Research Article

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Simply connected topological spaces of weighted composition operators

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Abstract: In this paper, we prove that the topological spaces of nonzero weighted composition operators acting on some Hilbert spaces of analytic functions on the unit open ball are simply connected.

Keywords: weighted composition operator, simple connectedness, deformation retraction, operator norm topology, Hilbert-Schmidt norm topology

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1 Introduction

Let $H(\mathbb{B}_N)$ be the space of analytic functions on the open unit ball

$$\mathbb{B}_N := \left\{ z = (z_1, \dots, z_N) \in \mathbb{C}^N : \|z\|^2 = \sum_{i=1}^N |z_i|^2 < 1 \right\}$$

and $H^\infty(\mathbb{B}_N)$ the space of bounded analytic functions on $\mathbb{B}_N$ with the supremum norm $\|\cdot\|_\infty$. When $N = 1$, the unit open ball reduces to the unit open disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let $\mathcal{S}(\mathbb{B}_N)$ be the set of analytic self-maps of $\mathbb{B}_N$. Every $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathcal{S}(\mathbb{B}_N)$ induces a composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{B}_N)$. If $u \in H(\mathbb{B}_N)$, the multiplication operator $M_u : H(\mathbb{B}_N) \rightarrow H(\mathbb{B}_N)$ is defined by $M_u(f) = u \cdot f$ for any $f \in H(\mathbb{B}_N)$. If $u \in H(\mathbb{B}_N)$ and $\varphi \in \mathcal{S}(\mathbb{B}_N)$, we define the weighted composition operator $W_{u,\varphi}$ to be the product $M_u C_\varphi$.

Much effort has been expended on characterizing those analytic maps which induce bounded or compact composition operators between classical spaces of analytic functions. Readers interested in this topic can refer to [1–4], all of which are excellent sources for the development of the theory of composition operators and function spaces.

An active topic is the topological structure of the space of composition operators acting on a given function space. When X is a Banach space of analytic functions, we write $C(X)$ for the topological space of composition operators on X under the operator norm topology. The investigation of the topological structure of $C(H^2(\mathbb{D}))$ was initiated by Berkson [5] in 1981. The central problem focuses on the relations between the structure of $C(H^2(\mathbb{D}))$ and the compactness of its members.

In 1990, Shapiro and Sundberg [6] gave results on compact difference and isolation. They posed the following fundamental question:

(*) Do the compact composition operators form a connected component of the set $C(H^2(\mathbb{D}))$?

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In [7], Gallardo-Gutiérrez et al. answered the question by demonstrating that, while the compact operators form a connected subset, there are non-compact operators belonging to the same component. In [8], Bourdon determined when two linear-fractional composition operators on the Hardy space $H^2(\mathbb{D})$ belong to the same component in $C(H^2(\mathbb{D}))$. He showed that two linear-fractional composition operators in the same component may fail to have compact difference. In [9], Hammond and MacCluer completely described the component structure of $C(H^p, H^q)$ when $0 < q < p \leq \infty$. Moorhouse and Toews [10] used Carleson measure techniques to give a sufficient condition for the difference of two composition operators on $A_\alpha^2(\mathbb{D})$ to be compact. In 2005, Moorhouse [11] answered the question of compact difference of composition operators acting on $A_\lambda^2(\mathbb{D})$, $\lambda > -1$, and gave a partial answer to the component structure of $C(A_\lambda^2(\mathbb{D}))$. Recently, Choe, Koo and Park [12,13] extended Moorhouse's characterization to the unit polydisk and unit ball in $\mathbb{C}^N$. In 2015, Hosokawa, Izuchi and Ohno [14] investigated the topological space of weighted composition operators acting between some Hilbert spaces on the unit disk in general, and they also considered the Hilbert-Schmidt norm topology. Inspired by the aforementioned studies, we are interested in studying the topological spaces of weighted composition operators further and giving a more detailed characterization for those topological spaces. We are concerned of not only the path connectedness but also the fundamental group of those topological spaces. We now state our main result.

Main Theorem 1.1 *If $\mathcal{H}_1$ and $\mathcal{H}$ are two Hilbert spaces satisfying conditions (c1)–(c5) in Section 2, and $\|z^\alpha\|_{\mathcal{H}} / \|z^\alpha\|_{\mathcal{H}_1} \rightarrow 0$ as $|\alpha| \rightarrow \infty$, then the topological space $C_w(\mathcal{H}_1, \mathcal{H})$ is simply connected.*

The paper is organized as follows: some basic definitions and lemmas are introduced in Section 2. Our main results are proved in Section 3; that is, we prove the topological spaces of nonzero weighted composition operators acting between certain Hilbert spaces of analytic functions on $\mathbb{B}_N$ are simply connected. We also prove that Hilbert-Schmidt topological spaces of nonzero weighted composition operators are simply connected in Section 4. Finally, some applications are listed in Section 5.

2 Preliminaries

Recall that a topological space is said to be *simply connected* if its fundamental group is $\{0\}$, or equivalently every closed continuous path can be homotopic to a point path. Inspired by the discussions in [14], we continue to investigate the simply connected topological spaces in higher dimensional case. One of the interesting problems is to characterize the topological spaces of weighted composition operators acting between Dirichlet spaces and Hardy spaces or weighted Bergman spaces.

Let dv denote the volume measure on $\mathbb{B}_N$, normalized so that $v(\mathbb{B}_N) = 1$. The surface measure on $\mathbb{S}_N$ will be denoted by $d\sigma$. Once again, we have normalized σ so that $\sigma(\mathbb{S}_N) = 1$. The normalizing constants are the volume of $\mathbb{B}_N$ and the surface area of $\mathbb{S}_N$. Let $H^2(\mathbb{B}_N)$ be the classical Hardy-Hilbert space on $\mathbb{B}_N$, namely, the space of holomorphic functions f on $\mathbb{B}_N$ such that

$$\|f\|_{H^2}^2 = \sup_{0 < r < 1} \int_{\mathbb{S}_N} |f(r\zeta)|^2 d\sigma(\zeta) < \infty.$$

