Research Article

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Evolution of the first eigenvalue of the Laplace operator and the p -Laplace operator under a forced mean curvature flow

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Abstract: In this paper, we discuss the monotonicity of the first nonzero eigenvalue of the Laplace operator and the p -Laplace operator under a forced mean curvature flow (MCF). By imposing conditions associated with the mean curvature of the initial hypersurface and the coefficient function of the forcing term of a forced MCF, and some special pinching conditions on the second fundamental form of the initial hypersurface, we prove that the first nonzero closed eigenvalues of the Laplace operator and the p -Laplace operator are monotonic under the forced MCF, respectively, which partially generalize Mao and Zhao's work. Moreover, we give an example to specify applications of conclusions obtained above.

Keywords: eigenvalue, Laplace operator, p -Laplace operator, monotonicity, forced mean curvature flow

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1 Introduction

In the past few decades, many interesting properties of eigenvalues of some self-adjoint elliptic operators such as the usual Laplace operator (also called Laplace-Beltrami operator), the p -Laplace operator (also called p -Laplacian), the biharmonic operator and so on have been investigated in fixed Riemannian metrics (see [1–3]). Motivated by the work of Perelman [4] and Cao [5], research on eigenvalues of the Laplace operator and some other deformations related to the Laplace operator such as p -Laplacian and Witten-Laplacian under various geometric flows such as the Ricci flow, the mean curvature flow (MCF), the Yamabe flow and the Gaussian curvature flow has always been an active area in the study of geometry and topology of manifolds during these years.

Some results associated with eigenvalue problems have been attained under the MCF and deformations of the MCF. For instance, Zhao [6] considered a compact, strictly convex two-dimensional surface without boundary smoothly immersed in $\mathbb{R}^3$ and proved that the first eigenvalue of the Laplace operator is non-increasing along the unnormalized powers of the MCF if the initial two-dimensional surface is totally umbilical. Subsequently, Zhao [7] obtained that the first eigenvalue of the p -Laplace operator is nondecreasing along the unnormalized powers of the m th MCF under some assumptions on the mean curvature and second fundamental form of initial given closed manifold. Furthermore, Zhao [8] also proved that the first eigenvalue of the p -Laplace operator is increasing along the unnormalized powers of the MCF under similar assumptions on mean curvatures. Mao [9] derived that the first eigenvalues of the Laplace and the

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p -Laplace operators are monotonic under the forced MCF by imposing some conditions on the mean curvature of the initial hypersurface and the coefficient function of the forcing term and an almost-umbilical pinching condition on the second fundamental form of the initial hypersurface. After going through these results, we wish to obtain some other conclusions for the case of eigenvalue problems related to the Laplace operator and the p -Laplace operator along the MCF and some other deformations associated with MCF, for example, the volume preserving MCF (see [10,11]), the area preserving MCF (see [12]), and more generally, MCF with a prescribed forcing term (see [9,13–15]) etc. In this paper, motivated by Mao's work [9], we consider the monotonicity of the first nonzero eigenvalue of the Laplace operator and the p -Laplace operator evolving along the forced MCF under more general pinching conditions imposed on the second fundamental form of the initial hypersurface.

2 Preliminaries

In this section, for the reader's convenience, we would like to give a sketch of the eigenvalue problem and recall some basic knowledge about the forced MCF.

The Laplace operator is defined as

$$\Delta = \operatorname{div} \nabla = \frac{1}{\sqrt{G}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{G} g^{ij} \frac{\partial}{\partial x_j} \right)$$

in a local coordinate system $\{x_1, x_2, \dots, x_n\}$, where the matrix (g^{ij}) is the inverse of the metric matrix (g_{ij}) , $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, $G = \det(g_{ij})$.

For a compact Riemannian manifold M without boundary, Δ is a self-adjoint operator, then it has discrete eigenvalues according to the spectral theory in functional analysis. Obviously, the smallest eigenvalue in the problem

$$\Delta u = \lambda u, \quad \text{on } M$$

is zero and the corresponding eigenfunctions should be constant functions. By Rayleigh's theorem and extreme principle, the first nonzero closed eigenvalue $\lambda_1(M)$ (λ_1 for short) can be defined by

$$\lambda_1 = \inf \left\{ \frac{\int_M (\nabla u)^2 d\mu}{\int_M u^2 d\mu} \mid u \neq 0, u \in W^{1,2}(M), \int_M u d\mu = 0 \right\},$$

where $W^{1,2}(M)$ denotes the Sobolev space given by the completion of $C^\infty(M)$ for the norm

$$\|u\|_{W^{1,2}(M)} = \left(\int_M u^2 d\mu + \int_M |\nabla u|^2 d\mu \right)^{\frac{1}{2}}.$$

Now let's recall some facts of the p -Laplace operator ($1 < p < \infty$). The p -Laplace operator is a natural generalization of the Laplace operator for the fact that the p -Laplace operator is the Laplace operator when $p = 2$. The p -Laplacian eigenvalue problem concerns the partial differential equation:

$$\Delta_p u + \lambda_{1,p}(t) |u|^{p-2} u = 0, \quad \text{on } M,$$

where $\Delta_p u$ is given by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \frac{1}{\sqrt{G}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{G} g^{ij} |\nabla u|^{p-2} \frac{\partial u}{\partial x_j} \right)$$

in a local coordinate system $\{x_1, x_2, \dots, x_n\}$, where the matrix (g^{ij}) is the inverse of the metric matrix (g_{ij}) , $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, $G = \det(g_{ij})$. As with the case of the Laplace operator, $\Delta_p u$ have discrete eigenvalues when

the Riemannian manifold M is closed. Furthermore, the first nonzero closed eigenvalue $\lambda_{1,p}(M)$ ($\lambda_{1,p}$ for short) can be defined by

$$\lambda_{1,p} = \inf \left\{ \frac{\int_M |\nabla u|^p d\mu}{\int_M |u|^p d\mu} \mid u \neq 0, u \in W^{1,p}(M), \int_M |u|^{p-2} u d\mu = 0 \right\},$$

where $W^{1,p}(M)$ denotes the Sobolev space given by the completion of $C^\infty(M)$ for the norm

$$\|u\|_{W^{1,p}(M)} = \left(\int_M |u|^p d\mu + \int_M |\nabla u|^p d\mu \right)^{\frac{1}{p}}.$$

