the characterizations and properties of these integral curves under which conditions they are general helix. Also, the relationships between these curves in terms of being some kinds of associated curves are introduced. Finally, an example is illustrated.

Keywords: general helix, associated curve, integral curve, Flc-frame

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1 Introduction

Differential geometry uses the technique of calculus to understand shapes and their properties. Curve theory is a significant subsection of differential geometry, in the sense of that shapes. Since general types of curves are needed to understand to go beyond this theory, researchers labour this field for a long time.

In the meantime of studying the curve theory, frames of curves are essential to investigate their characterizations. For a space curve α in three-dimensional Euclidean space E^3 , the adapted frames are the collection of triples $\{v_1, v_2, v_3\}$, where v_i form an orthonormal basis of E^3 such that v_1 is tangent to α and v_2, v_3 are picked permissively on the plane which is normal to v_1 . Several examples of the adapted frames have already been imputed. The most well-declared and used one is the Frenet frame, which is a moving frame. Although it is well-declared, in the case that the second derivative of the curve is zero, it is undefined and its rotation about the tangent of a general spine curve often leads to undesirable twist in motion design or sweep surface modelling. A moving frame that does not rotate about the instantaneous tangent of the spine curve is called rotating minimizing frame (RMF). RMF is defined continuously for any regular spine curve. Due to its minimal twist property and steady behaviour in the presence of inflection points, the RMF is preferred to the Frenet frame in many applications in computer graphics [1]. In spite of its preferability, RMF is tough to compute. Furthermore, an alternative frame which is called Bishop frame is defined by R. L. Bishop [2]. The Bishop frame is a parallel transport frame. A parallel transport frame is an alternative approach to define a moving frame that is well-defined even when the curve has a vanishing second derivative [3].

Recently, Dede [4] has defined a new frame which is called Frenet-like curve frame. This new frame is computed easily in terms of using the cross product of the first and highest order derivatives of the curve to calculate the binormal vector of the curve. The advantage of this frame is that it decreases the number of singular points where they cannot be defined in the Frenet frame. Also, it decreases the undesirable

* **Corresponding author: İlkay Arslan Güven**, Faculty of Arts and Science, Department of Mathematics, Gaziantep University, Gaziantep, Turkey, e-mail: iarslan@gantep.edu.tr

rotation around the tangent vector of the curve. Besides, Dede et al. [5] indicated the q -frame along a space curve but this frame is not invariant under the Euclidean rotation.

Associated curves have been studied by many researchers prevalently. An associated curve is constituted by means of another curve and the behaviour of it can be investigated with the link between the main curve and its associated curve. Some illustrations of these curves are involute-evolute mates, Bertrand mates, Mannheim pairs, spherical images and integral curves.

In [6], Choi and Kim introduced associated curves of a Frenet curve which they called principal (binormal)-direction curve and principal (binormal)-donor curve. They gave the relationship between the directional curves and the main curve. In [7], Macit and Döldül defined new associated curves that are W -direction curve and W -rectifying curve with the aid of Darboux vector of a Frenet curve, V -direction curve which is associated with a curve lying on an oriented surface in E^3 . Moreover, Körpınar et al. studied associated curves with the Bishop frame in E^3 [8]. Associated curves according to type-2 Bishop frame were given by Yılmaz in [9].

Integral curves are interesting topic since they are tangent at each point to the vector field which is used to form the curve. An integral curve is a parametric curve which depicts a particular solution to a differential equation or system of equations. In the case X is a vector field and $\alpha(t)$ is a parametric curve, $\alpha(t)$ is an integral curve of X , if it is a solution of the differential equation $\alpha'(t) = X(\alpha(t))$. There are various research studies in the literature about this matter (see [6–10]).

