Research Article

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Towards a homological generalization of the direct summand theorem

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Abstract: We present a more general homological characterization of the direct summand theorem (DST). Specifically, we state two new conjectures: the socle-parameter conjecture (SPC) in its weak and strong forms. We give a proof for the weak form by showing that it is equivalent to the DST. Furthermore, we prove the SPC in its strong form for the case when the multiplicity of the parameters is smaller than or equal to two. Finally, we present a new proof of the DST in the equicharacteristic case, based on the techniques thus developed.

Keywords: splitting extensions, direct summand theorem, multiplicity

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1 Introduction

Fundamental work of Hochster, Peskine, Szpiro, and Serre during the second half of the twentieth century allowed us to develop a theory of multiplicities, along with the introduction of powerful prime characteristic methods in commutative algebra [1–3]. As a by-product of this effort for understanding general properties of commutative rings and their prime spectra in terms of homological and algebraic invariants, a collection of *homological conjectures* emerged [1]. Further research of Hochster's showed that most of these conjectures were essentially new ways of describing a quite central algebraic splitting phenomenon for a particular kind of ring extensions: most of these homological questions turned out to be equivalent to the *direct summand conjecture* (DSC) [1,4–7].

Specifically, the DSC states that if $R \hookrightarrow S$ is a finite extension of rings and if R is a regular ring, then R is a direct summand of S as R -module. Or equivalently, this extension splits as a map of R -modules, i.e. there exists a retraction $\rho : S \rightarrow R$ sending 1_S to 1_R .

After many decades, the DSC (or direct summand theorem [DST]) was finally proved in the former general form by Y. André, by first reducing to the case of unramified complete regular local rings, as it had been suggested by the seminal work of Hochster's, and then by using Scholze's theory of perfectoid spaces [8,9]. Even though this fundamental question was settled, the former results (regarded as a whole) suggest some directions for future research in local homological algebra.

In this work, we present an equivalent form of the DSC given in terms of an estimate for the difference of the lengths of the first two Koszul homology groups of quotients of Gorenstein rings by principal zero-divisor ideals. Dutta and Griffith (see [10, Theorem 1.5]) had obtained similar results in an independent manner, although in a rather different context (for the case of complete and almost complete intersections).

In Sections 3 and 4, we state an equivalent form of the DSC conjecture in terms of the existence of annihilators of zero divisors on Gorenstein local rings not belonging to parameter ideals [11,12]. Based on

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these results, we find a new conjecture equivalent, in its weak form, to the DSC (Section 6). In its strong form, this conjecture states that if (T, η) is a Gorenstein local ring of dimension d and $\{x_1, \dots, x_d\} \subseteq T$ is any system of parameters, and if we denote by Q the ideal generated by these parameters, then for any zero divisor $z \in T$, and for any lifting $u \in T$ of a socle element in T/Q (i.e. $\text{Ann}_{T/Q}(\bar{\eta}) = (\bar{u})$) it must hold that $uz \in Q(z)$.

This rather technical condition allows for more flexibility when one tries to do computations in particular examples (see for instance the proof of Proposition 12). We call this conjecture the *socle-parameter conjecture, strong form* (SPCS). We obtain the weak form if we add the requirement that in the mixed-characteristic case $\text{char } T/\eta = p > 0$ and $x_1 = p$. The SPCS is at the same time equivalent to a very general and homological condition involving the lengths of the Koszul homology groups:

$$\ell(H_0(\underline{x}, T/(z))) - \ell(H_1(\underline{x}, T/(z))) > 0.$$

Evidently, this last condition only involves homological estimates. Then, a theorem of Ikeda (see [13, Corollary 1.4]) helps one to show that the SPCS is true when the multiplicities of the parameters are smaller than or equal to two, suggesting induction as a way of solving (Section 8). This approach has its origins in the work of the first author (J. D. Vélez). Specifically, the idea of proving the DSC by means of annihilators is first introduced in his thesis (see [14, Lemma 3.1.2.] and [15]). The reduction to the case $S = T/J$, where T is a Gorenstein local ring, and J is a principal ideal, was stated in a private communication from Vélez to Hochster, in 1996, and appears more explicitly in [11] and in [12]. Similar results related to the SPCS were obtained independently by Strooker and Stückrad [16].

Finally, it is worth noting that due to their simplicity and conceptual clarity, the results exposed in this paper are able to be used, in the context of modern artificial intelligence and pure mathematics, for creating pseudo-pre-code regarding fulfilling artificial mathematical intelligence in the specific mathematical sub-discipline of commutative algebra [17]. More specifically, this can be done based on the new cognitive foundations of mathematics program [17, Chapters 1–6]; the new (initial) global taxonomy of cognitive (metamathematical) mechanisms [17, Chapters 7–10]; and the computational realization and artificial generation of the artificial mathematical intelligence program [17, Chapters 11 and 12].

2 Preliminary results

Let $(R, m) \hookrightarrow S$ be a module finite extension. Let $s_1, \dots, s_n \in S$ be generators of S as an R -module. Then, there exist monic polynomials $f_i(y_i) \in R[y_i]$ such that $f_i(y_i) = y_i^{m_i} + a_{i1}y_i^{m_i-1} + \dots + a_{im_i}$, $a_{ij} \in R$, and $f_i(s_i) = 0$. One can define a homomorphism of R -algebras from

$$T = R[y_1, \dots, y_n]/(f_1(y_1), \dots, f_n(y_n))$$

to S sending each y_i to s_i . If J denotes the kernel of this map, $S \cong T/J$. Finally, since $\dim T = \dim R = \dim T/J$, by the reason of the finiteness of the extension, J should be contained in a minimal prime ideal of T , that means, $\text{ht } J = 0$. Later we develop all the necessary facts in order to prove that if the residue field is algebraically closed, then we can reduce the DST to the case where $a_{ij} \in m$.

Remark 1. Let $R \hookrightarrow S$ be a finite extension of Noetherian rings, where (R, m) is local. Then, it is an elementary fact to see that the maximal spectrum of S , $\text{Spec}_m S$, is equal to $V(mS) \subseteq \text{Spec } S$ (see for example [18, Corollary 5.8]).

Lemma 1. Let (R, m, k) be a local complete ring and $R \hookrightarrow S$ a finite extension. Assume that $\text{Spec } S = V(mS) = \{\eta_1, \dots, \eta_n\}$. Then S is naturally isomorphic, as a ring, to $S_{\eta_1} \times \dots \times S_{\eta_n}$.

Proof. This is a well-known consequence of the fact that a local complete ring is Henselian and one of the equivalence forms of being Henselian is essentially the statement of the lemma. See, for example [19, Lemma 10.152.3].¹ $\square$

¹ For a direct proof of this fact one can also see the arxiv previous version of this article (arxiv:1708.03393).

Corollary 2. Let (R, m, k) be a complete local ring with algebraically closed residue field k and $B = R[x]/(F(x))$, where $F(x)$ is a monic polynomial of degree n . Then there exist monic polynomials $G_i(x)$ of degree n_i such that $\sum_{i=1}^r n_i = n$, $B = \oplus_{i=1}^r R[x]/(G_i(x))$ as rings, and

$$G_i(x) = x^{n_i} + a_{i1}x^{n_i-1} + \cdots + a_{in_i},$$

where all $a_{ij} \in m$.

Proof. Since k is algebraically closed we can factor $f(x) := \bar{F}(x) = \prod_{i=1}^r (x - b_i)^{n_i} \in k[x]$, for some $b_i \in k$ and $n_i \in \mathbb{N}$, with $\sum_{i=1}^r n_i = n$. Besides, $B/mB \cong k[x]/(f(x))$ is an Artinian ring with maximal ideals $\bar{\eta}_i = (x - b_i)$ for $i = 1, \dots, r$; therefore, $B/mB \cong \oplus_{i=1}^r k[x]/((x - b_i)^{n_i})$ (see [18, Theorem 8.7 and proof]). Now, by Hensel's Lemma there exist monic polynomials $F_i(x) \in R[x]$ such that $F(x) = \prod_{i=1}^r F_i(x)$ and $\bar{F}_i(x) = (x - b_i)^{n_i}$. By Remark 1, $\text{Spec}_m B = \{\eta_1, \dots, \eta_r\} = V(mB)$.