Let M_0^n ($n \geq 2$) be a smooth, compact and strictly convex manifold without boundary of dimension n , smoothly embedded in the Euclidean space $\mathbb{R}^{n+1}$ and represented locally by a diffeomorphism $F_0 : \mathbb{R}^{n+1} \supset U \rightarrow F_0(U) \subset M_0 \subset \mathbb{R}^{n+1}$. M_0 evolves along the MCF, which is a one-parameter family of smooth immersions $F : M_0 \times [0, T) \rightarrow \mathbb{R}^{n+1}$ defined as follows:

$$\begin{cases} \frac{\partial F(x, t)}{\partial t} = -H(x, t) \vec{\nu}(x, t), & x \in M_0, \quad t > 0, \\ F(\cdot, 0) = F_0(\cdot), \end{cases} \quad (2.1)$$

where $H(x, t)$ is the mean curvature of the hypersurface $M_t = F(\cdot, t)(M_0)$ at $F(x, t)$ and $\vec{\nu}(x, t)$ is the outer unit normal vector of the hypersurface M_t at $F(x, t)$. The equation system (2.1) is also called unnormalized MCF (see [16,17] for more details) for which Huisken [18] showed that (2.1) is actually a quasilinear parabolic system with a smooth solution at least on some short time interval and also proved that the surfaces stay convex and contract to a point in finite time if the initial surface is convex. Li et al. [14] considered a more general MCF with a forcing term in the direction of its position vector which is called to be the forced MCF defined by the following evolution equation:

$$\begin{cases} \frac{\partial F(x, t)}{\partial t} = -H(x, t) \vec{\nu}(x, t) + \kappa(t) F(x, t), & x \in M_0, \quad t > 0, \\ F(\cdot, 0) = F_0(\cdot), \end{cases} \quad (2.2)$$

where $\kappa(t)$ is a continuous function which depends only on t , $H(x, t)$ is the mean curvature of the hypersurface $M_t = F(\cdot, t)(M_0)$ at $F(x, t)$ and $\vec{\nu}(x, t)$ is the outer unit normal vector of the hypersurface M_t at $F(x, t)$. They showed that this parabolic equation can be converted to a strictly parabolic equation and thus has a smooth solution on a maximal time interval $[0, T)$ for some $T > 0$ by the standard theory of parabolic equations. In fact, they have proved that different forcing term will lead to different maximal time interval. The forced MCF (2.2) is an extension of the MCF (see [9, Remark 2.2] for some of their difference) since the forced MCF coincides with the normalized MCF under which the evolving hypersurface M_t has constant total area when $\kappa(t) = \frac{h(t)}{n}$, where $h(t) = \frac{\int_{M_t} H^2 d\mu_t}{\int_{M_t} d\mu_t}$, and the well-known unnormalized MCF when $\kappa(t) \equiv 0$, respectively.

For the forced MCF (2.2), the evolution for g_{ij} , h_{ij} and H (see [14]) are given by

$$\frac{\partial g_{ij}}{\partial t} = -2Hh_{ij} + 2\kappa(t)g_{ij}, \quad (2.3)$$

$$\frac{\partial h_{ij}}{\partial t} = \Delta h_{ij} - 2Hh_{il}g^{mj}h_{ij} + |A|^2 h_{ij} + \kappa(t)h_{ij}, \quad (2.4)$$

$$\frac{\partial H}{\partial t} = \Delta H + |A|^2 H - \kappa(t)H. \quad (2.5)$$

Remark 2.1. From the facts about the forced MCF (2.2) introduced above, we know that the solutions to evolution equations for the forced MCF exist on a maximal time interval if the initial hypersurface is strictly

convex. Therefore, in what follows, we always assume $[0, T)$ ($T > 0$) to be the maximal time interval for the evolving hypersurface M_t .

Note that for the Laplace operator, many papers have pointed out that its differentiability under geometric flows from the perturbation theory (see [19] for the details), but we are not clear whether the first eigenvalue or the corresponding eigenfunction of the p -Laplace operator is C^1 -differentiable under the forced flow (2.2) for $p \neq 2$ till now. So we cannot use Ma's trick [20] to derive the monotonicity of the first eigenvalue of the p -Laplace operator. Fortunately, we can follow the arguments of Cao [5] and Wu [21]. Specifically, let M_t be an n -dimensional closed Riemannian manifold, and $g(t)$ be a smooth solution of the forced flow (2.2) on the time interval $[0, T)$ ($T > 0$). We now define a general smooth function $\lambda_{1,p}(u, t)$ as follows:

$$\lambda_{1,p}(u, t) := - \int_{M_t} \Delta_p u(x, t) \cdot u(x, t) d\mu_t = \int_{M_t} |\nabla u(x, t)|^p d\mu_t, \quad (2.6)$$

where $u(x, t)$ is any smooth function satisfying

$$\int_{M_t} |u(x, t)|^p d\mu_t = 1, \quad \int_{M_t} |u(x, t)|^{p-2} u(x, t) d\mu_t = 0. \quad (2.7)$$

For any time $t_0 \in [0, T)$, we assume $u(x, t_0)$ is the eigenfunction for the first eigenvalue $\lambda_{1,p}(t_0)$ of the p -Laplace operator. We define the smooth function

$$w(x, t) := u(x, t_0) \left[\frac{\det(g_{ij}(x, t_0))}{\det(g_{ij}(x, t))} \right]^{\frac{1}{2(p-1)}}$$

under the forced flow (2.2). Let

$$u(x, t) = \frac{w(x, t)}{\left(\int_{M_t} |w(x, t)|^p d\mu_t \right)^{\frac{1}{p}}},$$

then we can check that $u(x, t)$ defined here satisfies (2.7). Therefore, there exists a smooth function $u(x, t)$ satisfying (2.7) such that at time $t_0 \in [0, T)$, $u(x, t_0)$ is the eigenfunction corresponding to $\lambda_{1,p}(t_0)$.