It is well known that

$$\|z^\alpha\|_{H^2}^2 = \frac{(N-1)! \alpha!}{(N-1+|\alpha|)!},$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ and $\alpha! = \alpha_1! \cdots \alpha_N!$. The inner product of $H^2(\mathbb{B}_N)$ is given by

$$\langle f, g \rangle_{H^2} = \int_{\mathbb{S}_N} f^*(\zeta) \overline{g^*(\zeta)} d\sigma(\zeta),$$

where $f, g \in H^2$ and f^*, g^* are radial functions of f, g . The reproducing kernel of $H^2(\mathbb{B}_N)$ is given by

$$K_{H^2, w}(z) = \frac{1}{(1 - \langle z, w \rangle)^N}.$$

For $\lambda > -1$, the weighted Bergman space $A_\lambda^2(\mathbb{B}_N)$ on $\mathbb{B}_N$ is the space of $f \in H(\mathbb{B}_N)$ satisfying

$$\|f\|_{A_\lambda^2}^2 := \int_{\mathbb{B}_N} |f(z)|^2 d\nu_\lambda(z) < \infty,$$

where $d\nu_\lambda = \frac{\Gamma(N+\lambda+1)}{N! \Gamma(\lambda+1)} (1 - \|z\|^2)^\lambda d\nu(z)$. The inner product of $A_\lambda^2(\mathbb{B}_N)$ is given by

$$\langle f, g \rangle_{A_\lambda^2} = \int_{\mathbb{B}_N} f(z) \overline{g(z)} d\nu_\lambda(z),$$

where $f, g \in A_\lambda^2$. The reproducing kernel of $A_\lambda^2(\mathbb{B}_N)$ is given by

$$K_{A_\lambda^2, w}(z) = \frac{1}{(1 - \langle z, w \rangle)^{N+1+\lambda}}.$$

When $\lambda = 0$, $A_0^2(\mathbb{B}_N)$ is the classical Bergman space. It is well known that

$$\|z^\alpha\|_{A_0^2}^2 = \frac{\alpha! \Gamma(N + \lambda + 1)}{\Gamma(N + |\alpha| + \lambda + 1)},$$

where $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N$. We denote by $\mathcal{D}(\mathbb{B}_N)$ the Dirichlet space; that is, the space of holomorphic functions f on $\mathbb{B}_N$ such that all the functions

$$(1 - \|z\|^2)^{|\alpha|} \frac{\partial^\alpha f}{\partial z^\alpha}(z)$$

belong to $L^2\left(\mathbb{B}_N, \frac{d\nu(z)}{(1 - \|z\|^2)^{N+1}}\right)$ for any positive integer $n = |\alpha| > N/2$. It is known that holomorphic $f(z) = \sum_\alpha c_\alpha z^\alpha$ belongs to $\mathcal{D}(\mathbb{B}_N)$ if and only if $\sum_\alpha |\alpha| \frac{\alpha!}{|\alpha|!} |c_\alpha|^2 < \infty$, and

$$\|z^\alpha\|_{\mathcal{D}}^2 = |\alpha| \frac{\alpha!}{|\alpha|!}.$$

The inner product of $\mathcal{D}(\mathbb{B}_N)$ is given by

$$\langle f, g \rangle_{\mathcal{D}} = \sum_\alpha |\alpha| \frac{\alpha!}{|\alpha|!} a_\alpha \bar{b}_\alpha,$$

where both $f(z) = \sum_\alpha a_\alpha z^\alpha$ and $g(z) = \sum_\alpha b_\alpha z^\alpha$ are in $\mathcal{D}(\mathbb{B}_N)$. The reproducing kernel of $\mathcal{D}(\mathbb{B}_N)$ is given by

$$K_{\mathcal{D}, w}(z) = \log \frac{1}{1 - \langle z, w \rangle}.$$

We first study the topological spaces of weighted composition operators acting between general Hilbert spaces of analytic functions on $\mathbb{B}_N$. As applications, we give several simply connected topological spaces of weighted composition operators. Let $\mathcal{H}$ denote a Hilbert space of analytic functions on $\mathbb{B}_N$, with norm $\|\cdot\|_{\mathcal{H}}$, that satisfies the following conditions:

- (c1) The point evaluation $\varepsilon_w : f \mapsto f(w)$ is a bounded linear functional on $\mathcal{H}$ for any $w \in \mathbb{B}_N$, and $\sup_{\|w\| \leq r} \|\varepsilon_w\|_{\mathcal{H}} < \infty$ for every $0 < r < 1$.
- (c2) $H^\infty(\mathbb{B}_N) \cdot \mathcal{H} \subset \mathcal{H}$ and $\|fg\|_{\mathcal{H}} \leq \|f\|_\infty \|g\|_{\mathcal{H}}$ for every $f \in H^\infty(\mathbb{B}_N)$ and $g \in \mathcal{H}$.
- (c3) $\|1\|_{\mathcal{H}} = 1$ and $\{z^\alpha / \|z^\alpha\|_{\mathcal{H}} : \alpha = (\alpha_1, \dots, \alpha_N) \text{ with } \alpha_i \geq 0\}$ is an orthonormal basis in $\mathcal{H}$.
- (c4) For every $f \in \mathcal{H}$ and $0 \leq r \leq 1$, we have $f_r(z) := f(rz) \in \mathcal{H}$.

By (c3), $\mathcal{H}$ contains all polynomials of $z = (z_1, \dots, z_N)$. By (c2), we have $\|f\|_{\mathcal{H}} = \|f \cdot 1\|_{\mathcal{H}} \leq \|f\|_\infty \|1\|_{\mathcal{H}} = \|f\|_\infty$ for $f \in H^\infty$. We note that the aforementioned conditions are very common for many classical Hilbert spaces of analytic functions on $\mathbb{B}_N$. Let $C_w(\mathcal{H})$ denote the space of nonzero bounded weighted composition operators on $\mathcal{H}$ with the operator norm topology; that is,

$$C_w(\mathcal{H}) = \{W_{u, \varphi} : \mathcal{H} \rightarrow \mathcal{H} \text{ is bounded, } u \neq 0\}.$$

Let $\mathcal{H}_1$ denote another Hilbert space of analytic functions on $\mathbb{B}_N$, with $\mathcal{H}_1 \subset \mathcal{H}$, satisfying conditions (c1), (c3) and (c4). Hence, $\mathcal{H}_1$ contains all polynomials of z . Furthermore, we assume that $\mathcal{H}_1$ satisfies:

(c5) $\|f\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}_1}$ for every $f \in \mathcal{H}_1$.