Now, let $B_i \in R$ such that $\bar{B}_i = b_i$. Since $\bar{\eta}_i = (x - b_i)$ in B/mB , we see by the correspondence between the ideals of B/mB and the ideals of B containing mB that $\eta_i = (x - B_i) + mB$. Now, $(F(x))R[x]_{\eta_i} = (F_i(x))R[x]_{\eta_i}$, because $(F_j(x)) + \eta_i = R[x]$ for all $i \neq j$, since mod $mR[x]$ this ideal corresponds to $((x - b_j)^{n_j}) + (x - b_i) = \text{rad}(x - b_i, x - b_j) = k[x]$ for all $i \neq j$. Therefore, $F_j(x)$ is a unit in $R[x]_{\eta_i}$. Besides, the ring $R[x]/(F_i(x))$ is local with maximal ideal $(x - B_i) + m^e$, where m^e denotes the expansion of m in $R[x]$ (which we denote again by η_i). This is because any maximal ideal should contain the expansion of m (the extension $R \hookrightarrow R[x]/(F_i(x))$ is finite) and this ring module m^e is the local ring $(k[x]/((x - b_i)^{n_i}), (x - b_i)^e)$. So,

$$B_{\eta_i} = R[x]_{\eta_i}/(F(x))R[x]_{\eta_i} \cong R[x]_{\eta_i}/F_i(x)R[x]_{\eta_i} \cong (R[x]/(F_i(x)))_{\eta_i} \cong R[x]/(F_i(x)).$$

Thus, by the previous lemma, $B \cong \oplus_{i=1}^r B_{\eta_i} \cong \oplus_{i=1}^r R[x]/(F_i(x))$.

Finally, the isomorphism of rings $\theta_i : R[x] \rightarrow R[t]$, sending $x \mapsto t + B_i$, induces an isomorphism of rings $\bar{\theta}_i : R[x]/(F_i(x)) \rightarrow R[t]/(F_i(t + B_i))$. But the reduction of $F_i(x)$ mod $mR[x]$ is exactly $((t + B_i) - b_i)^{n_i} = t^{n_i}$, because translation and reduction mod m commutes. But that means exactly that $G_i(t) = t^{n_i} + a_{i1}t^{n_i-1} + \cdots + a_{in_i} \in R[t]$ with $a_{ij} \in m$, for any indices i, j . In conclusion, we get an isomorphism of rings between B and $\oplus_{i=1}^r R[x]/(G_i(t))$ satisfying the conditions of our corollary. $\square$

Remark 2. Let $i : R \hookrightarrow S = S_1 \times \cdots \times S_n$ be an extension of rings, where R is an integral domain. Then there exists an S_i such that $\pi_i \circ i : R \hookrightarrow S_i$ is also an extension, where π_i is the natural projection. Suppose by contradiction that for any i there exist $a_i \neq 0 \in R$ such that $\pi_i(i(a_i)) = 0$. Then, if $a = \prod_{i=1}^n a_i \neq 0$, for any j ,

$$\pi_j(a) = \prod_{i=1}^n \pi_j(i(a_i)) = \pi_j(i(a_j)) \prod_{i \neq j} \pi_j(i(a_i)) = 0.$$

Therefore, $i(a) = (\pi_1(i(a)), \dots, \pi_n(i(a))) = 0$, contradicting the hypothesis that i is an extension.

Theorem 3. Let (R, m, k) be a regular complete local ring with algebraically closed residue field k . Then, to prove the DSC for R it is enough to consider finite extensions $R \hookrightarrow S$, where $S = T/J$,

$$T = R[y_1, \dots, y_r]/(f_1(y_1), \dots, f_r(y_r)),$$

$f_i(y_i) = y_i^{n_i} + a_{i1}y_i^{n_i-1} + \cdots + a_{in_i}$ with $a_{ij} \in m$ and $J \subseteq T$ is an ideal of height zero.

Proof. Let us fix a finite extension $R \hookrightarrow S$. By the discussion above, we know that $S = T/J$, where $T = R[y_1, \dots, y_r]/(f_1(y_1), \dots, f_r(y_r))$ (the coefficients of the monic polynomials $f_i(y_i)$ are not necessarily in m), and $\text{ht}(J) = 0$. It is elementary to see that

$$T \cong \otimes_{i=1}^r R[y_i]/(f_i(y_i)).$$

Write $B_i = R[y_i]/(f_i(y_i))$, then by Corollary 2 $B_i \cong \oplus_{\alpha=1}^{m_i} R[y_i]/(f_{i\alpha}(y_i))$ with $f_{i\alpha}(y_i) = y_i^{n_{i\alpha}} + a_{i\alpha 1}y_i^{n_{i\alpha}-1} + \cdots + a_{i\alpha n_{i\alpha}}$ and $a_{i\alpha j} \in m$. Furthermore, by the distributive law between tensor products and direct sums (see [20]) we get

$$T \cong \otimes_{i=1}^r (\oplus_{\alpha=1}^{m_i} R[y_i]/(f_{i\alpha}(y_i))) \cong \oplus_w (\otimes_{i=1}^r R[y_i]/(f_{i\alpha_i}(y_i))) \cong \oplus_w T_w,$$

where $w = (\alpha_1, \dots, \alpha_r)$, $1 \leq \alpha_i \leq m_i$, and

$$T_w \cong \otimes_{i=1}^r R[y_i]/(f_{i\alpha_i}(y_i)) \cong R[y_1, \dots, y_r]/(f_{1\alpha_1}(y_1), \dots, f_{r\alpha_r}(y_r)),$$

where each $f_{i\alpha_i}(y_i)$ has lower coefficients in m , as desired. Besides, $J = \oplus_w J_w$ and then $S \cong \oplus_w T_w/J_w$.

Finally, by Remark 2, there is an α such that $R \hookrightarrow T_w/J_w$ is an extension. But if there is a retraction $\rho_w : T_w/J_w \hookrightarrow R$, then $\rho = \pi_w \circ \rho_w : S \rightarrow R$ is also a retraction. Besides $\text{ht}(J_w) = 0$, because $\dim T_w = \dim R = \dim T_w/J_w$. In conclusion, $S_w = T_w/J_w$ has the desired form of our proposition, and then it is enough to prove the DSC in this case. $\square$

Proposition 4. *Let (R, m, k) be a regular local ring of dimension d , and*

$$T = R[y_1, \dots, y_r]/(f_1(y_1), \dots, f_r(y_r)),$$

$f_i(y_i) = y_i^{n_i} + a_{i,1}y_i^{n_i-1} + \dots + a_{i,n_i}$, with $a_{i,j} \in m$. Then T is a Gorenstein local ring with maximal ideal $\eta = m + (\overline{y_1}, \dots, \overline{y_r})$.

Proof. First, we see that T is a local Cohen-Macaulay (C-M) ring. In fact, let m_1 be any maximal ideal of T . Then $m_1 \cap R = m$, because $\dim(R/(m_1 \cap R)) = \dim(T/m_1) = 0$ and $m_1 \cap R \in \text{Spec}(R)$. Therefore, $R/(m_1 \cap R)$ is a field. Thus, $y_i^{n_i} \in m_1$ and therefore $y_i \in m_1$. In conclusion, $m_1 = \eta$. Now, $R[y_1, \dots, y_r]$ is a C-M ring (see [21, Proposition 18.9]). Thus, $B = R[y_1, \dots, y_r]_\eta$ is also C-M. Let $m = (x_1, \dots, x_d)$, where $d = \dim R$. Then $\{f_1(y_1), \dots, f_r(y_r), x_1, \dots, x_d\} \subseteq B$ is a system of parameters because $\dim B = \text{ht}(\eta) = \dim R[y_1, \dots, y_r] = \dim R + r = d + r$, and since $R[y_1, \dots, y_r]$ is C-M, then by previous comments is equidimensional, and $\text{rad}(f_1, \dots, f_r, x_1, \dots, x_d) = \eta B$. Then $\{f_1, \dots, f_r, x_1, \dots, x_d\}$ is a regular sequence in B . Thus, $B/(f_1, \dots, f_r) \cong T$ is C-M, due to the fact that $\dim B/(f_1, \dots, f_r) = d$ and $\{\overline{x_1}, \dots, \overline{x_d}\} \subseteq B/(f_1, \dots, f_r)$ is a regular sequence.

