Research Article

Soon-Mo Jung* and Ginkyu Choi

Perturbation of the one-dimensional time-independent Schrödinger equation with a rectangular potential barrier

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Abstract: In *Applied Mathematics Letters* **74** (2017), 147–153, the Hyers-Ulam stability of the one-dimensional time-independent Schrödinger equation was investigated when the relevant system has a potential well of finite depth. As a continuous work, we prove in this paper a type of Hyers-Ulam stability of the one-dimensional time-independent Schrödinger equation with a rectangular potential barrier of V_0 in height and $2c$ in width, where V_0 is assumed to be greater than the energy E of the particle under consideration.

Keywords: perturbations, Hyers-Ulam stability, Schrödinger equation, time-independent Schrödinger equation, potential barrier

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1 Introduction

For an open subinterval $I = (a, b)$ of $\mathbb{R}$ with $-\infty \leq a < b \leq +\infty$ and for an integer $n > 0$, we are going to define the Hyers-Ulam stability of the linear differential equation of n th order

$$\mathcal{D}(y^{(n)}, y^{(n-1)}, \dots, y', y, x) = 0, \quad (1)$$

where $y : I \rightarrow \mathbb{C}$ is an n times continuously differentiable function.

We say that the differential equation (1) satisfies (or has) the Hyers-Ulam stability if the following statement is true for each $\varepsilon > 0$: for any n times continuously differentiable function $y : I \rightarrow \mathbb{C}$ satisfying the differential inequality

$$|\mathcal{D}(y^{(n)}, y^{(n-1)}, \dots, y', y, x)| \leq \varepsilon$$

for all $x \in I$, there exists a solution $y_0 : I \rightarrow \mathbb{C}$ to the differential equation (1) such that

$$|y(x) - y_0(x)| \leq K(x, \varepsilon)$$

for all $x \in I$, where the value of $K(x, \varepsilon)$ depends on x and ε , and regardless of the value of x , $\lim_{\varepsilon \rightarrow 0} K(x, \varepsilon) = 0$.

If the limit $\lim_{\varepsilon \rightarrow 0} K(x, \varepsilon)$ is affected by the value of x , then we say that the differential equation (1) satisfies a type of the Hyers-Ulam stability because this phenomenon is also interesting enough, but there is no proper formal terminology yet in this case.

We can refer [1–3] for more detailed definition of the Hyers-Ulam stability. As far as we know, Obloza [4,5] is the first mathematician who has proved the Hyers-Ulam stability of differential equations. In fact,

* **Corresponding author: Soon-Mo Jung**, Mathematics Section, College of Science and Technology, Hongik University, 30016 Sejong, Republic of Korea, e-mail: smjung@hongik.ac.kr

Ginkyu Choi: Department of Electronic and Electrical Engineering, College of Science and Technology, Hongik University, 30016 Sejong, Republic of Korea, e-mail: gkchoi@hongik.ac.kr

Obloza demonstrated the Hyers-Ulam stability of the linear differential equation $y'(x) + g(x)y(x) = r(x)$. Thereafter, a number of mathematicians have dealt with this subject (see [6–13]).

The Hyers-Ulam stability of the linear inhomogeneous differential equation of second order

$$y''(x) + \alpha y'(x) + \beta y(x) = r(x) \quad (2)$$

was studied in [8], where α and β are complex numbers (see also [14,15]). In a later paper [9], the Hyers-Ulam stability of the one-dimensional time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \quad (3)$$

was investigated when the relevant system has a potential well of finite depth. The ideas of papers [8,9] have a great influence on the present paper.

The Schrödinger equation based on postulates of quantum mechanics is a time-dependent equation which yields a time-independent equation that is useful for calculating energy eigenvalues. It is also possible to apply the one-dimensional Schrödinger equation to analyze the state associated with particles reflected by rectangular potential barriers, but this has some distance from the subject of this paper.

In this paper, we consider the one-dimensional time-independent Schrödinger equation (3), where $\psi : \mathbb{R} \rightarrow \mathbb{C}$ is the wave function, V is a rectangular potential barrier, $\hbar$ is the reduced Planck constant, m is the mass of the particle, and E is the energy of the particle. Indeed, we investigate a type of Hyers-Ulam stability of the one-dimensional time-independent Schrödinger equation (3) for the barrier potential with height V_0 and width $2c$, where $0 < E < V_0$.