From the viewpoint of differential geometry, a helix is a curve whose tangent vector makes a constant angle with a fixed direction which is called the axis of the helix. The well-known result indicated by Lancret in 1802 (see [11]) is that a necessary and sufficient condition that a curve be a general helix is that the ratio (harmonic curvature) $\frac{\tau}{\kappa}$ is constant where κ and τ denote the curvature and torsion of the curve, respectively. If both κ and τ are nonzero constants, then the curve is called circular helix. The helix was generalized, which was called “generalized helix” by Hayden [12]. Barros defined the general helix in three-dimensional real space form by using the concept of Killing vector field along a curve and proved the Lancret theorem for general helices in this space form [13]. Many research studies can be found in the literature about helices [14–18].

In this paper, we study some new integral curves that are associated curves. We construct them according to a new frame of a polynomial spatial curve. We give the properties of them via Frenet apparatus and new frame and curvatures. We also characterize them in terms of Bertrand pairs, Mannheim pairs, involute-evolute pairs and Salkowski curves.

2 Preliminaries

Let $\alpha(t)$ be a regular space curve and non-degenerate condition $\alpha' \times \alpha'' \neq 0$. Then three orthogonal vector fields which called Frenet frame are defined as

$$t = \frac{\alpha'}{\|\alpha'\|}, \quad \mathbf{n} = \mathbf{b} \times \mathbf{t}, \quad \mathbf{b} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \quad (2.1)$$

where t is the tangent, $\mathbf{n}$ is the principal normal and $\mathbf{b}$ is the binormal vector field. The well-known Frenet formulas are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \|\alpha'(t)\| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}, \quad (2.2)$$

where the curvature κ and torsion τ of the curve are (see [19–21])

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \quad \text{and} \quad \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \times \alpha''\|^2}. \quad (2.3)$$

A new frame which is called Flc-frame (Frenet-like curve frame) is defined by Dede in [4]. The concept of the Flc-frame is given below.

Let $\alpha(t)$ be a polynomial spatial curve of degree n (namely, α is a curve in space with polynomial components). We assume that $\alpha(t)$ is regular and $\alpha'(t) \neq 0$. The vector fields of the Flc-frame are as follows:

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \quad \mathbf{D}_1 = \frac{\alpha' \times \alpha^{(n)}}{\|\alpha' \times \alpha^{(n)}\|}, \quad \mathbf{D}_2 = \mathbf{D}_1 \times \mathbf{t}, \quad (2.4)$$

where t is the unit tangent vector, $\mathbf{D}_1$ is the binormal-like vector and $\mathbf{D}_2$ is the principal normal-like vector. Here the prime ' indicates the derivative with respect to t and (n) is the n th derivative.

The curvatures of the Flc-frame d_1 , d_2 and d_3 are given by

$$d_1 = \frac{\langle \mathbf{t}', \mathbf{D}_2 \rangle}{v}, \quad d_2 = \frac{\langle \mathbf{t}', \mathbf{D}_1 \rangle}{v} \quad \text{and} \quad d_3 = \frac{\langle \mathbf{D}_2', \mathbf{D}_1 \rangle}{v}, \quad (2.5)$$

where $v = \|\alpha'\|$.

Also, the Frenet-like formulas are expressed in [4] as

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{D}_2' \\ \mathbf{D}_1' \end{bmatrix} = v \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{D}_2 \\ \mathbf{D}_1 \end{bmatrix}. \quad (2.6)$$

The relations between the Frenet-like curve and Frenet curve apparatus are as follows:

$$\begin{aligned} d_1 &= \kappa \cos \theta, \\ d_2 &= -\kappa \sin \theta, \\ d_3 &= d\theta + \tau \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} \mathbf{D}_2 &= \cos \theta \mathbf{n} + \sin \theta \mathbf{b}, \\ \mathbf{D}_1 &= -\sin \theta \mathbf{n} + \cos \theta \mathbf{b}, \end{aligned} \quad (2.8)$$

where θ is the angle between the principal normal vector $\mathbf{n}$ and the principal normal-like vector $\mathbf{D}_2$.

Definition 2.1. Two curves that have a common principal normal vector are called *Bertrand curve pairs* [21].