If the weighted composition operator $W_{u,\varphi} : \mathcal{H}_1 \rightarrow \mathcal{H}$ is bounded, then $u \in \mathcal{H}$ by the fact that $1 \in \mathcal{H}_1$. Let $C_w(\mathcal{H}_1, \mathcal{H})$ denote the collection of bounded weighted composition operators $W_{u,\varphi} : \mathcal{H}_1 \rightarrow \mathcal{H}$ such that $W_{u,\varphi} \in C_w(\mathcal{H})$. By the closed graph theorem, it is well known that any bounded operator $W_{u,\varphi} : \mathcal{H} \rightarrow \mathcal{H}$ can be restricted to a bounded operator $W_{u,\varphi} : \mathcal{H}_1 \rightarrow \mathcal{H}$. We can see $C_w(\mathcal{H}_1, \mathcal{H}) = C_w(\mathcal{H})$ as sets. Denote the operator norm of a bounded linear operator $T : \mathcal{H}_1 \rightarrow \mathcal{H}$ by $\|T\|_{\mathcal{H}_1, \mathcal{H}}$. We equip the set $C_w(\mathcal{H}_1, \mathcal{H})$ with the topology induced by the norm $\|\cdot\|_{\mathcal{H}_1, \mathcal{H}}$. In the following context, we discuss the topological connectedness of the space $C_w(\mathcal{H}_1, \mathcal{H})$.

On the other hand, if $W_{u,\varphi} \in C_w(\mathcal{H})$, we can use (c5) to obtain that

$$\|W_{u,\varphi}f\|_{\mathcal{H}} \leq \|W_{u,\varphi}\|_{\mathcal{H}} \|f\|_{\mathcal{H}} \leq \|W_{u,\varphi}\|_{\mathcal{H}} \|f\|_{\mathcal{H}_1}$$

for every $f \in \mathcal{H}_1$. From this point of view, $W_{u,\varphi} : \mathcal{H}_1 \rightarrow \mathcal{H}$ is bounded and

$$\|W_{u,\varphi}\|_{\mathcal{H}_1, \mathcal{H}} \leq \|W_{u,\varphi}\|_{\mathcal{H}}, \quad (1)$$

which implies that the topology of $C_w(\mathcal{H})$ is stronger than the one of $C_w(\mathcal{H}_1, \mathcal{H})$.

We need the next lemma in Section 3 to prove the simple connectedness of $C_w(\mathcal{H}_1, \mathcal{H})$.

Lemma 2.1. *If $\varphi \in S(\mathbb{B}_N)$ and $0 < r < 1$ such that*

$$\sup_{z \in \mathbb{B}_N} \|\varphi(z)\| = r < 1,$$

then $C_\varphi f \in H^\infty(\mathbb{B}_N)$ for every $f \in \mathcal{H}$ and

$$\|C_\varphi f\|_\infty \leq \|f\|_{\mathcal{H}} \sup_{\|w\| \leq r} \|\varepsilon_w\|_{\mathcal{H}}.$$

Proof. For $f \in \mathcal{H}$ and $z \in \mathbb{B}_N$, by (c1) we have

$$|(C_\varphi f)(z)| = |f(\varphi(z))| \leq \|f\|_{\mathcal{H}} \|\varepsilon_{\varphi(z)}\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}} \sup_{\|w\| \leq r} \|\varepsilon_w\|_{\mathcal{H}},$$

which completes the proof. $\square$

3 Operator norm topological spaces

Let $\mathcal{H}$ and $\mathcal{H}_1$ be spaces satisfying the conditions given in Section 2. In this section, we study the simple connectedness of the topological space $C_w(\mathcal{H}_1, \mathcal{H})$ with operator norm topology.

Note that the composition operator induced by the self-mapping

$$0 : z \mapsto 0 \quad (\forall z \in \mathbb{B}_N)$$

is a bounded composition operator C_0 in $C_w(\mathcal{H}_1, \mathcal{H})$. If we denote by

$$\mathcal{M}_0(\mathcal{H}_1, \mathcal{H}) = \{W_{u,0} = M_u C_0 : \mathcal{H}_1 \rightarrow \mathcal{H} | u \in \mathcal{H}\},$$

it follows from the closed graph theorem that $\mathcal{M}_0(\mathcal{H}_1, \mathcal{H}) \subset C_w(\mathcal{H}_1, \mathcal{H})$.

A *deformation retraction* of a space X onto a subspace A is a family of maps $T_t : X \rightarrow X$, $t \in [0, 1]$ such that

- (a) $T_0 = \text{id}$,
- (b) $T_1(X) = A$,
- (c) $T_t|_A = \text{id}_A$ for all t ,

and the map $T : X \times [0, 1] \rightarrow X$, $(x, t) \mapsto T_t(x)$ is continuous.

Lemma 3.1. *If $\|z^\alpha\|_{\mathcal{H}}/\|z^\alpha\|_{\mathcal{H}_1} \rightarrow 0$ as $|\alpha| \rightarrow \infty$, then the space $\mathcal{M}_0(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$ with operator norm topology is a deformation retraction of $C_w(\mathcal{H}_1, \mathcal{H})$.*

Proof. Let $W_{u,\varphi} \in C_w(\mathcal{H}_1, \mathcal{H})$. Since $C_w(\mathcal{H}_1, \mathcal{H}) = C_w(\mathcal{H})$ as sets, we have $u \in \mathcal{H}$ and $\|W_{u,\varphi}\|_{\mathcal{H}} < \infty$. Let $0 \leq r < 1$. For $f \in \mathcal{H}$, by Lemma 2.1 we have $f \circ r\varphi = f(r\varphi_1, \dots, r\varphi_N) \in H^\infty$ and by (c2)

$$\|W_{u,r\varphi}f\|_{\mathcal{H}} = \|u(f \circ r\varphi)\|_{\mathcal{H}} \leq \|f \circ r\varphi\|_{\infty} \|u\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}} \|u\|_{\mathcal{H}} \sup_{\|w\| \leq r} \|\varepsilon_w\|_{\mathcal{H}}.$$

By (c1), $W_{u,r\varphi} \in C_w(\mathcal{H})$, so $W_{u,r\varphi} \in C_w(\mathcal{H}_1, \mathcal{H})$.