Since the main results of this paper do not fully meet the condition for the Hyers-Ulam stability, instead of saying that the one-dimensional time-independent Schrödinger equation (3) satisfies the Hyers-Ulam stability, we say that the Schrödinger equation with rectangular potential barrier satisfies a type of Hyers-Ulam stability.

2 Preliminaries

We denote by λ and μ the roots of characteristic equation, $x^2 + \alpha x + \beta = 0$, which is associated with Eq. (2).

We introduce in the following lemma a special version of [16, Theorem 3.2] or [17, Theorem 4] for the case when $\Re(\lambda) = \Re(\mu) = 0$ and $\Im(\lambda) = -\Im(\mu) \neq 0$, i.e., when $\alpha = 0$ and $\beta > 0$. Here, we note that this result also holds for the class of twice continuously differentiable functions $y : I \rightarrow \mathbb{C}$.

Lemma 2.1. *Let $I = (a, b)$ be an open interval and $x_0 \in I$, where $-\infty \leq a < b \leq +\infty$. Assume that $r : I \rightarrow \mathbb{C}$ is a continuous function and the characteristic equation, $x^2 + \beta = 0$, has a pair of complex conjugate roots λ and μ with $\Re(\lambda) = \Re(\mu) = 0$ and $\Im(\lambda) = -\Im(\mu) \neq 0$. The following statement is true for all $\varepsilon > 0$: For any twice continuously differentiable function $y : I \rightarrow \mathbb{C}$ satisfying the inequality*

$$|y''(x) + \beta y(x) - r(x)| \leq \varepsilon$$

for all $x \in I$, there exists a solution $y_0 : I \rightarrow \mathbb{C}$ to the differential equation (2) with $\alpha = 0$ and $\beta > 0$ such that

$$|y(x) - y_0(x)| \leq \frac{\varepsilon}{|\Im(\lambda)|} \left| \int_{x_0}^x |\sin \Im(\lambda)(x - t)| dt \right| = \frac{\varepsilon}{\sqrt{\beta}} \left| \int_0^{x-x_0} |\sin \sqrt{\beta} t| dt \right|$$

for all $x \in I$.

It is to be noted that x_0 can be one of the endpoints of the interval I in the previous lemma because $0 \leq |\sin \sqrt{\beta} t| \leq 1$ holds for any real number t .

Lemma 2.2. Let ω be a positive real constant. Then

$$\left| \int_0^x |\sin(\omega t)| dt \right| \leq \frac{2}{\pi} |x| + \frac{2}{\omega}$$

for all $x \in \mathbb{R}$.

Proof. For any $x \in \mathbb{R}$, we define $n(x) = \left[\frac{\omega x}{\pi} \right]$, where $\left[\frac{\omega x}{\pi} \right]$ denotes the largest integer not exceeding the number $\frac{\omega x}{\pi}$. Then, the integer $n(x)$ satisfies $n(x) \frac{\pi}{\omega} \leq x < (n(x) + 1) \frac{\pi}{\omega}$.

First, we assume that $x \geq 0$. Then

$$\begin{aligned} \left| \int_0^x |\sin(\omega t)| dt \right| &= \sum_{k=0}^{n(x)-1} \int_{k\pi/\omega}^{(k+1)\pi/\omega} |\sin(\omega t)| dt + \int_{n(x)\pi/\omega}^x |\sin(\omega t)| dt \\ &= \sum_{k=0}^{n(x)-1} \left| \int_{k\pi/\omega}^{(k+1)\pi/\omega} \sin(\omega t) dt \right| + \left| \int_{n(x)\pi/\omega}^x \sin(\omega t) dt \right| \leq \frac{2}{\pi} x + \frac{2}{\omega} \end{aligned}$$

for any $x \geq 0$.