Generally, $\lambda_{1,p}(u, t)$ is not equal to $\lambda_{1,p}(t)$. However, at time $t = t_0$, if $u(x, t_0)$ is the eigenfunction for the first eigenvalue $\lambda_{1,p}(t_0)$ of the p -Laplace operator, then we have that

$$\lambda_{1,p}(u(x, t_0), t_0) = \lambda_{1,p}(t_0).$$

3 Evolution equations for the first eigenvalue of the Laplace operator and the p -Laplace operator

In this section, by applying the aforementioned evolution equations, we obtain the following result.

Proposition 3.1. *Let $\lambda_1(t)$ be the first nonzero closed eigenvalue of the Laplace operator on an n -dimensional compact and strictly convex hypersurface $(M_t^n, g(t))$ ($n \geq 2$) under the flow (2.2). Let $u(x, t)$ be the normalized eigenfunction corresponding to $\lambda_1(t)$, namely, $\Delta u = -\lambda_1 u$ and $\int_{M_t} u^2 = 1$. Then we have*

$$\frac{d}{dt} \lambda_1(t) = \lambda_1(t) - 2\kappa(t) + \int_{M_t} H^2 u^2 d\mu_t + 2 \int_{M_t} H h^{ij} \nabla_i u \nabla_j u d\mu_t - \int_{M_t} H^2 |\nabla u|^2 d\mu_t. \quad (3.1)$$

Proof. Under the same assumptions of the aforementioned proposition, Mao [9] obtained

$$\frac{d}{dt}\lambda_1(t) = -2\lambda_1(t)\kappa(t) + 2 \int_{M_t} H h^{ij} \nabla_i u \nabla_j u d\mu_t + 2 \int_{M_t} u H \nabla_i h^{ij} \nabla_j u d\mu_t. \quad (3.2)$$

Since the hypersurface $M_t \subset \mathbb{R}^{n+1}$, the Codazzi equation $h_{ijk} = h_{ikj}$ holds for $i, j, k \in \{1, 2, \dots, n\}$, so $\nabla_i h^{ij} = \nabla^j H$. By integration by parts, the last term of (3.2) becomes

$$2 \int_{M_t} u H \nabla_i h^{ij} \nabla_j u d\mu_t = \lambda_1(t) \int_{M_t} H^2 u^2 d\mu_t - \int_{M_t} H^2 |\nabla u|^2 d\mu_t. \quad (3.3)$$

Inputting (3.3) into (3.2), we get the desired equality (3.1). $\square$

Proposition 3.2. Let $\lambda_{1,p}(t)$ be the first nonzero closed eigenvalue of the p -Laplace operator on an n -dimensional compact and strictly convex hypersurface $(M_t^n, g(t))$ ($n \geq 2$) under the forced flow (2.2). Assume that $u(x, t)$ is any smooth function satisfying (2.7) such that at time $t_0 \in [0, T)$, $u(x, t_0)$ is the eigenfunction corresponding to $\lambda_{1,p}(t_0)$. Let $\lambda_{1,p}(u, t)$ be a smooth function defined by (2.6). Then at time $t = t_0$, we have

$$\frac{d}{dt}\lambda_{1,p}(u, t) = \lambda_{1,p}(t)(-p\kappa(t) + \int_{M_t} H^2 |u|^p d\mu_t) + p \int_{M_t} |\nabla u|^{p-2} H h^{ij} \nabla_i u \nabla_j u d\mu_t - \int_{M_t} H^2 |\nabla u|^p d\mu_t. \quad (3.4)$$

Proof. Under the same assumptions of the aforementioned proposition, at time $t = t_0$, Mao [9] obtained

$$\frac{d}{dt}\lambda_{1,p}(u, t) = -p\kappa(t)\lambda_{1,p}(t) + p \int_{M_t} |\nabla u|^{p-2} H h^{ij} \nabla_i u \nabla_j u d\mu_t + 2 \int_{M_t} |\nabla u|^{p-2} u H \nabla_i h^{ij} \cdot \nabla_j u d\mu_t. \quad (3.5)$$

Since the hypersurface $M_t \subset \mathbb{R}^{n+1}$, the Codazzi equation $h_{ijk} = h_{ikj}$ holds for $i, j, k \in \{1, 2, \dots, n\}$, so $\nabla_i h^{ij} = \nabla^j H$. By integration by parts, the last term of (3.5) becomes

$$2 \int_{M_t} |\nabla u|^{p-2} u H \nabla_i h^{ij} \nabla_j u d\mu_t = \lambda_{1,p}(t) \int_{M_t} H^2 |u|^p d\mu_t - \int_{M_t} H^2 |\nabla u|^p d\mu_t. \quad (3.6)$$

Inputting (3.6) into (3.5), we get the desired equality (3.4). $\square$

4 Monotonicity of the first nonzero eigenvalue of the Laplace operator and the p -Laplace operator under the forced MCF

As applications of Propositions 3.1 and 3.2, we wish to obtain the monotonicity of the first nonzero eigenvalue of the Laplace operator and the p -Laplace operator under the forced MCF under some additional assumptions, respectively. Before we do that, several Lemmas are necessary.

Noting the evolutions for the second fundamental form h_{ij} and the mean curvature H under the forced MCF (2.2), that is, equalities (2.4) and (2.5), by applying Hamilton's maximal principle [22] to the evolution equations of h_{ij} and tensors constructed by h_{ij} , H , g_{ij} , under the forced MCF (2.2), Li et al. [14] proved the following results.

Lemma 4.1. [14, Corollary 2.5(i)] Let M_0 be a compact, strictly convex hypersurface of dimension $n \geq 2$ without boundary, smoothly embedded in $\mathbb{R}^{n+1}$. If $h_{ij} > 0$ on M_0 at $t = 0$, then it remains so on $0 \leq t < T$ under the forced MCF (2.2).

Remark 4.2. By Lemma 4.1, we know that the strict convexity is preserved under the powers of MCF and the forced MCF (2.2). Therefore, in what follows, the condition “strict convexity” imposed on the evolving hypersurface M_t is feasible.