Finally, let $Q = (y_1, \dots, y_r, x_1, \dots, x_d)$ be the ideal generated by the system of parameters $\{y_1, \dots, y_r, x_1, \dots, x_d\}$. Then $T/Q \cong k[y_1, \dots, y_r]/(y_1^{n_1}, \dots, y_r^{n_r}) = k[w_1, \dots, w_r]$, where $w_i = \overline{y_i}$. Let us see that

$$\text{Ann}_{T/Q}((w_1, \dots, w_r)) = \left(\prod_{i=1}^r w_i^{n_i-1} \right).$$

In fact, if $\bar{h} \in \text{Ann}_{T/Q}((w_1, \dots, w_r))$ and $c \prod_{i=1}^r w_i^{m_i} \neq 0$ is a monomial of $\bar{h}$ such that there exists $m_j \in \mathbb{N}$ with $m_j < n_j - 1$, then $w_j \bar{h} \neq 0 \in T/Q$, because the monomial term $w_j c \prod_{i=1}^r w_i^{m_i}$ has the power $m_j + 1 < n_j$ on w_j , which is a contradiction, by the reason that $\bar{h}$ is on the socle. Therefore, $m_i \geq n_i - 1$ for all i and so $\bar{h} \in (\prod_{i=1}^r w_i^{n_i})$. The other contention is clear, and then the socle has dimension one. In conclusion, (T, η) is a Gorenstein local ring.

Another approach for proving this proposition can be summarized as follows: since $f_i(y_i)$ involve different variables (for each $i = 1, \dots, r$), one can use this fact explicitly to see that $\{f_1(y_1), \dots, f_r(y_r)\}$ forms a regular sequence. Now, due to the fact that R is Gorenstein, we see that $R[y_1, \dots, y_r]$ so is. Moreover, one can also prove that when one mods out a Gorenstein ring by a regular sequence, the corresponding quotient ring remains Gorenstein. So, this proves that $T = R[y_1, \dots, y_r]/(f_1(y_1), \dots, f_r(y_r))$ is Gorenstein. Finally, T is local, due to the fact that T/mT so is (because it is isomorphic to $k[y_1, \dots, y_r]/(y_1^{n_1}, \dots, y_r^{n_r})$, with maximal ideal $(y_1, \dots, y_r)$) and due to the fact that all the maximal ideals of T contract to m by module-finiteness. $\square$

3 DSC in terms of annihilators

Now we make preparations for the proof of the following fact: let $h : (R, m) \rightarrow (T, \eta)$ be a finite homomorphism of local rings, i.e. $h(m) \subseteq \eta$, where T is a local R -free ring, with T/mT Gorenstein, and let $S = T/J$, for some ideal $J \subseteq T$. Then $h : R \rightarrow S$ splits if and only if $\text{Ann}_T J \not\subseteq mT$ (by abuse of notation we denote by h again its composition with the natural projection $\pi : T \rightarrow S$).

Proposition 5. Let (A, η, k) be a local Gorenstein ring of dimension zero (i.e. $\dim_k(\text{Ann}_A \eta) = 1$). Let $u \in \eta$ such that $(u) = \text{Ann}_A \eta$. Then $u \in I$ for any ideal $I \neq (0) \subseteq A$.

Proof. Clearly, we can assume that $I \subseteq \eta$. We know $\text{nil}(A) = \eta$; therefore, there exists $n \in \mathbb{N}$ such that $\eta^n = (0)$. Let $x \neq 0 \in I$. Then $\eta^{n-1}x \subseteq \eta^n = (0)$. Let $r \in \mathbb{N}$ be such that $\eta^r x = (0)$ but $\eta^{r-1}x \neq (0)$. Hence, $\eta(\eta^{r-1}x) = (0)$ and so $\eta^{r-1}x \subseteq \text{Ann}_A \eta = (u)$. Then (u) contains a nonzero element of the form bx , where $b \in \eta^{r-1}$. This means that there exists $c \in A \setminus \eta$ with $cu = xb$, so $u = (c^{-1}b)x \in I$. $\square$

Remark 3. If $h : R \hookrightarrow T$ is any homomorphism of rings, we can consider $T^* = \text{Hom}_R(T, R)$ as a T -module with the following action: fix $t \in T$ and define $(t \cdot \phi)(x) := \phi(tx)$, for $\phi \in \text{Hom}_R(T, R)$ and $x \in T$.

Remark 4. Let (R, m) be a local ring, T a finitely generated R -free module, and $\theta : T \rightarrow T$ an R -homomorphism. Then θ is an isomorphism of R -modules if and only if $\bar{\theta} : T/mT \rightarrow T/mT$ is an isomorphism of k -vector spaces. In fact, if $A \in M_{n \times n}(R)$ is the matrix defining θ , then θ is an isomorphism if and only if $\det A$ is a unit, which means that $\det A \notin m$. But that is equivalent to saying that $\det \bar{A} \neq 0 \in k$, where $\bar{A}$ is the reduction of $A \bmod m$. Finally, since $\bar{A}$ is the matrix defining $\bar{\theta}$, the last condition is equivalent to saying that $\bar{\theta}$ is an isomorphism of k -vector spaces.

Theorem 6. Let (R, m) and (T, η) be local rings. Assume that T/mT is Gorenstein. Let $h : R \rightarrow T$ be a finite homomorphism of local rings, such that T is R -free. Then there exists a T -isomorphism $\beta : T \rightarrow T^*$ such that for any ideal $J \subseteq T$, $\beta^{-1}((T/J)^*) = \text{Ann}_T J$.

Proof. We identify $(T/J)^*$ with $\{f \in T^* : f(J) = 0\}$. We know that $\dim T/mT = \dim R/m = 0$, since T/mT is a finitely generated R/m -module. So, fix $u_1 \in \eta$ such that $(\bar{u}_1) = \text{Ann}_{T/mT} \bar{\eta}$, since T/mT is Gorenstein. Let $\{u_2, \dots, u_d\} \subseteq T$ such that $T/mT = (\bar{u}_1, \dots, \bar{u}_d)$ (where $T \cong R^d$). Then, by the Lemma of Nakayama $T = (u_1, \dots, u_d)$, T is generated by $u_1, \dots, u_d$ as an R -free module. In fact, let $\{w_1, \dots, w_d\} \subseteq T$ be an R -basis for T . Define $\theta : T \rightarrow T$ by $w_i \rightarrow u_i$, then the induced $\bar{\theta} : T/mT \rightarrow T/mT$ is clearly an isomorphism of k -vector spaces. Since T is R -free, by Remark 4, θ is an isomorphism which means just that $\{u_1, \dots, u_d\} \subseteq T$ is an R -basis for T . Let $u_1^* \in T^*$ be the dual element and define $\beta : T \rightarrow T^*$ by $t \rightarrow t \cdot u_1^*$, where $(t \cdot u_1^*)(t_1) := u_1^*(tt_1)$, for all $t_1 \in T$. By definition, it is clear that β is a T -homomorphism. Now, we can make the natural identifications $T^* = \text{Hom}_R(R^d, R) \cong (\text{Hom}_R(R, R))^d = R^d$. Therefore, T^* is an R -free module of dimension d and β is an R -isomorphism if and only if $\bar{\beta} : T/mT \rightarrow T^*/mT^*$ is so, due to Remark 4 ($T \cong T^*$ as R -free modules). But $\bar{\beta}$ is an isomorphism of k -vector spaces if it is injective. Suppose by contradiction that $\ker \bar{\beta} \neq (0)$. Then, by Proposition 5, $\bar{u}_1 \in \ker \bar{\beta}$. This means that $\beta(u_1) \in mT^*$, so there exist $m_1, \dots, m_d \in m$ such that $\beta(u_1) = u_1 u_1^* = \sum_{i=1}^d m_i u_i^*$. But, evaluating at 1 we get: $1 = u_1 u_1^*(1) = \sum_{i=1}^d m_i u_i^*(1) \in m$, a contradiction. In conclusion, β is a T -isomorphism.