Definition 2.2. If there exists a corresponding relationship between the space curves α and α^* such that, at the corresponding points of the curves, the principal normal lines of α coincides with the binormal lines of α^* , then α is called *Mannheim curve* and α^* a *Mannheim partner curve* of α . The pair $\{\alpha, \alpha^*\}$ is said to be a *Mannheim pair* [22].

Definition 2.3. A curve which has constant curvature and non-constant torsion is called *Salkowski curve* [23,24].

Definition 2.4. An *involute* of a curve is the locus of a point on a piece of string as the string is either unwrapped from the curve or wrapped along the curve. The *evolute* of an involute is the original curve [25].

Also, Hacısalihoğlu [26] gave the involute-evolute curve pairs as: if the tangent vectors of two curves are orthogonal (namely, $\langle T, T^* \rangle = 0$, where T and T^* are tangent vectors of the curves), then they are called involute-evolute curves.

3 New integral curves

In this section, we define some new integral curves that are called “directional curves.” Then we give the Frenet apparatus of these curves by means of the new frame and curvatures.

Definition 1. Let α be a polynomial spatial curve and $\mathbf{D}_2$ be the principal normal-like vector field of α . The integral curve of $\mathbf{D}_2$ is called D_2 -direction curve of α . Namely, the D_2 -direction curve of α is given by

$$\lambda(t) = \int \mathbf{D}_2(t) \quad \text{or} \quad \mathbf{D}_2(t) = \lambda'(t). \quad (3.1)$$

Theorem 1. Let λ be the D_2 -direction curve of a polynomial spatial curve α with $d_1, d_3 \neq 0$. Then the Frenet vector fields, curvature and torsion of λ are given by

$$\begin{aligned} t_\lambda &= \mathbf{D}_2, \\ n_\lambda &= -\frac{d_1}{\sqrt{d_1^2 + d_3^2}} \mathbf{t} + \frac{d_3}{\sqrt{d_1^2 + d_3^2}} \mathbf{D}_1, \\ b_\lambda &= \frac{d_3}{\sqrt{d_1^2 + d_3^2}} \mathbf{t} + \frac{d_1}{\sqrt{d_1^2 + d_3^2}} \mathbf{D}_1 \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} \kappa_\lambda &= \nu \sqrt{d_1^2 + d_3^2}, \\ \tau_\lambda &= \left(\frac{d_3}{d_1} \right)' \frac{d_1^2}{d_1^2 + d_3^2} - \nu d_2. \end{aligned} \quad (3.3)$$

Proof. By using Eqs. (2.6) and (3.1) and taking the derivatives we have

$$\begin{aligned} \lambda' &= \mathbf{D}_2, \\ \lambda'' &= \nu(-d_1 \mathbf{t} + d_3 \mathbf{D}_1), \\ \lambda''' &= (-\nu' d_1 - \nu d_1' - \nu^2 d_2 d_3) \mathbf{t} - \nu^2 (d_1^2 + d_3^2) \mathbf{D}_2 + (\nu' d_3 + \nu d_3' - \nu^2 d_1 d_2) \mathbf{D}_1 \end{aligned}$$

and simple calculations accomplish the results:

$$\begin{aligned} \lambda' \times \lambda'' &= \nu d_3 \mathbf{t} + \nu d_1 \mathbf{D}_1 \\ \det(\lambda', \lambda'', \lambda''') &= \nu^2 (d_1 d_3' - d_1' d_3) - \nu^3 d_2 (d_1^2 + d_3^2). \end{aligned}$$

Then in deference to Eqs. (2.1) and (2.3), the results are obvious. $\square$

Theorem 2. Let α be a polynomial spatial curve and λ be the D_2 -direction curve of α . If α is the general helix and θ is the constant, then the D_2 -direction curve is the general helix.