We shall show that the family of maps

$$T_t : C_w(\mathcal{H}_1, \mathcal{H}) \rightarrow C_w(\mathcal{H}_1, \mathcal{H}), \quad W_{u,\varphi} \mapsto W_{u,(1-t)\varphi} \quad \forall t \in [0, 1]$$

is a deformation retraction of $C_w(\mathcal{H}_1, \mathcal{H})$ onto $\mathcal{M}_0(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$. Conditions (a), (b) and (c) are obvious. We just need to verify the continuity of T .

First of all, we prove $T_t : C_w(\mathcal{H}_1, \mathcal{H}) \rightarrow C_w(\mathcal{H}_1, \mathcal{H})$ is continuous for each $t \in [0, 1]$. If $W_{u,\varphi}, W_{v,\psi} \in C_w(\mathcal{H}_1, \mathcal{H})$, we note that $f_t(z) := f(tz) \in H^\infty$ for $f \in \mathcal{H}_1$, and

$$\begin{aligned} \|W_{u,t\varphi} - W_{v,t\psi}\|_{\mathcal{H}_1, \mathcal{H}} &= \sup_{\|f\|_{\mathcal{H}_1} \leq 1} \|(W_{u,t\varphi} - W_{v,t\psi})f\|_{\mathcal{H}} \\ &= \sup_{\|f\|_{\mathcal{H}_1} \leq 1} \|(W_{u,\varphi} - W_{v,\psi})f_t\|_{\mathcal{H}} \\ &\leq \sup_{\|f\|_{\mathcal{H}_1} \leq 1} \|W_{u,\varphi} - W_{v,\psi}\|_{\mathcal{H}_1, \mathcal{H}} \|f(tz)\|_{\infty} \\ &\leq \sup_{\|f\|_{\mathcal{H}_1} \leq 1} \|W_{u,\varphi} - W_{v,\psi}\|_{\mathcal{H}_1, \mathcal{H}} \|f\|_{\mathcal{H}} \sup_{\|w\| \leq t} \|\varepsilon_w\|_{\mathcal{H}} \\ &\leq \|W_{u,\varphi} - W_{v,\psi}\|_{\mathcal{H}_1, \mathcal{H}} \sup_{\|w\| \leq t} \|\varepsilon_w\|_{\mathcal{H}}. \end{aligned}$$

Then we prove that the map

$$T(\cdot, W_{u,\varphi}) : [0, 1] \rightarrow C_w(\mathcal{H}_1, \mathcal{H}), \quad t \mapsto W_{u,t\varphi}$$

is continuous. Fixing $0 \leq t_0 \leq 1$, we apply a similar method to that in [14] to show that $\|W_{u,t_0\varphi} - W_{u,t\varphi}\|_{\mathcal{H}_1, \mathcal{H}} \rightarrow 0$ as $t \rightarrow t_0$. Let $g(z) = \sum_{\alpha} c_{\alpha} z^{\alpha} \in \mathcal{H}_1$. For each $0 \leq t \leq 1$, let

$$g_t(z) = \sum_{\alpha} c_{\alpha} (t_0^{|\alpha|} - t^{|\alpha|}) z^{\alpha}.$$

Since $\mathcal{H}_1 \subset \mathcal{H}$ and $\mathcal{H}$ satisfies (c4), we have $g_t \in \mathcal{H}$. Hence,

$$\begin{aligned} \|(W_{u,t_0\varphi} - W_{u,t\varphi})g\|_{\mathcal{H}}^2 &= \left\| u \sum_{\alpha} c_{\alpha} (t_0^{|\alpha|} - t^{|\alpha|}) \varphi^{\alpha} \right\|_{\mathcal{H}}^2 = \|W_{u,\varphi} g_t\|_{\mathcal{H}}^2 \\ &\leq \|W_{u,\varphi}\|_{\mathcal{H}}^2 \|g_t\|_{\mathcal{H}}^2 = \|W_{u,\varphi}\|_{\mathcal{H}}^2 \sum_{\alpha} |c_{\alpha}|^2 |t_0^{|\alpha|} - t^{|\alpha|}|^2 \|z^{\alpha}\|_{\mathcal{H}}^2 \quad \text{by (c3)} \\ &\leq \|W_{u,\varphi}\|_{\mathcal{H}}^2 \sup_{|\alpha| > 0} \left(|t_0^{|\alpha|} - t^{|\alpha|}|^2 \frac{\|z^{\alpha}\|_{\mathcal{H}}^2}{\|z^{\alpha}\|_{\mathcal{H}_1}^2} \right) \sum_{\alpha} |c_{\alpha}|^2 \|z^{\alpha}\|_{\mathcal{H}_1}^2 \\ &\leq \|W_{u,\varphi}\|_{\mathcal{H}}^2 \sup_{|\alpha| > 0} \left(|t_0^{|\alpha|} - t^{|\alpha|}| \frac{\|z^{\alpha}\|_{\mathcal{H}}}{\|z^{\alpha}\|_{\mathcal{H}_1}} \right)^2 \|g\|_{\mathcal{H}_1}^2. \end{aligned}$$

Then

$$\|W_{u,t_0\varphi} - W_{u,t\varphi}\|_{\mathcal{H}_1, \mathcal{H}} \leq \|W_{u,\varphi}\|_{\mathcal{H}} \sup_{|\alpha| > 0} \left(|t_0^{|\alpha|} - t^{|\alpha|}| \frac{\|z^{\alpha}\|_{\mathcal{H}}}{\|z^{\alpha}\|_{\mathcal{H}_1}} \right).$$

For any positive integer n , we have

$$\sup_{|\alpha|>0} \left(|t_0^{|\alpha|} - t^{|\alpha|}| \frac{\|z^\alpha\|_{\mathcal{H}}}{\|z^\alpha\|_{\mathcal{H}_1}} \right) \leq \sum_{|\alpha|<n} \left(|t_0^{|\alpha|} - t^{|\alpha|}| \frac{\|z^\alpha\|_{\mathcal{H}}}{\|z^\alpha\|_{\mathcal{H}_1}} \right) + \sup_{|\alpha|\geq n} \frac{\|z^\alpha\|_{\mathcal{H}}}{\|z^\alpha\|_{\mathcal{H}_1}}.$$

Hence,

$$\limsup_{t \rightarrow t_0} \|W_{u,t_0\varphi} - W_{u,t\varphi}\|_{\mathcal{H}_1, \mathcal{H}} \leq \|W_{u,\varphi}\|_{\mathcal{H}} \sup_{|\alpha|\geq n} \frac{\|z^\alpha\|_{\mathcal{H}}}{\|z^\alpha\|_{\mathcal{H}_1}}.$$

Recall that $\|z^\alpha\|_{\mathcal{H}} / \|z^\alpha\|_{\mathcal{H}_1} \rightarrow 0$ as $|\alpha| \rightarrow \infty$, then we have $W_{u,t\varphi} \rightarrow W_{u,t_0\varphi}$ as $t \rightarrow t_0$ in $C_w(\mathcal{H}_1, \mathcal{H})$.