Now, let $x < 0$. Since $|\sin(\omega t)|$ is an even function with respect to t , we have

$$\begin{aligned} \left| \int_0^x |\sin(\omega t)| dt \right| &= \int_0^{-x} |\sin(\omega t)| dt = \sum_{k=0}^{n(-x)-1} \int_{k\pi/\omega}^{(k+1)\pi/\omega} |\sin(\omega t)| dt + \int_{n(-x)\pi/\omega}^{-x} |\sin(\omega t)| dt \\ &= \sum_{k=0}^{n(-x)-1} \left| \int_{k\pi/\omega}^{(k+1)\pi/\omega} \sin(\omega t) dt \right| + \left| \int_{n(-x)\pi/\omega}^{-x} \sin(\omega t) dt \right| \leq -\frac{2}{\pi} x + \frac{2}{\omega} \end{aligned}$$

for any $x < 0$. □

3 A type of Hyers-Ulam stability of the Schrödinger equation

In this section, we investigate a type of Hyers-Ulam stability of the one-dimensional time-independent Schrödinger equation (3), where the potential function $V : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$V(x) = \begin{cases} 0 & \text{(for } |x| \geq c), \\ V_0 & \text{(for } |x| < c), \end{cases} \quad (4)$$

where $0 < E < V_0$ (see the following picture).

The potential barrier $V(x)$

In quantum mechanics, the problem of rectangular potential barrier is a typical means of explaining the phenomenon of quantum tunneling and quantum reflection.

Throughout this paper, we investigate a type of Hyers-Ulam stability of the one-dimensional time-independent Schrödinger equation (3) with the potential function $V : \mathbb{R} \rightarrow \mathbb{R}$ defined by (4).

The characteristic roots λ and μ of the one-dimensional time-independent Schrödinger equation (3) are given by

$$(\lambda, \mu) = \begin{cases} (-i\omega_0, i\omega_0) & (\text{for } |x| \geq c), \\ (-\omega_1, \omega_1) & (\text{for } |x| < c), \end{cases}$$

where we set $\omega_0 = \sqrt{\frac{2mE}{\hbar^2}}$ and $\omega_1 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$.

In the following theorem, we assume that the function $\psi : \mathbb{R} \rightarrow \mathbb{C}$ is the piecewise twice continuously differentiable function, which means that each of $\psi|_{(-\infty, -c)}$, $\psi|_{(-c, c)}$, and $\psi|_{(c, +\infty)}$ is twice continuously differentiable.

As we see in Eq. (6), the inequality $|\psi(x) - \phi(x)| \leq K(x, \varepsilon)$ holds for any $x \in \mathbb{R}$ and $\varepsilon > 0$, where $K(x, \varepsilon)$ is strongly affected by the value of x . Hence, we call the phenomenon observed in the following theorem a type of Hyers-Ulam stability.

Theorem 3.1. *Let E and V_0 be real numbers with $0 < E < V_0$. Given any $\varepsilon > 0$, if a piecewise twice continuously differentiable function $\psi : \mathbb{R} \rightarrow \mathbb{C}$ satisfies the inequality*

$$\left| -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) - E\psi(x) \right| \leq \varepsilon \quad (5)$$

for all $x \in \mathbb{R} \setminus \{-c, c\}$, then there exists a piecewise twice continuously differentiable function $\phi : \mathbb{R} \rightarrow \mathbb{C}$, which is a solution to the one-dimensional time-independent Schrödinger equation (3) for all $x \in \mathbb{R} \setminus \{-c, c\}$, such that

$$|\psi(x) - \phi(x)| \leq \begin{cases} 2\left(\frac{\pi - c\omega_0}{\pi\omega_0^2} + \frac{|x|}{\pi\omega_0}\right)\varepsilon & (\text{for } |x| > c), \\ \frac{\varepsilon}{V_0 - E} & (\text{for } |x| < c). \end{cases} \quad (6)$$

Proof. It is to be proved that there exists a piecewise twice continuously differentiable solution $\phi : \mathbb{R} \setminus \{-c, c\} \rightarrow \mathbb{C}$ to the Schrödinger equation (3) that satisfies inequality (6).

First, in view of Lemmas 2.1 and 2.2, there exists a twice continuously differentiable function $\phi_1 : (-\infty, -c) \rightarrow \mathbb{C}$ such that

$$-\frac{\hbar^2}{2m}\phi_1''(x) = E\phi_1(x) \quad (7)$$

and

$$|\psi(x) - \phi_1(x)| \leq \frac{\varepsilon}{\omega_0} \left| \int_0^{x+c} |\sin(\omega_0 t)| dt \right| \leq 2\left(\frac{\pi - c\omega_0}{\pi\omega_0^2} - \frac{x}{\pi\omega_0}\right)\varepsilon \quad (8)$$

for all $x < -c$, where we set $\omega_0 = \sqrt{\frac{2mE}{\hbar^2}}$.