For the last part, let $a \in \text{Ann}_T J$. Then, for any $j \in J$, $\beta(a)(j) := (au_1^*)(j) = u_1^*(aj) = u_1^*(0) = 0$. Therefore, $\beta(a) \in (T/J)^*$ and thus $a = \beta^{-1}(\beta(a)) \in \beta^{-1}((T/J)^*)$.

Conversely, take $\phi_0 \in \beta^{-1}((T/J)^*)$. Then there exists a $\phi \in (T/J)^*$ such that $\phi_0 = \beta^{-1}(\phi)$, i.e. $\phi = \phi_0 u_1^*$ and $\phi(J) = 0$, which means that $u_1^*(\phi_0 j) = 0$ for all $j \in J$. Now, let us fix $j_0 \in J$. Then, for all $t \in T$, $\beta(\phi_0 j_0)(t) = u_1^*(\phi_0 j_0 t) = u_1^*(\phi_0(j_0 t)) = 0$, because $j_0 t \in J$. Therefore, $\beta(\phi_0 j_0) \equiv 0$ and then by the first part $\phi_0 j_0 = 0$, which means that $\phi_0 \in \text{Ann}_T J$. $\square$

Theorem 7. Let $h : R \rightarrow T/J$ be a finite homomorphism of local rings such that (T, η) is a local R -free ring, with T/mT Gorenstein, and $J \subseteq T$ an ideal. Assume that the structure of R -module of T/J inherited by the R -structure of T is the same as the one induced by h . Then $R \hookrightarrow T/J$ splits if and only if $\text{Ann}_T J \not\subseteq mT$, where m is the maximal ideal of m .

Proof. Assume that $\rho : T/J \hookrightarrow R$ is a splitting R -homomorphism. Then $\rho \in (T/J)^*$ and $\rho(1) = 1$. By the last theorem, there exists $\rho_0 \in \text{Ann}_T J$ such that $\rho_0 = \beta^{-1}(\rho)$, which means, in particular, that $1 = \rho(1) = \rho_0 \cdot u_1^*(1) = u_1^*(\rho_0) \notin m$. Then $\rho_0 \notin mT$, so $\text{Ann}_T J \not\subseteq mT$.

Conversely, take $a \in \text{Ann}_T J \setminus mT$. Then, by Proposition 5, $\overline{a_1} \in (\bar{a}) \subseteq T/mT$, which means that there exists $m_i \in m$, with $u_1 - a = \sum_{i=1}^d m_i u_i$, where $\{u_1, \dots, u_d\} \subseteq T$ is an R -basis for T as in the last Theorem. Then,

$$au_i^*(1) = (1 - m_1)u_1u_i^*(1) - \sum_{i=2}^d m_i u_i u_i^*(1) = (1 - m_1) - \sum_{i=2}^d m_i u_i^*(u_i) = (1 - m_1) + 0 = 1 - m_1 \notin m.$$

So $\rho = \beta((1 - m_1)^{-1}a)$ satisfies that $\rho \in (T/J)^*$ because $\beta^{-1}(\rho) = (1 - m_1)^{-1}a \in \text{Ann}_T J$, and by Theorem 6 $\beta^{-1}((T/J)^*) = \text{Ann}_T J$. Besides, $\rho_1 = (1 - m_1)^{-1}au_1^*(1) = (1 - m_1)^{-1}(1 - m_1) = 1$, which implies that $\rho : T/J \rightarrow R$ is the desired splitting R -homomorphism. $\square$

4 Reduction to the case where J is principal

In the next proposition, we prove that (a proof of) the DST can be reduced to (a proof for) the case where J is a principal ideal generated by an element in mT .

Proposition 8. Let (R_0, m_0, k_0) be a regular local ring and $R_0 \hookrightarrow T_0$ be a finite extension, where

$$T_0 = R_0[y_1, \dots, y_r]/(f_1(y_1), \dots, f_r(y_r)),$$

and each $f_i(y_i) = y_i^{n_i} + a_{i1}y_i^{n_i-1} + \dots + a_{in_i}$, with $a_{ij} \in m$, for all indices i, j . Let $S = T_0/J$, with $J = (g_1, \dots, g_s) \subseteq T_0$ such that $J \cap R_0 = (0)$. Let $x_1, \dots, x_s$ be new variables, and let R be $R_0[x_1, \dots, x_s]$, and let m be $m_0 + (x_1, \dots, x_s)$, a maximal ideal of R . Write

$$T = R \otimes_{R_0} T_0 \cong R[y_1, \dots, y_r]/(f_1(y_1), \dots, f_r(y_r)).$$

Let g be the element $x_1g_1 + \dots + x_sg_s$. Then:

1. (R_m, mR_m, k_0) is a regular local ring.
2. $(g)T_m \cap R_m = (0)$.
3. $R_m \hookrightarrow (T/(g))_m \cong T_m/(g)^e$ is a finite extension, and $g \in (mR_m)T_m$.
4. $R_m \hookrightarrow T_m/(g)$ splits if and only if $R_0 \hookrightarrow T_0/J$ splits.

Proof.

- (1) In general, if R is regular, then so is the polynomial ring $R[T]$ (see [20]). In particular, R_m is a regular local ring and $R_m/mR_m \cong R/m \cong R_0/m_0 = k_0$.
- (2) $R_0 \hookrightarrow R$ is an R -free extension, then, in particular, it is flat. Therefore, by tensoring $R_0 \hookrightarrow T_0/J$ with R , we see that $R \hookrightarrow R \otimes_{R_0} T_0/J \cong T/J^e$ is also an extension, and since localization is flat too, we get an extension $R_m \hookrightarrow T_m/J^e$. Because of $g \in J^e$, we get $gT_m \cap R_m = 0$.
- (3) Clearly, by definition $g \in (mR_m)T_m$. Now, by the previous paragraph $R_m \hookrightarrow T_m/(g)^e$ is an extension, and it is finite because $R_0 \hookrightarrow T_0$ is finite, in fact free. Then, after tensoring with R_m we get a module finite extension $R_m \hookrightarrow T_m$, so $T_m/(g)^e$ is also a finitely generated R_m -module.
- (4) Assume that $\rho_0 : T_0/J \rightarrow R_0$ is an R_0 -homomorphism such that $\rho_0(1) = 1$. Then, by tensoring with the flat R_0 -module R_m , we get an R_m -homomorphism $\rho : T_m/J \hookrightarrow R_m$, with $\rho(1) = 1$. Now, composing ρ with the natural map $T_m/(g) \rightarrow T_m/J$, we obtain a retraction from $T_m/(g)$ to R_m . $\square$

Conversely, it is clear that T_m satisfies the hypothesis of Proposition 4. Therefore, by Theorem 7, $\text{Ann}_{T_m}(g) \not\subseteq (mT_m)$. So let us choose

$$w = \sum_{\alpha} (h_{\alpha}(\underline{x})/k_{\alpha}(\underline{x}))y^{\alpha},$$

such that $wg = 0$, where $h_{\alpha} \in R$ and $k_{\alpha} \in R \setminus m$ (which is equivalent to saying that $k_{\alpha}(\underline{0}) \notin m_0$). Here y^{α} denotes $y_1^{\alpha_1} \dots y_r^{\alpha_r}$, $\alpha = (\alpha_1, \dots, \alpha_r)$, $0 \leq \alpha_i < \deg f_i$ and some $h_{\beta} \notin m$ (that means exactly $w \notin mT_m$). We have $T = R \otimes_{R_0} T \cong T_0[x_1, \dots, x_s]$. Now, multiplying by the product of the $k_{\alpha}(\underline{x})$, we can assume that $w = \sum_{\alpha} p_{\alpha}(\underline{x})y^{\alpha} \in T$, where some $p_{\alpha} \notin m$ and $0 = wg = \sum_i \sum_{\alpha} p_{\alpha}(\underline{x})y^{\alpha}x_i g_i(\underline{y})$ in T . Now, the coefficient of x_i , which is zero in T , is exactly $\sum_{\alpha} p_{\alpha}(\underline{0})y^{\alpha}g_i(\underline{y})$, because the terms $y^{\alpha}g_i(\underline{y})$ are constants in $T = T_0[x_1, \dots, x_s]$. Therefore, if $w_0 = \sum_{\alpha} p_{\alpha}(\underline{0})y^{\alpha} \in T_0$, we have $w_0 g_i = \sum_{\alpha} p_{\alpha}(\underline{0})y^{\alpha}g_i(\underline{y}) = 0$, and thus $w_0 \in \text{Ann}_{T_0} J$. But $p_{\beta}(\underline{0}) \notin m_0$, because $p_{\beta}(\underline{x}) \notin m$, so $w_0 \notin m_0 T_0$. In conclusion, $\text{Ann}_{T_0} J \not\subseteq m_0 T_0$, which is equivalent by Theorem 7 to the fact that $R_0 \hookrightarrow T_0/J$ splits.