Proof. The harmonic curvature of D_2 -direction curve is computed by Eq. (3.3) as follows:

$$\frac{\tau_\lambda}{\kappa_\lambda} = \left(\frac{d_3}{d_1} \right)' \frac{d_1^2}{\nu(d_1^2 + d_3^2)^{3/2}} - \frac{\frac{d_2}{d_1}}{\sqrt{1 + \left(\frac{d_3}{d_1} \right)^2}}.$$

Also from Eq. (2.7), we get

$$\begin{aligned} \frac{d_2}{d_1} &= -\tan \theta, \\ \frac{d_3}{d_1} &= \frac{d\theta + \tau}{\kappa \cos \theta}. \end{aligned}$$

If α is general helix, then $\frac{\tau}{\kappa}$ is constant and if θ is constant, the ratios $\frac{d_2}{d_1}$ and $\frac{d_3}{d_1}$ are constant. Thus, the result is clear. $\square$

Corollary 1. *If v , d_1 and d_3 of the polynomial spatial curve α are constant, then the D_2 -direction curve is the Salkowski curve.*

Definition 2. Let α be a polynomial spatial curve and $\mathbf{D}_1$ be the principal normal-like vector field of α . The integral curve of $\mathbf{D}_1$ is called D_1 -direction curve of α . Namely, the D_1 -direction curve of α is given by

$$\beta(t) = \int \mathbf{D}_1(t) \quad \text{or} \quad \mathbf{D}_1(t) = \beta'(t). \quad (3.4)$$

Theorem 3. *Let β be the D_1 -direction curve of a polynomial spatial curve α with $d_2, d_3 \neq 0$. Then the Frenet vector fields, curvature and torsion of β are given by*

$$\begin{aligned} t_\beta &= \mathbf{D}_1, \\ n_\beta &= -\frac{d_2}{\sqrt{d_2^2 + d_3^2}} \mathbf{t} - \frac{d_3}{\sqrt{d_2^2 + d_3^2}} \mathbf{D}_2, \\ b_\beta &= \frac{d_3}{\sqrt{d_2^2 + d_3^2}} \mathbf{t} - \frac{d_2}{\sqrt{d_2^2 + d_3^2}} \mathbf{D}_2 \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \kappa_\beta &= v\sqrt{d_2^2 + d_3^2}, \\ \tau_\beta &= \left(\frac{d_3}{d_2}\right)' \frac{d_2^2}{d_2^2 + d_3^2} + vd_1. \end{aligned} \quad (3.6)$$

Proof. Considering Eq. (3.4) and by differentiating, we possess:

$$\begin{aligned} \beta' &= \mathbf{D}_1, \\ \beta'' &= v(-d_2 \mathbf{t} - d_3 \mathbf{D}_2), \\ \beta''' &= (-v'd_2 - vd_2' + v^2 d_1 d_3) \mathbf{t} + (-v'd_3 - vd_3' - v^2 d_1 d_2) \mathbf{D}_2 - v^2(d_2^2 + d_3^2) \mathbf{D}_1 \end{aligned}$$

and then we calculate

$$\begin{aligned} \beta' \times \beta'' &= vd_3 \mathbf{t} - vd_2 \mathbf{D}_2, \\ \det(\beta', \beta'', \beta''') &= v^2(d_2 d_3' - d_2' d_3) + v^3 d_1 (d_2^2 + d_3^2). \end{aligned}$$

From the calculations above and Eqs. (2.1) and (2.3), we reach the results apparently. $\square$

Theorem 4. *Let α be a polynomial spatial curve and β be the D_1 -direction curve of α . If α is general helix and θ is constant, then the D_1 -direction curve is general helix.*

Proof. The harmonic curvature of β , from Eq. (3.6) is as follows:

$$\frac{\tau_\beta}{\kappa_\beta} = \left(\frac{d_3}{d_2}\right)' \frac{d_2^2}{v(d_2^2 + d_3^2)^{3/2}} + \frac{\frac{d_1}{d_2}}{\sqrt{1 + \left(\frac{d_3}{d_2}\right)^2}}.$$

Then we have $\frac{d_1}{d_2} = -\cot \theta$ and $\frac{d_3}{d_2} = \frac{d\theta + \tau}{-\kappa \sin \theta}$. Similar to the proof of Theorem 2, the consequence is clear. $\square$

Corollary 2. *If v , d_2 and d_3 of the polynomial spatial curve α are constant, then the D_1 -direction curve is the Salkowski curve.*

Now we will state theorems concerning both D_2 -direction curve and D_1 -direction curve.