Similarly, there exists a twice continuously differentiable function $\phi_3 : (c, +\infty) \rightarrow \mathbb{C}$ such that

$$-\frac{\hbar^2}{2m}\phi_3''(x) = E\phi_3(x) \quad (9)$$

and

$$|\psi(x) - \phi_3(x)| \leq \frac{\varepsilon}{\omega_0} \left| \int_0^{x-c} |\sin(\omega_0 t)| dt \right| \leq 2\left(\frac{\pi - c\omega_0}{\pi\omega_0^2} + \frac{x}{\pi\omega_0}\right)\varepsilon \quad (10)$$

for all $x > c$.

The solution ϕ_1 to (7) and the solution ϕ_3 to (9) are given by

$$\begin{aligned}\phi_1(x) &= A_1 \cos(\omega_0 x) + B_1 \sin(\omega_0 x), \\ \phi_3(x) &= A_3 \cos(\omega_0 x) + B_3 \sin(\omega_0 x),\end{aligned}\quad (11)$$

where A_1, B_1, A_3 , and B_3 are some complex numbers.

Also, by [8, Theorem 2.2, Remark 2.3], we have a twice continuously differentiable function $\phi_2 : (-c, c) \rightarrow \mathbb{C}$ such that

$$-\frac{\hbar^2}{2m}\phi_2''(x) + V_0\phi_2(x) = E\phi_2(x) \quad \text{and} \quad |\psi(x) - \phi_2(x)| \leq \frac{\varepsilon}{V_0 - E} \quad (12)$$

for all $x \in (-c, c)$. Indeed, ϕ_2 has the form

$$\phi_2(x) = A_2 \exp\{-\omega_1 x\} + B_2 \exp\{\omega_1 x\}, \quad (13)$$

where A_2 and B_2 are some complex numbers, where we let $\omega_1 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$.

Let us define the function $\phi : \mathbb{R} \rightarrow \mathbb{C}$ by

$$\phi(x) = \begin{cases} \phi_1(x) & (\text{for } x < -c), \\ \text{an arbitrary real number} & (\text{for } x = -c), \\ \phi_2(x) & (\text{for } -c < x < c), \\ \text{an arbitrary real number} & (\text{for } x = c), \\ \phi_3(x) & (\text{for } x > c). \end{cases}$$

Then ϕ is a piecewise twice continuously differentiable solution to the one-dimensional time-independent Schrödinger equation (3). Moreover, it follows from (8), (10), and (12) that inequality (6) holds for all $x \in \mathbb{R} \setminus \{-c, c\}$. $\square$

We recall that $\psi : \mathbb{R} \rightarrow \mathbb{C}$ is called a piecewise twice continuously differentiable function provided that each of $\psi|_{(-\infty, -c)}$, $\psi|_{(-c, c)}$, and $\psi|_{(c, +\infty)}$ is twice continuously differentiable. In the following theorem, under the conditions of continuity for the related function and its derivative, we prove a type of Hyers-Ulam stability of the one-dimensional time-independent Schrödinger equation (3) for the class of piecewise twice continuously differentiable functions. We note that in most problems in quantum mechanics, the continuity conditions of the wave function and its derivative are necessary.

Theorem 3.2. *Let E and V_0 be real numbers with $0 < E < V_0$. Assume that $c\omega_0 \neq \frac{\pi}{2} + n\pi$ for any $n \in \mathbb{Z}$. Given $\varepsilon > 0$, if a piecewise twice continuously differentiable function $\psi : \mathbb{R} \rightarrow \mathbb{C}$ satisfies inequality (5) for all $x \in \mathbb{R} \setminus \{-c, c\}$ and if ψ is continuous, then there exists a piecewise twice continuously differentiable function $\phi : \mathbb{R} \rightarrow \mathbb{C}$, which is a solution to the one-dimensional time-independent Schrödinger equation (3) for all $x \in \mathbb{R} \setminus \{-c, c\}$, such that ϕ satisfies each of the following boundary conditions*