5 Socle-parameter conjecture

In this section, and in the next section, we state two new conjectures the socle-parameter conjecture (SPC) in its strong (SPCS) and weak forms (SPCW), and we prove that the SPCW is equivalent to the DSC, and that the SPCS implies the SPCW. Besides, these two conjectures are equivalent in the equicharacteristic case and therefore both are equivalent to the DSC in the equicharacteristic case.

However, as far as we know, the mixed characteristic case remains open. The new approach shows that the DSC is, in essence, a problem concerning algebraic and homological properties of Gorenstein local rings.

Let us start by reviewing some elementary notions. Let R be an $\mathbb{N}$ -graded ring such that R_0 is an Artinian ring, and such that R is finitely generated as an R_0 -algebra. Let M be a finitely generated graded R -module of dimension d . Then, it is elementary to see that each homogeneous part M_n is a finitely generated R_0 -module, and therefore, it has finite length (see [18, Proposition 6.5.]). It is well known in this case that there exists a unique polynomial

$$p_M(t) = a_{d-1}t^{d-1} + \dots + a_0 \in \mathbb{Q}[t]$$

of degree $d - 1$, the *Hilbert polynomial*, such that for $n \gg 0$, $p_M(n) = \ell(M_n)$. The *multiplicity* of M , $e(M)$, is defined as $\ell(M)$ if $d = 0$, and as $(d - 1)!a_{d-1}$, if $d > 0$ (see [22, Definition 4.1.5.]). In particular, if M has positive dimension, then $e(M) > 0$, since $a_{d-1} > 0$. For the local case, assume that (R, m) is a local ring, M a finitely generated R -module of dimension d , and $I = (x_1, \dots, x_n)$ is an *ideal of definition* of M . This last condition means that $m^r M \subseteq IM$ for some $r > 0$, which is equivalent to saying that $x = x_1, \dots, x_n$ is a *multiplicity system* on I (i.e. $\ell(M/(x_1, \dots, x_n)M) < +\infty$) (see [22, p. 185]). We define the filtered graded ring $\text{gr}_I R = \bigoplus_{i=0}^{+\infty} I^i/I^{i+1}$, and the filtered graded module $\text{gr}_I M = \bigoplus_{i=0}^{+\infty} I^i M/I^{i+1} M$, where $I_0 = R$. Then, $\text{gr}_I R$ is in a natural way a graded ring (where R/I is Artinian, because after reducing to the case $\text{Ann}_R M = 0$, it is easy to see that $\ell(M/IM) < +\infty$ if and only if $\text{rad} I = m$). Therefore, we can define the multiplicity of M on I , $e(I, M) := e(\text{gr}_I M)$ (see [22, p. 180]) and the multiplicity of R , $e(R) := e(m, R)$ (see [18, p. 108]). In particular, we can define the multiplicity of M on $I = (x_1, \dots, x_n)$, where $x_1, \dots, x_n \in R$ is a *system of parameters* of M , i.e. $n = \dim M$ and M is Artinian, i.e. satisfied the descending chain condition for submodules (see [18, p. 74]). Besides, under the former hypothesis and assuming that $x = x_1, \dots, x_n$ is a multiplicity system of M , we can define the *Euler Characteristic* as

$$\chi(\underline{x}, M) = \sum_{i=0}^n (-1)^i \ell(H_i(\underline{x}, M)).$$

For a more technical reformulation of this notion due to Auslander and Buchsbaum, see [22]. Now, a theorem of Serre (see [22, Theorem 4.6.6.]) states that $\chi(\underline{x}, M) = e(I, M)$, if $\{x_1, \dots, x_n\} \subseteq R$ is a system of parameters for M , and zero otherwise. In particular, if $\{x_1, \dots, x_n\} \subseteq R$ is a system of parameters for M , and $\dim M > 0$, then $\chi(\underline{x}, M) = e(I, M) = e(\text{gr}_I M) > 0$, because $\dim M = \dim(\text{gr}_I M)$ (see [22, Theorem 4.4.6.]).

5.1 SPCS

Let (T, η) be a Gorenstein local ring of dimension d . Let $\{x_1, \dots, x_d\} \subseteq T$ be a system of parameters and write $Q = (x_1, \dots, x_d)$. Let $u \in T$ be any lifting of a socle element in T/Q , i.e. $\text{Ann}_{T/Q}(\bar{\eta}) = (\bar{u})$. Let $z \in T$ be a zero divisor. Then $u \cdot z \in Q \cdot (z)$.

Now, it holds that this strong form of the conjecture is equivalent to saying that $\ell(H_0(\underline{x}, T/(z))) - \ell(H_1(\underline{x}, T/(z))) > 0$.

We prove the last equivalence:

Proposition 9. *In the situation of the SPCS the following are equivalent.*

1. $u \cdot z \in Q \cdot (z)$.
2. $\text{Ann}_T(z) \not\subseteq Q$.
3. $\ell(H_0(\underline{x}, T/(z))) - \ell(H_1(\underline{x}, T/(z))) > 0$.

Proof. (1) $\Rightarrow$ (2) Consider the following natural short exact sequence

$$0 \rightarrow \text{Ann}_T(z) \rightarrow T \rightarrow (z) \rightarrow 0.$$

We know that $T/\text{Ann}_T(z) \cong (z)$, by the isomorphism sending $\bar{t}$ to tz . After tensoring with T/Q we get

$$(z)/(Q(z)) \cong T/Q \otimes (z) \cong T/Q \otimes T/\text{Ann}_T(z) \cong T/(\text{Ann}_T(z) + Q).$$

Now, let us see that $uz \in Q \cdot (z)$ implies $\text{Ann}_T(z) \not\subseteq Q$. Effectively, $uz \in Q \cdot (z)$ is equivalent to $\bar{u}\bar{z} = 0 \in (z)/(Q(z))$, and it is equivalent to $\bar{u} = 0 \in T/(\text{Ann}_T(z) + Q)$, under the last isomorphism. Therefore, there exists $w \in \text{Ann}_T(z)$ and $q \in Q$ such that $u = w + q$, and so $w = u - q \notin Q$ because $u \notin Q$. Then $\text{Ann}_T(z) \not\subseteq Q$.

(2) $\Rightarrow$ (1) If $\text{Ann}_T(z) \not\subseteq Q$, by Proposition 5, $\bar{u} \in \overline{\text{Ann}_T(z)} \subseteq T/Q$. Then there exists $w \in \text{Ann}_T(z)$ such that $\bar{u} = \bar{w}$, which means that there is a $q \in Q$ such that $u = w + q$. So,

$$uz = (w + q)z = wz + qz = qz \in Q \cdot (z).$$

(2) $\Leftrightarrow$ (3) $\text{Ann}_T(z) \not\subseteq Q$ if and only if $Q \subsetneq \text{Ann}_T(z) + Q$ if and only if $\ell(T/(\text{Ann}_T(z) + Q)) < \ell(T/Q)$. Besides, we have the natural short exact sequence

$$0 \rightarrow (z) \rightarrow T \rightarrow T/(z) \rightarrow 0.$$

Then, after considering the induced long exact sequence for Tor, and noting that $\text{Tor}_1^T(T, T/Q) = 0$, because T is a T -free module, and therefore flat, we get the following exact sequence

$$0 \rightarrow \text{Tor}_1^T(T/(z), T/Q) \rightarrow (z)/Q(z) \rightarrow T/Q \rightarrow T/(Q + (z)) \rightarrow 0.$$