Theorem 5. Let α be a polynomial spatial curve. α forms involute-evolute curve pair both with D_2 -direction curve β and D_1 -direction curve λ . Also, D_2 -direction curve β and D_1 -direction curve λ are involute-evolute pairs.

Proof. Since the tangent vectors of α , β and λ are $\mathbf{t}$, $t_\lambda = \mathbf{D}_2$ and $t_\beta = \mathbf{D}_1$, respectively, then:

$$\langle \mathbf{t}, t_\lambda \rangle = 0, \quad \langle \mathbf{t}, t_\beta \rangle = 0 \quad \text{and} \quad \langle t_\beta, t_\lambda \rangle = 0.$$

By using the definition of involute-evolute curve pairs, the proof is completed. $\square$

Now let us restate the Frenet vector fields of D_2 -direction curve β and D_1 -direction curve λ . By using Eqs. (3.2) and (3.5) and the relation between Frenet apparatus and new vector fields existing in Eq. (2.8), we possess

$$\begin{aligned} t_\lambda &= \cos \theta \mathbf{n} + \sin \theta \mathbf{b}, \\ n_\lambda &= -\frac{d_1}{\sqrt{d_1^2 + d_3^2}} \mathbf{t} - \frac{d_3 \sin \theta}{\sqrt{d_1^2 + d_3^2}} \mathbf{n} + \frac{d_3 \cos \theta}{\sqrt{d_1^2 + d_3^2}} \mathbf{b}, \\ b_\lambda &= \frac{d_3}{\sqrt{d_1^2 + d_3^2}} \mathbf{t} - \frac{d_1 \sin \theta}{\sqrt{d_1^2 + d_3^2}} \mathbf{n} + \frac{d_1 \cos \theta}{\sqrt{d_1^2 + d_3^2}} \mathbf{b} \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} t_\beta &= -\sin \theta \mathbf{n} + \cos \theta \mathbf{b}, \\ n_\beta &= -\frac{d_2}{\sqrt{d_2^2 + d_3^2}} \mathbf{t} - \frac{d_3 \cos \theta}{\sqrt{d_2^2 + d_3^2}} \mathbf{n} - \frac{d_3 \sin \theta}{\sqrt{d_2^2 + d_3^2}} \mathbf{b}, \\ b_\beta &= \frac{d_3}{\sqrt{d_2^2 + d_3^2}} \mathbf{t} - \frac{d_2 \cos \theta}{\sqrt{d_2^2 + d_3^2}} \mathbf{n} - \frac{d_2 \sin \theta}{\sqrt{d_2^2 + d_3^2}} \mathbf{b}. \end{aligned} \quad (3.8)$$

Theorem 6. Let α be a polynomial spatial curve, λ and β be the D_2 -direction curve and D_1 -direction curve of α , respectively. The following statements are satisfied:

- (1) If the third curvature of α is zero ($d_3 = 0$), then D_2 -direction and D_1 -direction curves of α are Bertrand curve pairs.
- (2) If $\theta = \frac{\pi}{2} + k\pi$; ($k \in \mathbb{Z}$), then the curve α and D_2 -direction curve of α are Bertrand curve pairs.
- (3) If $\theta = 2k\pi$; ($k \in \mathbb{Z}$), then the curve α and D_1 -direction curve of α are Bertrand curve pairs.

Proof.