- (i) ϕ is continuous on $\mathbb{R}$;
 - (ii) ϕ' is continuous on $\mathbb{R}$
- and such that

$$|\psi(x) - \phi(x)| \leq \begin{cases} 2 \left(\frac{\pi - c\omega_0}{\pi\omega_0^2} + \frac{|x|}{\pi\omega_0} \right) \varepsilon & (\text{for } |x| \geq c), \\ \frac{\varepsilon}{V_0 - E} + 2 \max\{|A_2 - A_2^*|, |B_2 - B_2^*|\} \cosh\{c\omega_1\} & (\text{for } |x| < c), \end{cases}$$

where B_1 and B_3 are given in (11), A_2 and B_2 are formulated in (13), and A_2^* and B_2^* are given in (21).

Proof. It is obvious that the proof of Theorem 3.1 is also valid for this theorem. We therefore use the proof of Theorem 3.1 as it is.

Moreover, we refer to (13) and consider the following function $\phi_2^* : (-c, c) \rightarrow \mathbb{C}$ given by

$$\phi_2^*(x) = A_2^* \exp\{-\omega_1 x\} + B_2^* \exp\{\omega_1 x\}, \quad (14)$$

where A_2^* and B_2^* are some complex numbers. We obviously know that ϕ_2^* is a particular solution to the Schrödinger equation (3) for $-c < x < c$.

Let us define the function $\phi : \mathbb{R} \rightarrow \mathbb{C}$ by

$$\phi(x) = \begin{cases} \phi_1(x) & (\text{for } x < -c), \\ \lim_{s \rightarrow -c^-} \phi_1(s) & (\text{for } x = -c), \\ \phi_2^*(x) & (\text{for } |x| < c), \\ \lim_{s \rightarrow c^+} \phi_3(s) & (\text{for } x = c), \\ \phi_3(x) & (\text{for } x > c). \end{cases} \quad (15)$$

Then ϕ is a piecewise twice continuously differentiable solution to the one-dimensional time-independent Schrödinger equation (3).

By the boundary condition (i), if we impose continuities on ϕ_1 , ϕ_2^* , and ϕ_3 at $x = -c$ and $x = c$, then it follows from (11) that

$$2 \cosh\{c\omega_1\}(A_2^* + B_2^*) = \cos(c\omega_0)(A_1 + A_3) - \sin(c\omega_0)(B_1 - B_3), \quad (16)$$

$$2 \sinh\{c\omega_1\}(A_2^* - B_2^*) = \cos(c\omega_0)(A_1 - A_3) - \sin(c\omega_0)(B_1 + B_3). \quad (17)$$

Moreover, by (11) and the boundary condition (ii), if we impose continuities on ϕ_1' , $\phi_2^{*'}$, and ϕ_3' at $x = -c$ and $x = c$, then we get

$$-2\omega_1 \cosh\{c\omega_1\}(A_2^* - B_2^*) = \omega_0 \sin(c\omega_0)(A_1 - A_3) + \omega_0 \cos(c\omega_0)(B_1 + B_3), \quad (18)$$

$$-2\omega_1 \sinh\{c\omega_1\}(A_2^* + B_2^*) = \omega_0 \sin(c\omega_0)(A_1 + A_3) + \omega_0 \cos(c\omega_0)(B_1 - B_3). \quad (19)$$

In the aforementioned four equations, if we perform a long calculation to eliminate A_2^* and B_2^* from them, we get the following:

$$A_1 = \frac{N}{D}B_1 + \frac{\omega_0\omega_1}{D}B_3 \quad \text{and} \quad A_3 = -\frac{\omega_0\omega_1}{D}B_1 - \frac{N}{D}B_3, \quad (20)$$

where we set

$$\begin{aligned} D &= (\omega_0^2 \sin^2(c\omega_0) + \omega_1^2 \cos^2(c\omega_0)) \sinh\{2c\omega_1\} + \omega_0\omega_1 \sin(2c\omega_0)(\sinh^2\{c\omega_1\} + \cosh^2\{c\omega_1\}), \\ N &= \frac{1}{2}(\omega_1^2 - \omega_0^2) \sin(2c\omega_0) \sinh\{2c\omega_1\} - \omega_0\omega_1 \cos(2c\omega_0)(\sinh^2\{c\omega_1\} + \cosh^2\{c\omega_1\}). \end{aligned}$$