Now, since Q is generated by a system of parameters, the T -modules $T/(Q + (z))$, T/Q and $T/(\text{Ann}_T(z) + Q) \cong (z)/Q(z)$ are Noetherian rings of dimension zero and therefore Artinian. In particular, they have finite length as T -modules. Then the submodule $\text{Tor}_1^T(T/(z), T/Q)$ has finite length too. By the additivity of $\ell(-)$, we have

$$\ell(\text{Tor}_1^T(T/(z), T/Q)) - \ell(T/(\text{Ann}_T(z) + Q)) + \ell(T/Q) - \ell(T/(Q + (z))) = 0.$$

Hence,

$$\ell(T/Q) - \ell(T/(\text{Ann}_T(z) + Q)) = \ell(T/(Q + (z))) - \ell(\text{Tor}_1^T(T/(z), T/Q)).$$

Then, $\text{Ann}_T(z) \not\subseteq Q$ if and only if $\ell(T/(Q + (z))) - \ell(\text{Tor}_1^T(T/(z), T/Q)) > 0$. But T is C-M and then $\{x_1, \dots, x_n\}$ is a regular sequence. Hence, the Koszul Complex is a free (and then projective) resolution of T/Q :

$$\cdots \rightarrow K_2 \rightarrow K_1 \rightarrow K_0 \rightarrow T/Q \rightarrow 0.$$

Hence, after tensoring this resolution with $T/(z)$, and taking homology, we find that $H_1(\underline{x}, T/(z)) \cong \text{Tor}_1^T(T/Q, T/(z))$ and $H_0(\underline{x}, T/(z)) \cong (T/(z))/\bar{Q} \cong T/((z) + Q)$. In conclusion, $\text{Ann}_T(z) \not\subseteq Q$ is equivalent to the condition

$$\ell(H_0(\underline{x}, T/(z))) - \ell(H_1(\underline{x}, T/(z))) > 0. \quad \square$$

Proposition 10. Assume the same hypothesis as in SPCS and, in addition, that $\text{depth}(\bar{\eta}, T/(z)) \geq d - 1$, then SPCS holds.

Proof. Write $a = \text{depth}(\bar{\eta}, T/(z))$. It is a well-known fact that if $H_r(\underline{x}, T/(z))$ denotes the Koszul homology and $q = \sup\{r : H_r(\underline{x}, T/(z)) \neq 0\}$, then $a = d - q$, therefore, $q = d - a \leq 1$. In the case that $d = 0$, $Q = 0$, and $\dim_T(\eta) = 1$, thus for any element $u \in T$ holds $uz = 0 \in Q \cdot (z)$. Then assume $d \geq 1$. Besides, $\{x_1, \dots, x_n\} \subseteq T$ is a system of parameters for the T -module $T/(z)$, because $\dim T = \dim T/(z)$ and $\dim((T/(z))/(x_1, \dots, x_d)T/(z)) = 0$. So, $(T/(z))/(x_1, \dots, x_d)T/(z)$ is an Artinian ring. Hence, by previous results, we get

$$\ell(H_0(\underline{x}, T/(z))) - \ell(H_1(\underline{x}, T/(z))) = \chi(\underline{x}, T/(z)) = e(\bar{Q}, T/(z)).$$

Now, by previous comments and the fact that $d \geq 1$ we see that $e(\bar{Q}, T/(z)) > 0$. $\square$

5.2 SPCW

Let (T, η) be a Gorenstein local ring of dimension d . Let $\{x_1, \dots, x_d\}$ be any system of parameters (if T is mixed characteristic $[\text{char} T/\eta = p > 0]$ we assume that $x_1 = p$). Let $Q = (x_1, \dots, x_d)$, and let $u \in T$ be any lifting of a socle element in T/Q , i.e. $\text{Ann}_{T/Q}(\bar{\eta}) = (\bar{u})$. Let z be a zero divisor. Then $u \cdot z \in Q \cdot (z)$, which is equivalent to the inequality $\ell(H_0(\underline{x}, T/(z))) - \ell(H_1(\underline{x}, T/(z))) > 0$.

Note that between the two forms of the SPC the only difference is the fact that in the mixed characteristic case one assume that $x_1 = p$. One needs this last condition in order to apply Cohen's structure theorem in mixed characteristic.

Remark 5. For proving any of the two versions of the SPC, it is enough to assume that (T, η) is complete.

Proof. Let $\tau : T \rightarrow \hat{T}$ be the natural homomorphism to the completion. Then τ is a faithfully flat extension and $\hat{T} \cap T = I$ for any ideal I of T (see [20, p. 63]). Besides, another elementary consequence of faithfully flatness is that for any ideals I, J of T , $(J : I)\hat{T} = (J\hat{T} : I\hat{T})$. Now, assume by contradiction that there exists a system of parameters $\{x_1, \dots, x_d\}$ (for the SPCW assume $x_1 = p$), a zero divisor $z \in T$ and $u \in T$ a lifting of a socle element for T/Q such that $uz \notin Q(z)$. Let us write $\tau(y) = y'$. Then $\{x'_1, \dots, x'_d\} \subseteq \hat{T}$ is a system of parameters in $\hat{T}$, because $\hat{\eta} = \eta\hat{T} = \text{rad}((x_1, \dots, x_d)\hat{T}) \subseteq \text{rad}(x'_1, \dots, x'_d) \subseteq \hat{\eta}$. Besides, $(Q \cdot (z) : z)\hat{T} = (Q\hat{T} \cdot (z') : z') ; \hat{T}/Q\hat{T} \cong (\hat{T}/\bar{Q}) = T/Q$, due to the fact that $\bar{\eta}^n = (0)$ for some $n > 0$; therefore, $(u') = \text{Ann}_{\hat{T}/Q\hat{T}}(\hat{\eta})$ and so u' is a socle element. Note that $p = \text{char}(\hat{T}/Q\hat{T}) = \text{char}(T/Q)$. Furthermore, $\hat{T}$ is also a Gorenstein ring (see [20, Theorem 18.3]).

Finally, $((Q\hat{T} : z') + (u'))/(Q\hat{T} : z) \cong ((Q : z) + u)/(Q : z) \otimes \hat{T} \neq 0$, because $(Q : z) + u/(Q : z) \neq 0$, since $uz \notin Q(z)$. Then $u'z' \notin Q\hat{T} \cdot (z')$ which contradicts SPC in the complete case. $\square$

6 Equivalence to the DSC

First, we review the notion of a *coefficient ring*: If (R, m, k) is equicharacteristic, a *coefficient field* is a field $K_0 \subseteq R$ such that the natural projection $\pi : K_0 \subseteq R \rightarrow R/m = k$ is an isomorphism. On the other hand, let (R, m, k) be a complete quasi-local (that means with a unique maximal ideal m but not necessarily Noetherian), mixed characteristic, and *separated* ring (i.e. $\cap_{n \in \mathbb{N}} m^n = (0)$). Then a *coefficient ring* for R is a sub-ring $(D, \eta) \hookrightarrow R$ such that it is a local complete discrete valuation ring such that $m \cap D = \eta$, and so that the inclusion induces an isomorphism $D/\eta \cong R/m$. It is elementary to see that if $\text{char} R = 0$ and $\text{char} k = p > 0$, then D is a domain, and therefore one dimensional, which means exactly that (D, η) is a discrete valuation domain (DVD). A theorem of I. S. Cohen states that for complete local rings there always exists a coefficient ring (see [23, Theorem 9, Theorem 11] and [24, p. 24]). In fact, in the mixed characteristic

case ($\text{char } k = p > 0$), there exist coefficient rings which are DVD-s (D, pD, k), with local parameter p , or $D/p^m D$, when $\text{char } R = p^m$ (see [24, Corollary p. 24]).