- (1) If $d_3 = 0$, then by considering Eq. (3.7) and (3.8), $n_\lambda = a \cdot n_\beta$, which infers that n_λ and n_β are linearly dependent. Also, the following equations are provided in the case of $d_3 = 0$:

$$\begin{aligned} \langle t_\beta, n_\lambda \rangle &= 0 \quad \text{and} \quad \langle t_\lambda, n_\beta \rangle = 0, \\ \langle b_\beta, n_\lambda \rangle &= 0 \quad \quad \langle b_\lambda, n_\beta \rangle = 0. \end{aligned}$$

Hence, this means that D_2 -direction and D_1 -direction curves are Bertrand curve pairs.

- (2) Taking the fact that $\theta = \frac{\pi}{2} + k\pi$, which also infers that $d_1 = 0$, we get $n = c \cdot n_\lambda$. Then the equations

$$\begin{aligned} \langle \mathbf{t}, n_\lambda \rangle &= 0 \quad \text{and} \quad \langle t_\lambda, \mathbf{n} \rangle = 0, \\ \langle \mathbf{b}, n_\lambda \rangle &= 0 \quad \quad \langle b_\lambda, \mathbf{n} \rangle = 0 \end{aligned}$$

are satisfied. So the result is obvious.

- (3) By similar approach to the former proof, if $\theta = 2k\pi$, which means $d_2 = 0$, we have $\mathbf{n} = d \cdot n_\beta$. Then it is clear to see that:

$$\begin{aligned}\langle \mathbf{t}, n_\beta \rangle &= 0 \quad \text{and} \quad \langle t_\beta, \mathbf{n} \rangle = 0, \\ \langle \mathbf{b}, n_\beta \rangle &= 0 \quad \langle b_\beta, \mathbf{n} \rangle = 0.\end{aligned}$$

□

Theorem 7. Let α be a polynomial spatial curve, λ and β be the D_2 -direction curve and D_1 -direction curve of α , respectively. D_2 -direction curve and D_1 -direction curve, α and D_2 -direction curve, α and D_1 -direction curve never constitute Mannheim curve pairs with the Flc-like frame of α .

Proof. If the D_2 -direction curve and D_1 -direction curve are thought to be Mannheim curve pairs, then we must have $n_\beta = b_\lambda$ and the equations $\langle t_\beta, b_\lambda \rangle = 0$, $\langle b_\beta, b_\lambda \rangle = 0$, $\langle t_\lambda, n_\beta \rangle = 0$ and $\langle n_\lambda, n_\beta \rangle = 0$ must be satisfied. They are satisfied in the condition that $d_1 = 0$ and $d_3 = 0$. But in this case the Frenet vector fields of the D_2 -direction curve do not exist. So the uppermost claim is not true.

Similarly, for the couples α and D_2 -direction curve to be satisfied $\mathbf{n} = b_\lambda$ and $\langle \mathbf{t}, b_\lambda \rangle = 0$, $\langle \mathbf{b}, b_\lambda \rangle = 0$, $\langle t_\lambda, \mathbf{n} \rangle = 0$ and $\langle n_\lambda, \mathbf{n} \rangle = 0$, we must have the condition $\theta = \frac{\pi}{2} + k\pi$; $k \in \mathbb{Z}$ (which means $d_1 = 0$) and $d_3 = 0$.

For the couples α and D_1 -direction curve to be satisfied $\mathbf{n} = b_\beta$ and $\langle \mathbf{t}, b_\beta \rangle = 0$, $\langle \mathbf{b}, b_\beta \rangle = 0$, $\langle t_\beta, \mathbf{n} \rangle = 0$, $\langle n_\beta, \mathbf{n} \rangle = 0$, we must have the condition $\theta = 2k\pi$; $k \in \mathbb{Z}$ (which means $d_2 = 0$) and $d_3 = 0$.