If we substitute the expressions in (20) for A_1 and A_3 in (16) and (17), then we get

$$\begin{aligned} A_2^* + B_2^* &= \frac{1}{2} \left(\frac{N - \omega_0\omega_1}{D} \cos(c\omega_0) - \sin(c\omega_0) \right) \operatorname{sech}\{c\omega_1\} (B_1 - B_3), \\ A_2^* - B_2^* &= \frac{1}{2} \left(\frac{N + \omega_0\omega_1}{D} \cos(c\omega_0) - \sin(c\omega_0) \right) \operatorname{csch}\{c\omega_1\} (B_1 + B_3). \end{aligned}$$

Furthermore, by using the last two equations, we obtain

$$\begin{aligned} A_2^* &= \frac{1}{4} \left(\frac{N - \omega_0\omega_1}{D} \cos(c\omega_0) - \sin(c\omega_0) \right) \operatorname{sech}\{c\omega_1\} (B_1 - B_3) \\ &\quad + \frac{1}{4} \left(\frac{N + \omega_0\omega_1}{D} \cos(c\omega_0) - \sin(c\omega_0) \right) \operatorname{csch}\{c\omega_1\} (B_1 + B_3), \\ B_2^* &= \frac{1}{4} \left(\frac{N - \omega_0\omega_1}{D} \cos(c\omega_0) - \sin(c\omega_0) \right) \operatorname{sech}\{c\omega_1\} (B_1 - B_3) \\ &\quad - \frac{1}{4} \left(\frac{N + \omega_0\omega_1}{D} \cos(c\omega_0) - \sin(c\omega_0) \right) \operatorname{csch}\{c\omega_1\} (B_1 + B_3). \end{aligned} \quad (21)$$

So far, we have proved that the one-dimensional time-independent Schrödinger equation (3) has the piecewise twice continuously differentiable solution ϕ that satisfies the boundary conditions (i) and (ii).

Next, by the continuity of ψ at $x = -c$, it follows from (8), (11), and (15) that

$$\begin{aligned} |\psi(-c) - \phi(-c)| &= |\psi(-c) - \lim_{x \rightarrow -c^-} \phi_1(x)| = \lim_{x \rightarrow -c^-} |\psi(x) - \phi_1(x)| \\ &= |\psi(-c) - A_1 \cos(c\omega_0) + B_1 \sin(c\omega_0)| \\ &\leq \lim_{x \rightarrow -c^-} 2 \left(\frac{\pi - c\omega_0}{\pi\omega_0^2} - \frac{x}{\pi\omega_0} \right) \varepsilon = \frac{2\varepsilon}{\omega_0^2}. \end{aligned} \quad (22)$$

Similarly, by the continuity of ψ at $x = c$, it follows from (10), (11), and (15) that

$$\begin{aligned} |\psi(c) - \phi(c)| &= |\psi(c) - \lim_{x \rightarrow c^+} \phi_3(x)| = \lim_{x \rightarrow c^+} |\psi(x) - \phi_3(x)| \\ &= |\psi(c) - A_3 \cos(c\omega_0) - B_3 \sin(c\omega_0)| \\ &\leq \lim_{x \rightarrow c^+} 2 \left(\frac{\pi - c\omega_0}{\pi\omega_0^2} + \frac{x}{\pi\omega_0} \right) \varepsilon = \frac{2\varepsilon}{\omega_0^2}. \end{aligned}$$

Finally, when $-c < x < c$, it follows from (12)–(15) and (21) that

$$\begin{aligned} |\psi(x) - \phi(x)| &= |\psi(x) - \phi_2^*(x)| \leq |\psi(x) - \phi_2(x)| + |\phi_2(x) - \phi_2^*(x)| \\ &= |\psi(x) - \phi_2(x)| + |(A_2 - A_2^*)\exp\{-\omega_1 x\} + (B_2 - B_2^*)\exp\{\omega_1 x\}| \\ &\leq \frac{\varepsilon}{V_0 - E} + 2 \max\{|A_2 - A_2^*|, |B_2 - B_2^*|\} \cosh\{c\omega_1\}, \end{aligned}$$

where B_1 and B_3 are given in (11), A_2 and B_2 are formulated in (13), and A_2^* and B_2^* are given in (21). $\square$

By adding the evenness condition to the set of boundary conditions in Theorem 3.2, we achieve the following corollary.