Theorem 11. *The SPCW is equivalent to the DSC.*

Proof. $\text{SPCW} \Rightarrow \text{DSC}$. By previous comments, we may assume that $R \hookrightarrow S$ is a finite extension and R is a unramified regular local mixed characteristic ring ($\text{char}(R/m) = p > 0$) with algebraically closed residue field k . Besides, by Theorem 3 we can assume that $S = T/(J)$ such that $T = R[y_1, \dots, y_r]/(f_1(y_1), \dots, f_r(y_r))$; $f_i(y_i) = y_i^{n_i} + a_{i1}y_i^{n_i-1} + \dots + a_{in_i}$, where $a_{ik} \in m$, for all indices i, k , and $\text{ht}(J) = 0$. Now, T is a Gorenstein local ring by Proposition 4. Moreover, by Proposition 8 $R \hookrightarrow S$ splits if and only if $R_1 := R[x_1, \dots, x_d]_{m_1} \hookrightarrow S_1 = T_1/(z)$ splits, where $m_1 = m + (x_1, \dots, x_d)$, $T_1 = R_1[y_1, \dots, y_r]/(f_1(y_1), \dots, f_r(y_r))$, and $z \in T_1$.

Besides, since R is regular, $R[x_1, \dots, x_d]$ so is. In particular, R_1 is regular. Furthermore, R_1 is unramified, otherwise there exist elements $a_i, b_i \in m_1$ and $s \in R_1 \setminus m_1$ such that $sp = \sum_{i=1}^c a_i b_i$, and then evaluating in $(0, \dots, 0)$ we get $p = s(\underline{0})^{-1} \sum_{i=1}^c a_i(\underline{0}) b_i(\underline{0})$, where $a_i(\underline{0}), b_i(\underline{0}) \in m$ and $s(\underline{0}) \in R \setminus m$, so $p \in m^2$, which is a contradiction.

Hence, $p \notin m^2$ and then $\bar{p} \neq 0 \in m/m^2$ is a part of a basis of m/m^2 as k -vector space, which is equivalent by the Lemma of Nakayama to the fact that p is a part of a minimal set of generators of m , say, $\{w_1 = p, \dots, w_n\} \subseteq m$. Now, $\{w_1 = p, \dots, w_n\} \subseteq T_1$ is a system of parameters in T_1 , because $\dim(T_1/m_1 T_1) = \dim(R_1/m_1) = 0$ and $\dim T_1 = \dim R_1$, since $R_1 \hookrightarrow S_1$ is a finite extension.

On the other hand, z is a zero divisor in T_1 because $\dim T_1/(z) = \dim S_1 = \dim R_1 = \dim T_1$ and therefore z is contained in a minimal prime of T_1 , since T is C-M.

Since T_1 is Gorenstein, choose $u \in T_1$ such that $(\bar{u}) = \text{Ann}_{T_1/m_1 T_1}(\bar{\eta})$. By SPCW, $uz \in m_1 T_1 \cdot (z)$, so there exists $a \in m_1 T_1$ such that $uz = az$, hence $(u - a)z = 0$. But $u - a \notin m_1 T_1$, because $u \notin m_1 T_1$. Therefore, $\text{Ann}_{T_1}(z) \not\subseteq m_1 T_1$, and then, by Theorem 7, $R_1 \hookrightarrow S_1$ splits.

$\text{DSC} \Rightarrow \text{SPCW}$. Let (T, η) be a Gorenstein local ring and $\{x_1, \dots, x_d\} \subseteq T$ a system of parameters ($x_1 = p$ in the mixed characteristic case). By Remark 5, we can assume that T is complete. Let D be a coefficient ring for T (which always exists for any complete local ring). Then, due to Cohen's structure theorem (see [23, Lemma 16]), the ring generated (after completion) as D -algebra by the parameters $R = D[[x_1, \dots, x_d]]$ is a complete regular local ring with maximal ideal $Q = (x_1, \dots, x_d)$ such that the extension $R \hookrightarrow T$ is finite. Since R is regular, by Serre's theorem (see [20] Theorem 19.2) $pd_R(T)$ is finite. Hence, for the Auslander-Buchsbaum formula and the fact that $\text{depth}(Q, T) = \text{depth}(\eta, T)$ (see [22, Exercise 1.2.26]), we know that

$$pd_R(T) = \text{depth}(Q, R) - \text{depth}(Q, T) = \dim R - \text{depth}(\eta, T) = d - \dim T = d - d = 0.$$

So T is a free R -module. Furthermore, z is contained in an associated prime of T because it is a zero divisor. Since T is C-M, any associated prime is, in fact, a minimal prime. Thus, z is contained in a minimal prime $P \in \text{Spec } T$. Moreover, since T is C-M, T is equidimensional, which means, in particular, that $\dim T/(z) \geq \dim T/P = \dim T$. In conclusion,

$$\dim R/((z) \cap R) = \dim T/(z) = \dim T = \dim R,$$

so $(z) \cap R = (0)$, because R is a domain (R is regular!). Now, to see that $uz \in Q \cdot (z)$, it is enough to see that $\text{Ann}_T(z) \not\subseteq Q$. In fact, if $\text{Ann}_T(z) \subseteq Q$, then $\overline{\text{Ann}_T(z)} \neq 0$ in T/Q . Therefore, by Proposition 5, $\bar{u} \in \overline{\text{Ann}_T(z)}$, there exists $w \in \text{Ann}_T(z)$ such that $u - w \in Q$. Thus, $uz = (u - w)z \in Q \cdot (z)$, because $wz = 0$. By Theorem 7, $R \hookrightarrow T/(z)$ splits if and only if $\text{Ann}_T(z) \not\subseteq mT = Q$, where $m = (x_1, \dots, x_d) \subseteq R$. Hence, the DSC for $R \hookrightarrow T/(z)$ implies $\text{Ann}_T(z) \not\subseteq QT$, and then $uz \in Q \cdot (z)$. Another way of proving this is using directly Proposition 9. $\square$

7 SPCS for small multiplicities

A natural way one could attack the SPCS would be by induction on $e(T)$, the multiplicity of T . We note first that by Remark 5 one may assume that T is complete. In the case $e(T) = 1$, then, since T is a complete C-M

ring, it is equicharacteristic and therefore unmixed. Hence, by the *Criterion for multiplicity one* (see [25]) T must be a regular local ring; in particular, this ring is an integral domain, which implies $z = 0$, from which the SPCS follows directly.

Suppose now that $e(T) = 2$. Since T is C-M, it must satisfy the S_2 condition of Serre: for any $P \in \text{Spec } T$, $\text{depth}(P, T) \geq \min(2, \dim T_P)$, due to the fact that $\text{depth}(P, T) = \dim(T_P)$. Hence, by a Theorem of Ikeda (see [13, Corollary 1.3.]) T is a hypersurface of the form $B/(f)$, where B is a complete regular local ring. Now, we prove a more general result, namely, that the SPCS holds for residue class ring of local Gorenstein rings which are UFD and C-M, which implies, in particular, the case of multiplicity two because regular local rings are UFD and C-M (see [21]).

Proposition 12. *The SPCS holds for Gorenstein rings of the form $T = B/(f)$, where B is a local C-M ring which is an UFD and $f \neq 0 \in B$.*

Proof. Let $z \neq 0 \in T$ be a zero divisor and $\{\bar{y}_1, \dots, \bar{y}_d\} \subseteq T$ a system of parameters. We will see that $\ell(H_0(\bar{y}, T/(z))) - \ell(H_1(\bar{y}, T/(z))) > 0$. The minimal prime ideals of T are just the principal ideals generated by the prime factors of $f = \prod f_i^{c_i}$, i.e. $P_i = (f_i)$, since B is an UFD. Besides, it is enough to prove SPCS for $z = \bar{f}_i$, because each zero divisor is a multiple of one of these, i.e. $z = a\bar{f}_i$ for some $a \in B$, and thus if $u\bar{f}_i \in Q \cdot (\bar{f}_i)$, where $Q = (\bar{y}_1, \dots, \bar{y}_d)$ and $(\bar{u}) = \text{Ann}_{T/Q}(\bar{\eta})$, then $uz = u\bar{f}_i = a\bar{f}_i \in Q \cdot (\bar{f}_i) = Q \cdot (z)$. Let us fix some f_j , then $T/(\bar{f}_j) = B/(f_j)$ is a C-M ring, because it is a quotient of a C-M ring by an ideal generated by a regular element (B is an integral domain) (see [21, Proposition 18.13]). Since T is equidimensional, $\dim T = \dim T/(f_j)$ and $\dim(T/(f_j)/(\bar{y}_1, \dots, \bar{y}_d)) = \dim B/(f_j, y_1, \dots, y_d) \leq \dim B/(f, y_1, \dots, y_d) = \dim T/(\bar{y}_1, \dots, \bar{y}_d) = 0$, because $(f_j, y_1, \dots, y_d) \subseteq (f, y_1, \dots, y_d) \subseteq B$. Hence, $(\bar{y}) = \{\bar{y}_1, \dots, \bar{y}_d\} \subseteq T/(f_j)$ is a system of parameters and so it is a regular sequence. Thus, $H_1(\bar{y}, T/(\bar{f}_j)) = 0$ (see [20, Theorem 16.5]). In conclusion, $\ell(H_0((\bar{y}), T/(\bar{f}_j))) - \ell(H_1((\bar{y}), T/(\bar{f}_j))) = \ell(H_0((\bar{y}), T/(\bar{f}_j))) = \ell((T/(\bar{f}_j))/(\bar{y}_1, \dots, \bar{y}_d)) \geq 1 > 0$, because $T/(\bar{f}_j)/(\bar{y}_1, \dots, \bar{y}_d) \neq (0)$. This proves the SPCS for T . $\square$