Now we are ready to prove that the map

$$T : C_w(\mathcal{H}_1, \mathcal{H}) \times [0, 1] \rightarrow C_w(\mathcal{H}_1, \mathcal{H})$$

is jointly continuous, where the topological space $C_w(\mathcal{H}_1, \mathcal{H})$ consists of bounded nonzero weighted composition operators from $\mathcal{H}_1$ to $\mathcal{H}$ with the operator norm topology, and the closed interval $[0, 1]$ is equipped with the usual Euclidean topology.

The product topology of the space $C_w(\mathcal{H}_1, \mathcal{H}) \times [0, 1]$ is the weakest topology that makes the natural projections bounded; that is, both

$$\text{Projection}_1 : C_w(\mathcal{H}_1, \mathcal{H}) \times [0, 1] \rightarrow C_w(\mathcal{H}_1, \mathcal{H}), \quad (W_{u,\varphi}, t) \mapsto W_{u,\varphi}$$

and

$$\text{Projection}_2 : C_w(\mathcal{H}_1, \mathcal{H}) \times [0, 1] \rightarrow [0, 1], \quad (W_{u,\varphi}, t) \mapsto t$$

are continuous. It follows immediately that $(W_{u,\varphi}, t) \rightarrow (W_{u_0,\varphi_0}, t_0)$ implies both $|t - t_0| \rightarrow 0$ and

$$\|W_{u,\varphi} - W_{u_0,\varphi_0}\|_{\mathcal{H}_1, \mathcal{H}} \rightarrow 0$$

in $C_w(\mathcal{H}_1, \mathcal{H})$. Keeping the separate continuity of T in our mind, we can use the triangle inequality to see that

$$\|W_{u,t\varphi} - W_{u_0,t_0\varphi_0}\|_{\mathcal{H}_1, \mathcal{H}} \leq \|W_{u,t\varphi} - W_{u_0,t\varphi_0}\|_{\mathcal{H}_1, \mathcal{H}} + \|W_{u_0,t\varphi_0} - W_{u_0,t_0\varphi_0}\|_{\mathcal{H}_1, \mathcal{H}}$$

converges to 0 as $(W_{u,\varphi}, t) \rightarrow (W_{u_0,\varphi_0}, t_0)$.

The discussions above demonstrate that $\mathcal{M}_0(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$ is a deformation retraction of $C_w(\mathcal{H}_1, \mathcal{H})$. $\square$

We now obtain the following theorem, which is the main result of this paper.

Theorem 3.2. *If $\mathcal{H}_1$ and $\mathcal{H}$ are two Hilbert spaces satisfying conditions (c1)–(c5) in Section 1, and $\|z^\alpha\|_{\mathcal{H}} / \|z^\alpha\|_{\mathcal{H}_1} \rightarrow 0$ as $|\alpha| \rightarrow \infty$, then the topological space $C_w(\mathcal{H}_1, \mathcal{H})$ is simply connected.*

Proof. In view of Lemma 3.1, the proof will be complete if we can show that the space $\mathcal{M}_0(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$ is simply connected. It is obvious that $\mathcal{M}_0(\mathcal{H}_1, \mathcal{H})$ is a Banach space containing $\{W_{u,0} : u \in H^\infty(\mathbb{B}_N)\}$, which is an infinite-dimensional subspace. Hence, one can deform a closed curve in $\mathcal{M}_0(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$ to a polygonal closed curve in some finite-dimensional (greater than 2) subspace of $\mathcal{M}_0(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$ by linear homotopy. Then the polygonal curve can be retracted to a nonzero point by direct application of van Kampen's theorem. See more details of the fundamental group and van Kampen's theorem in Chapter 9 in [15] and Chapter 1 in [16]. $\square$

4 Hilbert-Schmidt norm topological spaces

We now consider the topological space of Hilbert-Schmidt-weighted composition operators. If X is a separable Hilbert space with orthonormal base $\{e_m\}$ and X' is another Hilbert space, recall that a linear operator $T : X \rightarrow X'$ is said to be Hilbert-Schmidt if the Hilbert-Schmidt norm of T

$$\|T\|_{X,X',HS} := \left(\sum_{m=0}^{\infty} \|Te_m\|_{X'}^2 \right)^{\frac{1}{2}}$$

is finite.

By condition **(c3)**, $W_{u,\varphi} \in C_w(\mathcal{H})$ is Hilbert-Schmidt if and only if

$$\|W_{u,\varphi}\|_{\mathcal{H},HS}^2 := \sum_{|\alpha| \geq 0} \frac{\|W_{u,\varphi}(z^\alpha)\|_{\mathcal{H}}^2}{\|z^\alpha\|_{\mathcal{H}}^2} = \sum_{|\alpha| \geq 0} \frac{\|u\varphi_1^{\alpha_1} \cdots \varphi_N^{\alpha_N}\|_{\mathcal{H}}^2}{\|z_1^{\alpha_1} \cdots z_N^{\alpha_N}\|_{\mathcal{H}}^2} < \infty.$$

We denote by $C_{w,HS}(\mathcal{H})$ the space of Hilbert-Schmidt operators $W_{u,\varphi}$ in $C_w(\mathcal{H})$ with the Hilbert-Schmidt norm topology.