Corollary 3.3. *Let E and V_0 be real numbers with $0 < E < V_0$. Assume that $c\omega_0 \neq \frac{\pi}{2} + n\pi$ for any $n \in \mathbb{Z}$. Given $\varepsilon > 0$, if a piecewise twice continuously differentiable function $\psi : \mathbb{R} \rightarrow \mathbb{C}$ satisfies inequality (5) for all $x \in \mathbb{R} \setminus \{-c, c\}$ and if ψ is even and continuous, then there exists a piecewise twice continuously differentiable function $\phi : \mathbb{R} \rightarrow \mathbb{C}$, which is a solution to the one-dimensional time-independent Schrödinger equation (3) for all $x \in \mathbb{R} \setminus \{-c, c\}$, such that ϕ satisfies each of the following conditions*

- (i) ϕ is continuous on $\mathbb{R}$;
- (ii) ϕ' is continuous on $\mathbb{R}$;
- (iii) ϕ is an even function and such that

$$|\psi(x) - \phi(x)| \leq \begin{cases} 2 \left(\frac{\pi - c\omega_0}{\pi\omega_0^2} + \frac{|x|}{\pi\omega_0} \right) \varepsilon & (\text{for } |x| \geq c), \\ \frac{\varepsilon}{V_0 - E} + 2|A_2 - A_2^*| \cosh\{c\omega_1\} & (\text{for } |x| < c), \end{cases} \quad (23)$$

where A_2 is formulated in (13) and A_2^* is given in (21).

Proof. As we see in (11), let $\phi_1 : (-\infty, -c) \rightarrow \mathbb{C}$ be the twice continuously differentiable function which satisfies the one-dimensional time-independent Schrödinger equation (7) and inequality (8) for all $x < -c$. Let us define the twice continuously differentiable function $\phi_3 : (c, \infty) \rightarrow \mathbb{C}$ by $\phi_3(x) = \phi_1(-x)$ for all $x > c$. If we put $z = -x$, then we have $\phi_3(x) = \phi_1(z)$ and

$$\begin{aligned} \frac{d}{dx} \phi_3(x) &= \frac{d}{dz} \phi_1(z) \cdot \frac{dz}{dx} = -\frac{d}{dz} \phi_1(z), \\ \frac{d^2}{dx^2} \phi_3(x) &= -\frac{d^2}{dz^2} \phi_1(z) \cdot \frac{dz}{dx} = \frac{d^2}{dz^2} \phi_1(z) \end{aligned}$$

for all $x > c$ and equivalently for all $z < -c$. In addition, it follows from (7) and the last equality that

$$\frac{d^2}{dx^2}\phi_3(x) = \frac{d^2}{dz^2}\phi_1(z) = -\frac{2mE}{\hbar^2}\phi_1(z) = -\frac{2mE}{\hbar^2}\phi_3(x)$$

for all $x > c$, i.e., ϕ_3 satisfies the one-dimensional time-independent Schrödinger equation (9) and inequality (10) for any $x > c$ because of the evenness of ψ and by (8).

By (11), (14), and (20), we get

$$A_1 - A_3 = \tan(\omega_0 x)(B_1 + B_3) = \frac{N + \omega_0 \omega_1}{D}(B_1 + B_3)$$

for all $x > c$. Hence, by (21) and the last equality, and by imposing the evenness on ψ and ϕ_2^* , we obtain that $A_1 = A_3$, $B_1 = -B_3$, and $A_2^* = B_2^*$. In addition, we set

$$\begin{cases} \phi_1(x) = A_1 \cos(\omega_0 x) + B_1 \sin(\omega_0 x) & (\text{for } x < -c), \\ \phi_2^*(x) = \left(\frac{N - \omega_0 \omega_1}{D} \cos(c\omega_0) - \sin(c\omega_0) \right) \operatorname{sech}\{c\omega_1\} B_1 \cosh\{\omega_1 x\} & (\text{for } |x| < c), \\ \phi_3(x) = A_1 \cos(\omega_0 x) - B_1 \sin(\omega_0 x) & (\text{for } x > c). \end{cases} \quad (24)$$

As we did in the proof of Theorem 3.2, we define the function $\phi : \mathbb{R} \rightarrow \mathbb{C}$ by using formula (15). Then ϕ is a piecewise twice continuously differentiable solution to the one-dimensional time-independent Schrödinger equation (3) and ϕ satisfies inequality (23) and conditions (i), (ii), and (iii). $\square$

4 Discussion

We assume that there is an isolated system, which can be described by the one-dimensional time-independent Schrödinger equation (3). Knowing the general solution and initial conditions of the Schrödinger equation determines the past, present, and future of this system completely. So we can say that this system is a predictable one.