Lemma 4.3. [14, Corollary 2.5(ii)] *Let M_0 be a compact, strictly convex hypersurface of dimension $n \geq 2$ without boundary, smoothly embedded in $\mathbb{R}^{n+1}$. If $\varepsilon H g_{ij} \leq h_{ij} \leq \beta H g_{ij}$, and $H > 0$ at the beginning for some constants $0 < \varepsilon \leq \frac{1}{n} \leq \beta < 1$, then it remains so under the forced MCF (2.2) on $0 \leq t < T$.*

Remark 4.4. Similar conclusions about the preservation of the pinching conditions on h_{ij} under the forced MCF (2.2) can be found in [9, Lemma 4.3] where the pinching condition on h_{ij} is actually an almost-umbilical condition as [9] says. In addition, once we let $\varepsilon = \beta = \frac{1}{n}$ in Lemma 4.3, then we know that the property of being totally umbilical is preserved under the forced MCF (2.2) if the initial closed hypersurface is strictly convex. Since a well-known result states that a totally umbilical hypersurface of $\mathbb{R}^{n+1}$ which is not totally geodesic is a round sphere, it would be very interesting to discuss how a hypersurface with an almost-umbilical condition is close to a round sphere (see [9, Remark 1.2] for the discussion). In general, “strictly convex” means that the second fundamental form $h_{ij} > 0$, which implies that $H > 0$, namely, mean convex. When the hypersurface is totally umbilical, then $h_{ij} = \frac{1}{n} H g_{ij}$, so the convexity is equivalent to mean convexity in this case.

Observing the evolution equations for the first nonzero eigenvalue of the Laplace operator and the p -Laplace operator in Section 3, by applying Proposition 3.1, Proposition 3.2 and the aforementioned lemmas, we have the following.

Theorem 4.5. *Let $\hat{\lambda}_1(t)$, $\tilde{\lambda}_{1,p}(t)$ ($p \in (1, \infty)$, $p \neq 2$) be the first nonzero closed eigenvalue of the Laplace operator and the p -Laplace operator on an n -dimensional compact and strictly convex hypersurface $(M_t^n, g(t))$ ($n \geq 2$) which evolves under the forced MCF (2.2), $t \in [0, T)$, respectively. Denote by $H_{\max}(0)$ and $H_{\min}(0)$ the maximum and minimum values of the mean curvature of the initial hypersurface M_0 . If the initial hypersurface M_0 satisfies the pinching condition*

$$\varepsilon H g_{ij} \leq h_{ij} \leq \beta H g_{ij}, \quad (4.1)$$

for some constants $0 < \varepsilon \leq \frac{1}{n} \leq \beta < 1$, then we have that

- (i) If $\kappa(t) \geq \frac{(1+2\beta)\psi^2(t) - \sigma^2(t)}{2}$ for $t \in [0, T)$, then $\hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) for $t \in [0, T)$.
Furthermore, $e^{\int_0^t (2\kappa(\tau) - (1+2\beta)\psi^2(\tau) + \sigma^2(\tau)) d\tau} \hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) for $t \in [0, T)$.
- (ii) If $\kappa(t) \leq \frac{(1+2\varepsilon)\sigma^2(t) - \psi^2(t)}{2}$ for $t \in [0, T)$, then $\hat{\lambda}_1(t)$ is nondecreasing under the forced MCF (2.2) for $t \in [0, T)$.
Furthermore, $e^{\int_0^t (2\kappa(\tau) - (1+2\varepsilon)\sigma^2(\tau) + \psi^2(\tau)) d\tau} \hat{\lambda}_1(t)$ is nondecreasing under the forced MCF (2.2) for $t \in [0, T)$.
- (iii) If $\kappa(t) \geq \frac{(1+p\varepsilon)\psi^2(t) - \sigma^2(t)}{p}$ for $t \in [0, T)$, then $\tilde{\lambda}_{1,p}(t)$ is nonincreasing and differentiable almost everywhere under the forced MCF (2.2) for $t \in [0, T)$.
- (iv) If $\kappa(t) \leq \frac{(1+p\varepsilon)\sigma^2(t) - \psi^2(t)}{p}$ for $t \in [0, T)$, then $\tilde{\lambda}_{1,p}(t)$ is nondecreasing and differentiable almost everywhere under the forced MCF (2.2) for $t \in [0, T)$, where

$$\sigma(t) = e^{-\int_0^t \kappa(\tau) d\tau} H_{\min}(0) \left[1 - \frac{2}{n} H_{\min}^2(0) \int_0^t e^{-2\int_0^\tau \kappa(s) ds} d\tau \right]^{-\frac{1}{2}},$$

$$\psi(t) = e^{-\int_0^t \kappa(\tau) d\tau} H_{\max}(0) \left[1 - 2\beta H_{\max}^2(0) \int_0^t e^{-2\int_0^\tau \kappa(s) ds} d\tau \right]^{-\frac{1}{2}}.$$

Proof. For the forced MCF (2.2), since the initial hypersurface M_0 satisfies the pinching condition (4.1), by Lemma (4.3), we know that the evolving hypersurface M_t also satisfies the pinching condition (4.1), that is, $\varepsilon H g_{ij} \leq h_{ij} \leq \beta H g_{ij}$, for some constants $0 < \varepsilon \leq \frac{1}{n} \leq \beta < 1$, for any $t \in [0, T)$. Locally, we can choose an ortho-

gonal frame $\{e_{ij}\}_{i,j=1}^n$ such that $g_{ij} = \delta_{ij}$, $i, j = 1, 2, \dots, n$, then let $h_{ij} = \mu_i \delta_{ij}$, $i, j = 1, 2, \dots, n$, where $\mu_i(x, t)$ is the principal curvature at $F(x, t)$, so for every $i \in \{1, 2, \dots, n\}$,

$$\varepsilon H \leq \mu_i \leq \beta H. \quad (4.2)$$

Since the evolving hypersurface M_t is strictly convex, by (4.2) and the Cauchy-Schwartz inequality, we have

$$\frac{1}{n} H^2 \leq |A|^2 = \sum_i \mu_i^2 \leq (\max_i \mu_i) \sum_i \mu_i \leq \beta H^2, \quad (4.3)$$