We have that $W_{u,\varphi} \in C_w(\mathcal{H}_1, \mathcal{H})$ is Hilbert-Schmidt if and only if

$$\|W_{u,\varphi}\|_{\mathcal{H}_1,\mathcal{H},HS}^2 := \sum_{\alpha} \frac{\|u\varphi^\alpha\|_{\mathcal{H}}^2}{\|z^\alpha\|_{\mathcal{H}_1}^2} < \infty.$$

We denote by $C_{w,HS}(\mathcal{H}_1, \mathcal{H})$ the space of all Hilbert-Schmidt-weighted composition operators $W_{u,\varphi} \in C_w(\mathcal{H}_1, \mathcal{H})$. We consider $C_{w,HS}(\mathcal{H}_1, \mathcal{H})$ with the Hilbert-Schmidt norm topology, which is stronger than the operator norm topology. Hence, a path connected set in $C_{w,HS}(\mathcal{H}_1, \mathcal{H})$ is also a path connected set in $C_w(\mathcal{H}_1, \mathcal{H})$. Since

$$\|W_{u,\varphi}\|_{\mathcal{H}_1,\mathcal{H},HS}^2 \leq \sum_{\alpha} \frac{\|u\varphi^\alpha\|_{\mathcal{H}}^2}{\|z^\alpha\|_{\mathcal{H}}^2} = \|W_{u,\varphi}\|_{\mathcal{H},HS}^2$$

by **(c5)**, we have $C_{w,HS}(\mathcal{H}) \subset C_{w,HS}(\mathcal{H}_1, \mathcal{H})$. The next lemma plays an important role.

Lemma 4.1. *Let $0 \leq t \leq 1$. If the operator $W_{u,\varphi} : \mathcal{H}_1 \rightarrow \mathcal{H}$ is a Hilbert-Schmidt operator for some nonzero $u \in \mathcal{H}$ and $\varphi \in \mathcal{S}(\mathbb{B}_N)$, then $W_{u,t\varphi}$ is also a Hilbert-Schmidt operator from $\mathcal{H}_1$ to $\mathcal{H}$.*

Proof. The Lemma follows immediately from the computations that

$$\|W_{u,t\varphi}\|_{\mathcal{H}_1,\mathcal{H},HS}^2 = \sum_{\alpha \geq 0} \frac{\|W_{u,t\varphi}(z^\alpha)\|_{\mathcal{H}}^2}{\|z^\alpha\|_{\mathcal{H}_1}^2} = \sum_{\alpha \geq 0} t^{2|\alpha|} \frac{\|u\varphi^\alpha\|_{\mathcal{H}}^2}{\|z^\alpha\|_{\mathcal{H}_1}^2} \leq \|W_{u,\varphi}\|_{\mathcal{H}_1,\mathcal{H},HS}^2. \quad \square$$

We are going to study the topological space $C_{w,HS}(\mathcal{H}_1, \mathcal{H})$ with the Hilbert-Schmidt norm topology. The strategy is analogous to that in Section 3. We need to consider the topological space

$$\mathcal{M}_{0,HS}(\mathcal{H}_1, \mathcal{H}) = \{W_{u,0} : u \in \mathcal{H}\},$$

in which open sets are induced by the Hilbert-Schmidt norm.

Lemma 4.2. *If $\mathcal{H}_1$ and $\mathcal{H}$ are two Hilbert spaces satisfying conditions **(c1)**–**(c5)** in Section 1, then $C_{0,HS}(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$ with the Hilbert-Schmidt norm topology is a deformation retraction of $C_{w,HS}(\mathcal{H}_1, \mathcal{H})$.*

Proof. If $W_{u,\varphi} \in C_{w,HS}(\mathcal{H}_1, \mathcal{H})$, we recall that, by **(c3)**,

$$\sum_{|\alpha| \geq 0} \frac{\|u\varphi^\alpha\|_{\mathcal{H}}^2}{\|z^\alpha\|_{\mathcal{H}_1}^2} = \|W_{u,\varphi}\|_{\mathcal{H}_1,\mathcal{H},HS}^2 < \infty. \quad (2)$$

Besides, $W_{u,t\varphi} \in C_{w,HS}(\mathcal{H}_1, \mathcal{H})$ for every $0 \leq t \leq 1$ by Lemma 4.1.

We need to show that the family of maps

$$T_t : C_{w,HS}(\mathcal{H}_1, \mathcal{H}) \rightarrow C_{w,HS}(\mathcal{H}_1, \mathcal{H}), \quad M_u C_\varphi \mapsto M_u C_{(1-t)\varphi} \quad \forall t \in [0, 1]$$

is a deformation retraction of $C_{w,HS}(\mathcal{H}_1, \mathcal{H})$ onto $\mathcal{M}_{0,HS}(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$. Conditions (a), (b) and (c) are obvious; all that remains is to show that T is continuous.

We first prove that T is separately continuous in each variable. For any fixed $t \in [0, 1]$, it follows from the direct computation that

$$\|W_{u,t\varphi} - W_{v,t\psi}\|_{HS}^2 = \sum_{\alpha} t^{|\alpha|} \frac{\|u\varphi^{\alpha} - v\psi^{\alpha}\|_{\mathcal{H}_1}^2}{\|z^{\alpha}\|_{\mathcal{H}_1}^2} \leq \|W_{u,\varphi} - W_{v,\psi}\|_{HS}^2.$$

If we fix $0 \leq t_0 \leq 1$, we can also prove $\|W_{u,t_0\varphi} - W_{u,t\varphi}\|_{\mathcal{H}_1, \mathcal{H}, HS} \rightarrow 0$ as $t \rightarrow t_0$ as follows. For any positive integer n , we have

$$\begin{aligned} \|W_{u,t_0\varphi} - W_{u,t\varphi}\|_{\mathcal{H}_1, \mathcal{H}, HS}^2 &= \sum_{|\alpha| \geq 0} \frac{\|u(t_0^{|\alpha|} - t^{|\alpha|})\varphi^{\alpha}\|_{\mathcal{H}_1}^2}{\|z^{\alpha}\|_{\mathcal{H}_1}^2} \\ &\leq \sum_{|\alpha| \leq n} |t_0^{|\alpha|} - t^{|\alpha|}|^2 \frac{\|u\varphi^{\alpha}\|_{\mathcal{H}_1}^2}{\|z^{\alpha}\|_{\mathcal{H}_1}^2} + \sum_{|\alpha| > n} \frac{\|u\varphi^{\alpha}\|_{\mathcal{H}_1}^2}{\|z^{\alpha}\|_{\mathcal{H}_1}^2}. \end{aligned}$$

Take $\varepsilon > 0$ arbitrary. Then by (2), we may take n large enough so that

$$\sum_{|\alpha| > n} \frac{\|u\varphi^{\alpha}\|_{\mathcal{H}_1}^2}{\|z^{\alpha}\|_{\mathcal{H}_1}^2} < \varepsilon.$$