However, sometimes due to some external disturbances, the system is not determined by Schrödinger equation (3), but it can only be explained by using inequality such as (5). In this case, we cannot accurately predict the future of the disturbed system.

Although the system cannot be accurately predicted due to external disturbances, we say that if the future of the disturbed system follows the exact solution of the Schrödinger equation within a certain error bound, the system (or the Schrödinger equation) has the Hyers-Ulam stability. But if the error bound depends on the value of x , such as inequality (6), we say that the system (or the Schrödinger equation) does not have the Hyers-Ulam stability. The main result obtained in this paper is an example of this phenomenon and the resonance is a typical example.

The above argument is a quotation from the paper [18], and it illustrates how important the Hyers-Ulam concept is.

In this paper, we have not demonstrated the Hyers-Ulam stability of the Schrödinger equation. Instead, we could prove a type of Hyers-Ulam stability, a weaker result than the Hyers-Ulam stability. We have hereby demonstrated that there is an exact solution to the Schrödinger equation near every approximate solution to the equation.

5 Conclusion

In Section 3, we proved a type of Hyers-Ulam stability of the one-dimensional time-independent Schrödinger equation (3). Because the difference between the approximate solution ψ and the exact solution ϕ of

the one-dimensional time-independent Schrödinger equation (3) is strongly influenced by x , we have not proved in Theorem 3.1, Theorem 3.2, or in Corollary 3.3 the Hyers-Ulam stability of the one-dimensional time-independent Schrödinger equation in a strict sense when the potential is a rectangular barrier and $0 < E < V_0$. Therefore, instead of saying that we have proved the Hyers-Ulam stability in this paper, we say that we have proved a type of Hyers-Ulam stability.

It is necessary to improve Lemma 2.1 first to demonstrate the exact Hyers-Ulam stability of the Schrödinger equation (3) with a rectangular potential barrier. But it seems that this work will be very difficult. Therefore, instead of attempting this improvement in this paper, we intend to leave this improvement as an open problem.

We now introduce the detailed process of constructing the exact solution ϕ to the Schrödinger equation, which repeatedly appears in the main theorems of this paper.

Let E and V_0 be real numbers with $0 < E < V_0$. Assume that the potential function $V : \mathbb{R} \rightarrow \mathbb{R}$ is given by formula (4) and that $c\omega_0 \neq \frac{\pi}{2} + n\pi$ for any $n \in \mathbb{Z}$. We define

$$\begin{aligned} \mathcal{P}_c = \{ \varphi : \mathbb{R} \rightarrow \mathbb{C} \mid & \varphi \text{ is continuous and even; } \varphi|_{(-\infty, -c)}, \varphi|_{(-c, c)}, \text{ and} \\ & \varphi|_{(c, +\infty)} \text{ are twice continuously differentiable; and} \\ & \varphi' \text{ is continuous} \}. \end{aligned}$$

Assume that $\psi \in \mathcal{P}_c$ and ψ satisfies inequality (5) for all $x \in \mathbb{R} \setminus \{-c, c\}$. Then, due to Theorem 3.1, we can choose a twice continuously differentiable solution $\phi_1 : (-\infty, -c) \rightarrow \mathbb{C}$ to the differential equation (7), i.e., we have determined coefficients A_1 and B_1 .

Now, we define the twice continuously differentiable function $\phi_3 : (c, +\infty) \rightarrow \mathbb{C}$ by $\phi_3(x) = \phi_1(-x)$ for each $x > c$. Then it is easy to see that ϕ_3 is a solution to the differential equation (9). Furthermore, we define the function $\phi : \mathbb{R} \rightarrow \mathbb{C}$ by using (15) and (24). We know that $\phi \in \mathcal{P}_c$ and this ϕ satisfies all the requirements stated in Corollary 3.3.

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