that is,

$$\frac{H^2}{n} \leq |A|^2 \leq \beta H^2. \quad (4.4)$$

As Mao [9, Theorem 5.1] does, in view of (4.4), by applying the maximum principle to (2.5), we obtain the following estimates for $H(x, t)$ of the general n -dimensional compact and strictly convex hypersurface evolving along the forced MCF (2.2), that is,

$$\sigma(t) \leq H(x, t) \leq \psi(t), \quad (4.5)$$

for $t \in [0, T)$, where

$$\begin{aligned} \sigma(t) &= e^{-\int_0^t \kappa(\tau) d\tau} H_{\min}(0) \left[1 - \frac{2}{n} H_{\min}^2(0) \int_0^t e^{-2 \int_0^\tau \kappa(s) ds} d\tau \right]^{-\frac{1}{2}}, \\ \psi(t) &= e^{-\int_0^t \kappa(\tau) d\tau} H_{\max}(0) \left[1 - 2\beta H_{\max}^2(0) \int_0^t e^{-2 \int_0^\tau \kappa(s) ds} d\tau \right]^{-\frac{1}{2}}. \end{aligned}$$

For the case of the Laplace operator, let $u(x, t)$ be the normalized eigenfunction corresponding to $\hat{\lambda}_1(t)$, namely, $\Delta u = -\hat{\lambda}_1 u$ and $\int_{M_t} u^2 = 1$. By (3.1) in Proposition 3.1, we have

$$\frac{d}{dt} \hat{\lambda}_1(t) \leq \hat{\lambda}_1(t) [-2\kappa(t) + (1 + 2\beta)\psi^2(t) - \sigma^2(t)] \quad (4.6)$$

and

$$\frac{d}{dt} \hat{\lambda}_1(t) \geq \hat{\lambda}_1(t) [-2\kappa(t) + (1 + 2\varepsilon)\sigma^2(t) - \psi^2(t)], \quad (4.7)$$

then $\frac{d}{dt} \hat{\lambda}_1(t) \leq 0$ provided $\kappa(t) \geq \frac{(1 + 2\beta)\psi^2(t) - \sigma^2(t)}{2}$ for $t \in [0, T)$ and $\frac{d}{dt} \hat{\lambda}_1(t) \geq 0$ provided $\kappa(t) \leq \frac{(1 + 2\varepsilon)\sigma^2(t) - \psi^2(t)}{2}$ for $t \in [0, T)$. Then we conclude that $\hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) provided $\kappa(t) \geq \frac{(1 + 2\beta)\psi^2(t) - \sigma^2(t)}{2}$ for $t \in [0, T)$ and $\hat{\lambda}_1(t)$ is nondecreasing under the forced MCF (2.2) provided $\kappa(t) \leq \frac{(1 + 2\varepsilon)\sigma^2(t) - \psi^2(t)}{2}$ for $t \in [0, T)$. Furthermore, from (4.6) and (4.7), we infer that $e^{\int_0^t (2\kappa(\tau) - (1 + 2\beta)\psi^2(\tau) + \sigma^2(\tau)) d\tau} \hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) for $t \in [0, T)$ and $e^{\int_0^t (2\kappa(\tau) - (1 + 2\varepsilon)\sigma^2(\tau) + \psi^2(\tau)) d\tau} \hat{\lambda}_1(t)$ is nondecreasing under the forced MCF (2.2) for $t \in [0, T)$.

Now for the case of the p -Laplace operator, we assume that $u(x, t)$ is any smooth function satisfying (2.7) such that at time $t_0 \in [0, T)$, $u(x, t_0)$ is the eigenfunction corresponding to $\tilde{\lambda}_{1,p}(t_0)$. Let $\tilde{\lambda}_{1,p}(u, t)$ be a smooth function defined by (2.6). Similar to the aforementioned process, by (3.4) in Proposition (3.2), at time $t = t_0$, we obtain

$$\frac{d}{dt} \tilde{\lambda}_{1,p}(u, t) \leq \tilde{\lambda}_{1,p}(t) [-p\kappa(t) + (1 + p\beta)\psi^2(t) - \sigma^2(t)] \quad (4.8)$$

and

$$\frac{d}{dt} \tilde{\lambda}_{1,p}(u, t) \geq \tilde{\lambda}_{1,p}(t) [-p\kappa(t) + (1 + p\varepsilon)\sigma^2(t) - \psi^2(t)], \quad (4.9)$$

then $\frac{d}{dt}\tilde{\lambda}_{1,p}(u, t)|_{t=t_0} \leq 0$ provided $\kappa(t) \geq \frac{(1+p\beta)\psi^2(t) - \sigma^2(t)}{p}$ for $t \in [0, T)$ and $\frac{d}{dt}\tilde{\lambda}_{1,p}(u, t)|_{t=t_0} \geq 0$, provided $\kappa(t) \leq \frac{(1+p\epsilon)\sigma^2(t) - \psi^2(t)}{p}$ for $t \in [0, T)$. Since $\tilde{\lambda}_{1,p}(u, t)$ is a smooth function with respect to t -variable, we have $\frac{d}{dt}\tilde{\lambda}_{1,p}(u, t) \leq 0$ in any sufficiently small neighbourhood of t_0 provided $\kappa(t) \geq \frac{(1+p\beta)\psi^2(t) - \sigma^2(t)}{p}$ for $t \in [0, T)$. Integrating the inequality with respect to t on a sufficiently small time interval $[t_0, t_1]$ ($t_0 \leq t_1$), we get $\tilde{\lambda}_{1,p}(u(x, t_0), t_0) \geq \tilde{\lambda}_{1,p}(u(x, t_1), t_1)$. Note that $\tilde{\lambda}_{1,p}(u(x, t_0), t_0) = \tilde{\lambda}_{1,p}(t_0)$, $\tilde{\lambda}_{1,p}(u(x, t_1), t_1) \geq \tilde{\lambda}_{1,p}(t_1)$, therefore, $\tilde{\lambda}_{1,p}(t_0) \geq \tilde{\lambda}_{1,p}(t_1)$ for any $t_1 \geq t_0$. As $t_0 \in [0, T)$ is arbitrary, then we conclude that $\tilde{\lambda}_{1,p}(t)$ is nonincreasing under the forced MCF (2.2) provided $\kappa(t) \geq \frac{(1+p\beta)\psi^2(t) - \sigma^2(t)}{p}$ for $t \in [0, T)$. By the similar process, we can obtain $\tilde{\lambda}_{1,p}(t)$ is nondecreasing under the forced MCF (2.2) provided $\kappa(t) \leq \frac{(1+p\epsilon)\sigma^2(t) - \psi^2(t)}{p}$ for $t \in [0, T)$. The assertion that $\tilde{\lambda}_{1,p}(t)$ is differentiable almost everywhere for $t \in [0, T)$ can be derived by the classical Lebesgue's theorem which states that a monotonous continuous function is differentiable almost everywhere. $\square$