Hence,

$$\|W_{u,t_0\varphi} - W_{u,t\varphi}\|_{\mathcal{H}_1, \mathcal{H}, HS}^2 < \varepsilon + \sum_{|\alpha| \leq n} |t_0^{|\alpha|} - t^{|\alpha|}|^2 \frac{\|u\varphi^{\alpha}\|_{\mathcal{H}_1}^2}{\|z^{\alpha}\|_{\mathcal{H}_1}^2}.$$

By letting $t \rightarrow t_0$, we have

$$\limsup_{t \rightarrow t_0} \|W_{u,t_0\varphi} - W_{u,t\varphi}\|_{\mathcal{H}_1, \mathcal{H}, HS}^2 < \varepsilon.$$

Thus, the map

$$T(W_{u,\varphi}, \cdot) : [0, 1] \rightarrow C_{w,HS}(\mathcal{H}_1, \mathcal{H}), \quad t \mapsto W_{u,t\varphi}$$

is continuous.

Now we are ready to prove that the map T is jointly continuous. The product topology of $C_{w,HS}(\mathcal{H}_1, \mathcal{H}) \times [0, 1]$ is the weakest topology that makes the natural projections bounded; that is, both

$$\text{Projection}_1 : C_{w,HS}(\mathcal{H}_1, \mathcal{H}) \times [0, 1] \rightarrow C_{w,HS}(\mathcal{H}_1, \mathcal{H}), \quad (W_{u,\varphi}, t) \mapsto W_{u,\varphi}$$

and

$$\text{Projection}_2 : C_{w,HS}(\mathcal{H}_1, \mathcal{H}) \times [0, 1] \rightarrow [0, 1], \quad (W_{u,\varphi}, t) \mapsto t$$

are continuous. It follows immediately that the pair $(W_{u,\varphi}, t)$ converges to (W_{u_0,φ_0}, t_0) in $C_{w,HS}(\mathcal{H}_1, \mathcal{H}) \times [0, 1]$ implies both $|t - t_0| \rightarrow 0$ and

$$\|W_{u,\varphi} - W_{u_0,\varphi_0}\|_{\mathcal{H}_1, \mathcal{H}, HS} \rightarrow 0$$

in $C_{w,HS}(\mathcal{H}_1, \mathcal{H})$. Then we can use the triangle inequality to see that

$$\|W_{u,t\varphi} - W_{u_0,t_0\varphi_0}\|_{\mathcal{H}_1, \mathcal{H}, HS} \leq \|W_{u,t\varphi} - W_{u_0,t\varphi_0}\|_{\mathcal{H}_1, \mathcal{H}, HS} + \|W_{u_0,t\varphi_0} - W_{u_0,t_0\varphi_0}\|_{\mathcal{H}_1, \mathcal{H}, HS}$$

converges to 0 as $(W_{u,\varphi}, t) \rightarrow (W_{u_0,\varphi_0}, t_0)$ in $C_{w,HS}(\mathcal{H}_1, \mathcal{H}) \times [0, 1]$. Hence, $\mathcal{M}_{0,HS}(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$ is a deformation retraction of $C_{w,HS}(\mathcal{H}_1, \mathcal{H})$. $\square$

The following theorem is the main result of this section.

Theorem 4.3. If $\mathcal{H}_1$ and $\mathcal{H}$ are two Hilbert spaces satisfying conditions (c1)–(c5) in Section 1, then the topological space $C_{w,HS}(\mathcal{H}_1, \mathcal{H})$ is simply connected.

Proof. By Lemma 4.2, we see that $\mathcal{M}_{0,HS}(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$ is a deformation retraction of $C_{w,HS}(\mathcal{H}_1, \mathcal{H})$. For any $u \in \mathcal{H}$, it is clear that

$$\|W_{u,0}\|_{\mathcal{H}_1, \mathcal{H}, HS} = \frac{\|u\|_{\mathcal{H}}}{\|1\|_{\mathcal{H}_1}} < \infty.$$

Consequently, the topological space $\mathcal{M}_{0,HS}(\mathcal{H}_1, \mathcal{H})$ contains $\{W_{u,0} : u \in \mathcal{H}\}$, which is an infinite-dimensional subspace. Hence, $\mathcal{M}_{0,HS}(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$ is simply connected by the same argument as that in Theorem 3.2. It follows immediately that $\mathcal{M}_{0,HS}(\mathcal{H}_1, \mathcal{H}) \setminus \{0\}$ is simply connected. $\square$

5 Conclusion

On the Hilbert spaces of analytic functions satisfying conditions (c1)–(c5), the topological spaces of nonzero weighted composition operators are simply connected. In this section, we discuss two applications of our main results together with a direct corollary on the compactness of the weighted composition operators.

5.1 Application 1

By Stirling's formula $\Gamma(s+1) \sim \sqrt{2\pi s} \cdot s^s \cdot e^{-s}$ as $s \rightarrow \infty$, we have for $\lambda > 0$

$$\begin{aligned} \frac{\Gamma(n+\lambda)}{n!} &= \frac{\Gamma(n+\lambda)}{\Gamma(n+1)} \sim \frac{\sqrt{2\pi(n+\lambda-1)}}{\sqrt{2\pi n}} \left(\frac{n+\lambda-1}{e}\right)^{n+\lambda-1} \left(\frac{e}{n}\right)^n \\ &\sim \left(\frac{n+\lambda-1}{n}\right)^n (n+\lambda-1)^{\lambda-1} \sim \left(1 + \frac{\lambda-1}{n}\right)^n n^{\lambda-1} \sim n^{\lambda-1} \end{aligned}$$

as $n \rightarrow \infty$.