But in both of the conditions Frenet apparatus of curves λ and β cannot be formed. Hence, we have the result apparently. □

Remark 1. The adjoint curve as an integral curve of binormal vector in the Frenet frame was studied by Nurkan et al. [27]. We can give a comparison with our new integral curve and that adjoint curve. The adjoint curve is compared with our integral curve D_1 -direction curve, because adjoint curve and D_1 -direction curve are the integral curves of binormal vector in the Frenet frame and binormal-like vector in the Frenet-like frame, respectively. If α is a general helix, then adjoint curve of α is general helix [27], whereas if α is general helix and θ is constant, then the D_1 -direction curve is general helix (Theorem 4). The condition “ θ is constant” is different. Besides, a curve α and its adjoint curve are Bertrand curve pairs and involute-evolute curves [27]. Also, the same result is given between α and D_1 -direction curves, which are involute-evolute curve pairs (Theorem 6). But if $\theta = 2k\pi$, ($k \in \mathbb{Z}$), then the curve α and D_1 -direction curve of α are Bertrand curve pairs (Theorem 7).

Example. A polynomial spatial curve of degree 4 is

$$\alpha(t) = \left(t^2, \frac{2t^3}{3}, \frac{t^4}{4} \right).$$

The curvature and torsion of $\alpha(t)$ are

$$\kappa = \frac{2}{t(t^2 + 2)^2} \quad \text{and} \quad \tau = \frac{2}{t(t^2 + 2)^2}.$$

Since $\frac{\tau}{\kappa} = 1$ (constant), $\alpha(t)$ is the general helix.

The tangent, principal normal-like and binormal-like vector fields of α are as follows:

$$\begin{aligned}\mathbf{t} &= \left(\frac{2}{t^2 + 2}, \frac{2t}{t^2 + 2}, \frac{t^2}{t^2 + 2} \right), \\ \mathbf{D}_2 &= \left(-\frac{t^2}{(t^2 + 2)\sqrt{t^2 + 1}}, -\frac{t^3}{(t^2 + 2)\sqrt{t^2 + 1}}, \frac{2\sqrt{t^2 + 1}}{t^2 + 2} \right), \\ \mathbf{D}_1 &= \left(\frac{t}{\sqrt{t^2 + 1}}, -\frac{1}{\sqrt{t^2 + 1}}, 0 \right).\end{aligned}$$

D_1 -direction curve of $\alpha(t)$ is as follows:

$$\beta(t) = \left(\sqrt{t^2 + 1} + c_1, -\ln(\sqrt{t^2 + 1} + t) + c_2, c_3 \right),$$

where c_1, c_2 and c_3 are real constants.

D_2 -direction curve of $\alpha(t)$ is as follows:

$$\begin{aligned} \lambda(t) = & \left(\sqrt{2} \operatorname{arctan} h \left(\frac{t}{\sqrt{2} \sqrt{t^2 + 1}} \right) - \operatorname{arcsinh}(t) + d_1, \right. \\ & 2 \operatorname{arctan}(\sqrt{t^2 + 1}) - \sqrt{t^2 + 1} + d_2, \\ & \left. 2 \operatorname{arcsinh}(t) - \sqrt{2} \operatorname{arctan} h \left(\frac{t}{\sqrt{2} \sqrt{t^2 + 1}} \right) + d_3 \right), \end{aligned}$$

where d_1, d_2 and d_3 are real constants.

The curvatures of $\alpha(t)$ with respect to the Flc-frame are as follows:

$$\begin{aligned} d_1 &= \frac{2}{(t^2 + 2)^2 \sqrt{t^2 + 1}}, \\ d_2 &= -\frac{2}{t(t^2 + 2)^2 \sqrt{t^2 + 1}}, \\ d_3 &= \frac{t}{(t^2 + 2)^2 (t^2 + 1)}. \end{aligned}$$

By using Eq. (2.7), the angle θ is obtained as

$$\theta = -\frac{1}{4}(\cos^2 \theta + \ln(2 - \cos^2 \theta) - 2 \ln(\cos \theta) + 1) + c,$$

where c is the real constant.

Figure 1: The curve α .

Figure 2: D_1 -direction curve of α .

Figure 3: D_2 -direction curve of α .

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