Remark 4.6. In terms of Theorem 4.5, when $\beta = \frac{1}{n}$, by (4.4), we know $|A|^2 = \frac{H^2}{n}$, so M_0 is totally umbilical. By Remark 4.4 and the fact that M_0 is strictly convex and closed, so M_0 is a round sphere, $H_{\max}(0) = H_{\min}(0)$, then $\sigma(t) = \psi(t)$ by their definition. Under additional condition that $\kappa(t) \leq 0$, for $t \in [0, T)$, obviously $\kappa(t) \leq 0 < \frac{(1+2\epsilon)\sigma^2(t) - \psi^2(t)}{2}$ for any $t \in [0, T)$ and also $\kappa(t) \leq 0 < \frac{(1+p\epsilon)\sigma^2(t) - \psi^2(t)}{p}$ for any $t \in [0, T)$. Particularly, when $\kappa(t) \equiv 0$, for $t \in [0, T)$, the forced MCF (2.2) reduces to the classical MCF, then from the proof of Theorem 4.5, we conclude that the first eigenvalue of the Laplace operator and the p -Laplace operator are strictly increasing under the forced MCF (2.2) if the initial hypersurface is a round sphere and $\kappa(t) \leq 0$, for $t \in [0, T)$. Regarding the normalized MCF case which corresponds to the case when $\kappa(t) = \frac{h(t)}{n}$, where $h(t) = \frac{\int_{M_t} H^2 d\mu_t}{\int_{M_t} d\mu_t}$, by Theorem 4.5, we infer that the first nonzero eigenvalue of the Laplace operator and the p -Laplace operator are constants under the normalized MCF if the initial hypersurface is a round sphere. In fact, we know that the evolving hypersurface M_t preserves the property of being totally umbilical, so if the initial hypersurface is a round sphere, then the evolving hypersurface M_t is also a round sphere. As we know, the evolving hypersurface M_t preserves the total area, so the evolving hypersurface M_t as a round sphere has the same radius as the initial round sphere, then the first eigenvalue is a constant, for $t \in [0, T)$. Moreover, the second assertion of (ii) in Theorem 4.5 coincides with [6, Theorem 1.2] ($k = 1$) when $\kappa(t) \equiv 0$, for $t \in [0, T)$.

In the following, we consider some special cases of Theorem 4.5.

Since the constant β satisfies $\frac{1}{n} \leq \beta < 1$, we restrict the constant β such that $\frac{1}{n} \leq \beta \leq \frac{1}{2}$ or $\frac{1}{2} \leq \beta < 1$, then we can get the following conclusions sharper than assertions (i),(ii) in Theorem 4.5.

Theorem 4.7. Let $\hat{\lambda}_1(t)$ be the first nonzero closed eigenvalue of the Laplace operator on an n -dimensional compact and strictly convex hypersurface $(M_t^n, g(t))$ ($n \geq 2$) which evolves under the forced MCF (2.2), $t \in [0, T)$. Denote by $H_{\max}(0)$ and $H_{\min}(0)$ the maximum and minimum values of the mean curvature of the initial hypersurface M_0 . If the initial hypersurface M_0 satisfies the pinching condition (4.1) for some constants $0 < \epsilon \leq \frac{1}{n} \leq \beta < 1$, then we have that

- (i) If $\kappa(t) \leq \frac{\sigma^2(t) + (2\epsilon - 1)\psi^2(t)}{2}$ for $t \in [0, T)$, and $\frac{1}{n} \leq \beta \leq \frac{1}{2}$, then $\hat{\lambda}_1(t)$ is nondecreasing under the forced MCF (2.2) for $t \in [0, T)$. Furthermore, $e^{\int_0^t (2\kappa(\tau) - \sigma^2(\tau) - (2\epsilon - 1)\psi^2(\tau)) d\tau} \hat{\lambda}_1(t)$ is nondecreasing under the forced MCF (2.2) for $t \in [0, T)$.
- (ii) If $\kappa(t) \geq \frac{\psi^2(t) + (2\beta - 1)\sigma^2(t)}{2}$ for $t \in [0, T)$ and $\frac{1}{n} \leq \beta \leq \frac{1}{2}$, then $\hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) for $t \in [0, T)$. Furthermore, $e^{\int_0^t (2\kappa(\tau) - \psi^2(\tau) - (2\beta - 1)\sigma^2(\tau)) d\tau} \hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) for $t \in [0, T)$.

(iii) If $\kappa(t) \geq \beta\psi^2(t)$ for $t \in [0, T)$ and $\frac{1}{2} \leq \beta < 1$, then $\hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) for $t \in [0, T)$. Furthermore, $e^{\int_0^t (2\kappa(\tau) - 2\beta\psi^2(\tau)) d\tau} \hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) for $t \in [0, T)$, where $\sigma(t), \psi(t)$ are defined as in Theorem 4.5.