Applying the norm formulas for the monomial z^α in the Hardy space, Bergman spaces and Dirichlet space on the unit ball $\mathbb{B}_N$ ($N \geq 1$), and employing the above estimate repeatedly, we have

$$\frac{\|z^\alpha\|_{H^2}^2}{\|z^\alpha\|_{\mathcal{D}}^2} = \frac{(N-1)! \alpha!}{(N-1+|\alpha|)!} = \frac{\Gamma(|\alpha|+1)}{|\alpha| \Gamma(|\alpha|+N)} \sim \frac{1}{|\alpha|^N} \rightarrow 0 \quad \text{as } |\alpha| \rightarrow \infty.$$

For $-1 < \lambda < \infty$,

$$\frac{\|z^\alpha\|_{A_\lambda^2}^2}{\|z^\alpha\|_{\mathcal{D}}^2} = \frac{\frac{\alpha! \Gamma(N+\lambda+1)}{\Gamma(N+|\alpha|+\lambda+1)}}{|\alpha| \frac{\alpha!}{|\alpha|!}} = \frac{|\alpha|! \Gamma(N+\lambda+1)}{|\alpha| \Gamma(N+|\alpha|+\lambda+1)} \sim |\alpha|^{-N-\lambda-1} \rightarrow 0 \quad \text{as } |\alpha| \rightarrow \infty$$

and

$$\frac{\|z^\alpha\|_{A_\lambda^2}^2}{\|z^\alpha\|_{H^2}^2} = \frac{\frac{\alpha! \Gamma(N+\lambda+1)}{\Gamma(N+|\alpha|+\lambda+1)}}{\frac{(N-1)! \alpha!}{(N-1+|\alpha|)!}} = \frac{(N-1+|\alpha|)!}{(N-1)! |\alpha|!} \frac{|\alpha|! \Gamma(N+\lambda+1)}{\Gamma(N+|\alpha|+\lambda+1)} \sim |\alpha|^{N-1} |\alpha|^{-N-\lambda} \sim \frac{1}{|\alpha|^{\lambda+1}} \rightarrow 0 \quad \text{as } |\alpha| \rightarrow \infty.$$

For $-1 < \lambda_1 < \lambda_2 < \infty$, we also have

$$\frac{\|z^\alpha\|_{A_{\lambda_2}^2}^2}{\|z^\alpha\|_{A_{\lambda_1}^2}^2} = \frac{\frac{\alpha! \Gamma(N+\lambda_2+1)}{\Gamma(N+|\alpha|+\lambda_2+1)}}{\frac{\alpha! \Gamma(N+\lambda_1+1)}{\Gamma(N+|\alpha|+\lambda_1+1)}} = \frac{|\alpha|! \Gamma(N+\lambda_2+1)}{|\alpha|! \Gamma(N+\lambda_1+1)} \sim (|\alpha|+1)^{\lambda_1-\lambda_2} \rightarrow 0 \quad \text{as } |\alpha| \rightarrow \infty.$$

Hence, by Theorems 3.2 and 4.3, we obtain the following corollaries:

Corollary 5.1. *The topological spaces $C_w(\mathcal{D}, H^2)$, $C_w(\mathcal{D}, A_\lambda^2)$, $C_w(H^2, A_\lambda^2)$ for $-1 < \lambda < \infty$, and $C_w(A_{\lambda_1}^2, A_{\lambda_2}^2)$ for $-1 < \lambda_1 < \lambda_2 < \infty$ are simply connected.*

Corollary 5.2. *The topological spaces $C_{w,HS}(\mathcal{D}, H^2)$, $C_{w,HS}(\mathcal{D}, A_\lambda^2)$, $C_{w,HS}(H^2, A_\lambda^2)$ for $-1 < \lambda < \infty$, and $C_{w,HS}(A_{\lambda_1}^2, A_{\lambda_2}^2)$ for $-1 < \lambda_1 < \lambda_2 < \infty$ are simply connected.*

5.2 Application 2

In [17], the authors proved that the topological space of weighted composition operators on the ball algebra $\mathcal{A}$ is path connected. The strategy is similar to that of Lemma 3.1. Hence, by the same arguments, we can show that $C_w(\mathcal{A})$ is simply connected.

Corollary 5.3. *The topological space of nonzero weighted composition operators $C_w(\mathcal{A})$ is simply connected.*

5.3 Compactness

It is well known that the compact (weighted) composition operators on many classical spaces form a path connected subset, e.g., Proposition 9.9 in [1]. We prove that the topology space $C_w(\mathcal{H}_1, \mathcal{H})$ in Theorem 3.2 consists of compact-weighted composition operators.

Lemma 5.4. *If $\sup_{z \in \mathbb{B}_N} \|\varphi(z)\| < 1$ and $u \in \mathcal{H}$, then $W_{u,\varphi} \in C_w(\mathcal{H})$ is compact.*

Proof. By the first paragraph of the proof in Lemma 3.1, we have $W_{u,\varphi} \in C_w(\mathcal{H})$. To show that $W_{u,\varphi}$ is compact, let $\{f_n\}$ be a sequence in $\mathcal{H}$ such that there is a positive constant K satisfying $\|f_n\|_{\mathcal{H}} < K$ for every n . By (c1), we may assume that f_n converges to some $f \in H(\mathbb{B}_N)$ uniformly on compact subsets of $\mathbb{B}_N$. It follows from this assumption that $f_n \circ \varphi$ converges to $f \circ \varphi$ in $H^\infty(\mathbb{B}_N)$. Hence, it follows from (c2) that both $u(f_n \circ \varphi)$ and $u(f \circ \varphi)$ belong to $\mathcal{H}$, with

$$\|W_{u,\varphi}f_n - u(f \circ \varphi)\|_{\mathcal{H}} \leq \|u\|_{\mathcal{H}} \|f_n \circ \varphi - f \circ \varphi\|_{\infty} \rightarrow 0, \quad n \rightarrow \infty.$$

Thus, $W_{u,\varphi} \in C_w(\mathcal{H})$ is compact. □

Proposition 5.5. *If $\|z^\alpha\|_{\mathcal{H}} / \|z^\alpha\|_{\mathcal{H}_1} \rightarrow 0$ as $|\alpha| \rightarrow \infty$, then any $W_{u,\varphi} \in C_w(\mathcal{H}_1, \mathcal{H})$ is compact.*

Proof. For $0 < t < 1$, Lemma 5.4 dictates that $W_{u,\varphi} \in C_w(\mathcal{H}_1, \mathcal{H})$ is compact. By (c5), $\text{id}: \mathcal{H}_1 \rightarrow \mathcal{H}$ is bounded. Hence, $W_{u,t\varphi}: \mathcal{H}_1 \rightarrow \mathcal{H}$ is compact. Since the algebra of compact operators is closed in the norm topology, we see that $W_{u,\varphi}$ is compact since it can be approximated by compact operators $W_{u,t\varphi}$. □

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