Typical examples of local C-M rings which are UFD are localizations of polynomial rings in prime ideals, or rings of formal power series over DVD or fields. More generally, regular local rings fulfil these conditions (see [21, Corollary 18.7, Theorem 19.19]).

8 A new proof of DSC in the positive characteristic case

Now, we present a new proof of the DSC in the positive characteristic case by proving some particular case of the SPCW. The new key ingredient is the following Lemma. We refer to [26, Proposition 5.2.6] for a proof.

Lemma 13. *Let R be a Noetherian ring, M an R -module, and $x_1, \dots, x_n$ be a sequence of elements in R such that $M/(x_1, \dots, x_n)M$ has finite length. Let $i \in \{1, \dots, n\}$. Then, there exists a constant c such that the length of the Koszul homology module $\ell(H_i(x_1^q, \dots, x_n^q; M)) \leq cq^{n-i}$, for all positive integers q .*

Proposition 14. *Let (R, m, k) be an equicharacteristic regular local ring, with $\text{char } R = p > 0$, and $R \hookrightarrow S$ a finite extension. Then $R \hookrightarrow S$ splits.*

Proof. After tensoring with the completion of R , which is faithfully flat, we can assume, by previous comments, that R is complete. By Cohen's structure theorem (see [24, Theorem, p. 26]), $R \cong k[[x_1, \dots, x_n]]$, where $\text{char } k = p > 0$. Now, we can assume that k is perfect (i.e. $k^p = k$), because each extension of the tower $R \hookrightarrow \bar{k} \otimes_k R \hookrightarrow \bar{k}[[x_1, \dots, x_n]]$ is faithfully flat, where $\bar{k}$ denotes an algebraic closure of k . Effectively, $R \hookrightarrow \bar{k} \otimes_k R$ is R -free and therefore faithfully flat. Besides, we can identify $\bar{k} \otimes_k R$ with $\cup_{i \in \mathbb{N}} E_i[[x_1, \dots, x_n]]$,

where E_i runs over all field extensions $k \subseteq E_i \subseteq \bar{k}$, such that $[E : k] < +\infty$. From this, we see that the completion of the local ring $\bar{k} \otimes_k R$ is exactly $\bar{k}[[x_1, \dots, x_n]]$ and so $\bar{k} \otimes_k R \hookrightarrow \bar{k}[[x_1, \dots, x_n]]$ is faithfully flat.

Again, by Theorem 3 and Proposition 4 we can assume that $S \cong T/J$, where T is a Gorenstein local ring and $\text{ht} J = 0$. Moreover, by Proposition 8 and after tensoring with the completion of $R_1 = R[[w_1, \dots, w_r]]_{(m+(w_1, \dots, w_r))}$, which is isomorphic to $\bar{k}[[X_1, \dots, X_m]]$ (for some $m \geq n$), we see that $\widehat{R_1} \otimes_{R_1} (R_1 \otimes_R T)$ has exactly the same form as in Proposition 4. But, now we can assume that J is a principal ideal generated by a zero divisor. In conclusion, we can assume that $R = k[[x_1, \dots, x_m]]$, where k is a perfect field and $S = T/(z)$, where T is a Gorenstein local ring and z is a zero divisor.

Now, we set $P_q = (x_1^q, \dots, x_n^q) \subseteq T$, where q is a power of p . Note that $R^q = k[[x_1^q, \dots, x_n^q]] \hookrightarrow R$ is finite and thus by the proof of Theorem 11 $R^q \hookrightarrow S$ splits if and only if

$$\delta = \delta_{R^q}(x_1^q, \dots, x_n^q; T/(z)) := \ell_{R^q}(H_0(x_1^q, \dots, x_n^q; T/(z))) - \ell_{R^q}(H_1(x_1^q, \dots, x_n^q; T/(z))) > 0.$$

Besides, it is elementary to see that

$$\delta = \delta_R(x_1^q, \dots, x_n^q; T/(z)),$$

because the degree of the residue field extension of $R^q \hookrightarrow R$ is one and then

$$\ell_R(H_i^R(x_1^q, \dots, x_n^q; T/(z))) = \ell_{R^q}(H_i^R(x_1^q, \dots, x_n^q; T/(z))) = \ell_{R^q}(H_i^{R^q}(x_1^q, \dots, x_n^q; T/(z))).$$

The last equality holds because the last two Koszul homology groups are isomorph as R^q -modules.

We prove that $\lim_{q \rightarrow +\infty} \delta_R(x_1^q, \dots, x_n^q; T/(z)) = +\infty$. In fact, since $\{x_1^q, \dots, x_n^q\} \subseteq T$ is a system of parameters for the R -module $T/(z)$, we know that

$$\chi(x_1^q, \dots, x_n^q; T/(z)) = q^n \chi(x_1, \dots, x_n; T/(z))$$

(see Corollary 5.2.4 [26]) and by previous comments,

$$\chi(x_1, \dots, x_n; T/(z)) = e((x_1, \dots, x_n), T/(z)) > 0,$$

because $\dim R > 0$ (if $\dim R = 0$, then R is a field and the DSC is trivial). Besides, by the previous corollary we know that there is a constant c such that $\ell_R(H_i(x_1^q, \dots, x_n^q; T/(z))) < cq^{n-i}$ for each $i = 1, \dots, n$ and each q . Combining this we get the following estimate:

$$\delta_R(x_1^q, \dots, x_n^q; T/(z)) = q^n e + \sum_{i=2}^n (-1)^{i+1} \ell_R(H_i^R(x_1^q, \dots, x_n^q; T/(z))) \geq q^n e - \sum_{i=2}^n cq^{n-i},$$

where $e = e(x_1^q, \dots, x_n^q; T/(z)) > 0$. Let us write $f(q) := q^n e - \sum_{i=2}^n cq^{n-i}$. Then, the polynomial $f(q) \rightarrow +\infty$, when $q \rightarrow +\infty$, because it has positive leading coefficient. Therefore, $\delta_R(x_1^q, \dots, x_n^q; T/(z)) = +\infty$. Let us fix $b = p^h > 0$. Then by the proof of Theorem 11 $R^b \hookrightarrow T/(z)$ splits. Denote by $\rho_1 : T/(z) \rightarrow R^b$ a splitting R^b -homomorphism. Finally, the Frobenius homomorphism $F_b : R \rightarrow R^b$ sending $x \rightarrow x^b$ is an isomorphism of rings. Hence, we can define $\rho : T/(z) \rightarrow R$, by $\rho(x) := F_b^{-1}(\rho_1(x^b))$. Clearly, $\rho(1) = 1$ and if $x \in T/(z)$ and $r \in R$, then

$$\rho(rx) = F_b^{-1}(\rho_1((rx)^b)) = F_b^{-1}(\rho_1(r^b x^b)) = F_b^{-1}(r^b \rho_1(x^b)) = F_b^{-1}(r^b) F_b^{-1}(\rho_1(x^b)) = r \rho(x).$$

So ρ is R -linear. In view of that $R \hookrightarrow T/(z)$ splits. □

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