Proof. The proof is similar to that of Theorem 4.5. Let $u(x, t)$ be the normalized eigenfunction corresponding to $\hat{\lambda}_1(t)$, namely, $\Delta u = -\hat{\lambda}_1 u$ and $\int_{M_t} u^2 = 1$. Let's deal with the following terms in (3.1):

$$\gamma(t) := 2 \int_{M_t} H h^{ij} \nabla_i u \nabla_j u d\mu_t - \int_{M_t} H^2 |\nabla u|^2 d\mu_t.$$

Obviously,

$$\gamma(t) = 2 \int_{M_t} H \left(h^{ij} - \frac{1}{2} H g^{ij} \right) \nabla_i u \nabla_j u d\mu_t. \quad (4.10)$$

As is shown in the proof of Theorem 4.5, the evolving hypersurface M_t satisfies the pinching condition (4.1) for some constants $0 < \varepsilon \leq \frac{1}{n} \leq \beta < 1$, then $0 < \varepsilon H g^{ij} \leq h^{ij} \leq \beta H g^{ij}$, therefore, if $\frac{1}{n} \leq \beta \leq \frac{1}{2}$, by (4.5), (4.10), we have

$$(2\varepsilon - 1)\psi^2(t)\hat{\lambda}_1(t) \leq \gamma(t) \leq (2\beta - 1)\sigma^2(t)\hat{\lambda}_1(t). \quad (4.11)$$

Inputting (4.11) into (3.1), by (4.5), we have

$$\frac{d}{dt} \hat{\lambda}_1(t) \geq \hat{\lambda}_1(t) [-2\kappa(t) + \sigma^2(t) + (2\varepsilon - 1)\psi^2(t)] \quad (4.12)$$

and

$$\frac{d}{dt} \hat{\lambda}_1(t) \leq \hat{\lambda}_1(t) [-2\kappa(t) + \psi^2(t) + (2\beta - 1)\sigma^2(t)]. \quad (4.13)$$

If $\frac{1}{2} \leq \beta \leq 1$, by (4.5), (4.10), we have

$$\gamma(t) \leq (2\beta - 1)\psi^2(t)\hat{\lambda}_1(t). \quad (4.14)$$

Inputting (4.14) into (3.1), by (4.5), we have

$$\frac{d}{dt} \hat{\lambda}_1(t) \leq \hat{\lambda}_1(t) [-2\kappa(t) + 2\beta\psi^2(t)], \quad (4.15)$$

then by (4.12), (4.13), if $\frac{1}{n} \leq \beta < \frac{1}{2}$, we conclude that $\hat{\lambda}_1(t)$ is nondecreasing under the forced MCF (2.2) provided $\kappa(t) \leq \frac{\sigma^2(t) + (2\varepsilon - 1)\psi^2(t)}{2}$ for $t \in [0, T)$, and $e^{\int_0^t (2\kappa(\tau) - \sigma^2(\tau) - (2\varepsilon - 1)\psi^2(\tau)) d\tau} \hat{\lambda}_1(t)$ is nondecreasing under the forced MCF (2.2) for $t \in [0, T)$, and also $\hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) provided $\kappa(t) \geq \frac{\psi^2(t) + (2\beta - 1)\sigma^2(t)}{2}$ for $t \in [0, T)$, and $e^{\int_0^t (2\kappa(\tau) - \psi^2(\tau) - (2\beta - 1)\sigma^2(\tau)) d\tau} \hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) for $t \in [0, T)$. If $\frac{1}{2} \leq \beta < 1$, by (4.15), then we conclude that $\hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) provided $\kappa(t) \geq \beta\psi^2(t)$, for $t \in [0, T)$, and $e^{\int_0^t (2\kappa(\tau) - 2\beta\psi^2(\tau)) d\tau} \hat{\lambda}_1(t)$ is nonincreasing under the forced MCF (2.2) for $t \in [0, T)$. $\square$

As with Theorem 4.7, since the constant β satisfies $\frac{1}{n} \leq \beta < 1$, we restrict the constant β such that $\frac{1}{n} \leq \beta \leq \frac{1}{p}$ or $\max\left\{\frac{1}{p}, \frac{1}{n}\right\} \leq \beta < 1$, then we can get the following conclusions sharper than assertions (iii) and (vi) in Theorem 4.5.

Theorem 4.8. Let $\tilde{\lambda}_{1,p}(t)$ ($p \in (1, \infty)$, $p \neq 2$) be the first nonzero closed eigenvalue of the p -Laplace operator on an n -dimensional compact and strictly convex hypersurface $(M_t^n, g(t))$ ($n \geq 2$) which evolves under the forced MCF (2.2), $t \in [0, T)$. Denote by $H_{\max}(0)$ and $H_{\min}(0)$ the maximum and minimum values of the mean

curvature of the initial hypersurface M_0 . If the initial hypersurface M_0 satisfies the pinching condition (4.1) for some constants $0 < \varepsilon \leq \frac{1}{n} \leq \beta < 1$, then we have that

- (i) If $\kappa(t) \leq \frac{\sigma^2(t) + (p\varepsilon - 1)\psi^2(t)}{p}$ for $t \in [0, T)$ and $\frac{1}{n} \leq \beta \leq \frac{1}{p}$ ($n \geq \max\{p, 2\}$), then $\tilde{\lambda}_{1,p}(t)$ is nondecreasing and differentiable almost everywhere under the forced MCF (2.2) for $t \in [0, T)$.
- (ii) If $\kappa(t) \geq \frac{\psi^2(t) + (p\beta - 1)\sigma^2(t)}{p}$ for $t \in [0, T)$ and $\frac{1}{n} \leq \beta \leq \frac{1}{p}$ ($n \geq \max\{p, 2\}$), then $\tilde{\lambda}_{1,p}(t)$ is nonincreasing and differentiable almost everywhere under the forced MCF (2.2) for $t \in [0, T)$.
- (iii) If $\kappa(t) \geq \beta\psi^2(t)$ for $t \in [0, T)$ and $\max\left\{\frac{1}{p}, \frac{1}{n}\right\} \leq \beta < 1$, then $\tilde{\lambda}_{1,p}(t)$ is nonincreasing and differentiable almost everywhere under the forced MCF (2.2) for $t \in [0, T)$, where $\sigma(t), \psi(t)$ are defined as in Theorem 4.5.

Proof. The proof is similar to that of Theorem 4.7. We assume that $u(x, t)$ is any smooth function satisfying (2.7) such that at time $t_0 \in [0, T)$, $u(x, t_0)$ is the eigenfunction corresponding to $\tilde{\lambda}_{1,p}(t_0)$. Let $\tilde{\lambda}_{1,p}(u, t)$ be a smooth function defined by (2.6). Let's deal with the following terms in (3.4):

$$\delta(t) := p \int_{M_t} |\nabla u|^{p-2} H h^{ij} \nabla_i u \nabla_j u d\mu_t - \int_{M_t} H^2 |\nabla u|^p d\mu_t.$$

Obviously,

$$\delta(t) = p \int_{M_t} |\nabla u|^{p-2} H \left(h^{ij} - \frac{1}{p} H g^{ij} \right) \nabla_i u \nabla_j u d\mu_t. \quad (4.16)$$

As is shown in the proof of Theorem 4.5, the evolving hypersurface M_t satisfies the pinching condition (4.1) for some constants $0 < \varepsilon \leq \frac{1}{n} \leq \beta < 1$, then $0 < \varepsilon H g^{ij} \leq h^{ij} \leq \beta H g^{ij}$, thus, at time $t = t_0$, by (4.5), (4.16), if $\frac{1}{n} \leq \beta \leq \frac{1}{p}$, we have

$$(p\varepsilon - 1)\psi^2(t)\tilde{\lambda}_{1,p}(t) \leq \delta(t) \leq (p\beta - 1)\sigma^2(t)\tilde{\lambda}_{1,p}(t). \quad (4.17)$$

Inputting (4.17) into (3.4), by (4.5), at time $t = t_0$, we have

$$\frac{d}{dt} \tilde{\lambda}_{1,p}(u, t) \geq \tilde{\lambda}_{1,p}(t) [-p\kappa(t) + \sigma^2(t) + (p\varepsilon - 1)\psi^2(t)] \quad (4.18)$$

and

$$\frac{d}{dt} \tilde{\lambda}_{1,p}(u, t) \leq \tilde{\lambda}_{1,p}(t) [-p\kappa(t) + \psi^2(t) + (p\beta - 1)\sigma^2(t)]. \quad (4.19)$$

If $\max\left\{\frac{1}{p}, \frac{1}{n}\right\} \leq \beta < 1$, by (4.5), (4.16), we have

$$\delta(t) \leq (p\beta - 1)\psi^2(t)\tilde{\lambda}_{1,p}(t). \quad (4.20)$$

Inputting (4.20) into (3.4), by (4.5), at time $t = t_0$, we have

$$\frac{d}{dt} \tilde{\lambda}_{1,p}(u, t) \leq \tilde{\lambda}_{1,p}(t) [-p\kappa(t) + p\beta\psi^2(t)]. \quad (4.21)$$

Following similar arguments of the proof of Theorem 4.5, if $\frac{1}{n} \leq \beta \leq \frac{1}{p}$, then by (4.18), (4.19), we conclude that $\tilde{\lambda}_{1,p}(t)$ is nondecreasing and differentiable almost everywhere under the forced MCF (2.2) provided $\kappa(t) \leq \frac{\sigma^2(t) + (p\varepsilon - 1)\psi^2(t)}{p}$ for $t \in [0, T)$ and also $\tilde{\lambda}_{1,p}(t)$ is nonincreasing and differentiable almost everywhere under the forced MCF (2.2) provided $\kappa(t) \geq \frac{\psi^2(t) + (p\beta - 1)\sigma^2(t)}{p}$. If $\max\left\{\frac{1}{p}, \frac{1}{n}\right\} \leq \beta < 1$, then by (4.21), we conclude that $\tilde{\lambda}_{1,p}(t)$ is nonincreasing and differentiable almost everywhere under the forced MCF (2.2) provided $\kappa(t) \geq \beta\psi^2(t)$. $\square$

Since the constant ε satisfies $0 < \varepsilon < \frac{1}{n}$, we restrict the constant ε such that $0 < \varepsilon \leq \frac{1}{p}$ or $\frac{1}{p} \leq \varepsilon \leq \frac{1}{n}$ ($p \geq n$), then we can get the following conclusions sharper than assertions (iii) and (vi) in Theorem 4.5.

Theorem 4.9. Let $\tilde{\lambda}_{1,p}(t)$ ($p \in (1, \infty)$, $p \neq 2$) be the first nonzero closed eigenvalue of the p -Laplace operator on an n -dimensional compact and strictly convex hypersurface $(M_t^n, g(t))$ ($n \geq 2$) which evolves under the forced MCF (2.2), $t \in [0, T)$. Denote by $H_{\max}(0)$ and $H_{\min}(0)$ the maximum and minimum values of the mean curvature of the initial hypersurface M_0 . If the initial hypersurface M_0 satisfies the pinching condition (4.1) for some constants $0 < \varepsilon \leq \frac{1}{n} \leq \beta < 1$, then we have that

- (i) If $\kappa(t) \geq \beta\psi^2(t)$ for $t \in [0, T)$ and $\frac{1}{p} \leq \varepsilon \leq \frac{1}{n}$ ($p \geq n$), then $\tilde{\lambda}_{1,p}(t)$ is nonincreasing and differentiable almost everywhere under the forced MCF (2.2) for $t \in [0, T)$.
- (ii) If $\kappa(t) \leq \varepsilon\sigma^2(t)$ for $t \in [0, T)$ and $\frac{1}{p} \leq \varepsilon \leq \frac{1}{n}$ ($p \geq n$), then $\tilde{\lambda}_{1,p}(t)$ is nondecreasing and differentiable almost everywhere under the forced MCF (2.2) for $t \in [0, T)$.
- (iii) If $\kappa(t) \leq \frac{\sigma(t) + (p\varepsilon - 1)\psi^2(t)}{p}$ for $t \in [0, T)$ and $0 < \varepsilon \leq \frac{1}{p}$, then $\tilde{\lambda}_{1,p}(t)$ is nondecreasing and differentiable almost everywhere under the forced MCF (2.2) for $t \in [0, T)$, where $\sigma(t)$, $\psi(t)$ are defined as in Theorem 4.5.

Proof. The proof is similar to that of Theorem 4.7. We assume that $u(x, t)$ is any smooth function satisfying (2.7) such that at time $t_0 \in [0, T)$, $u(x, t_0)$ is the eigenfunction corresponding to $\tilde{\lambda}_{1,p}(t_0)$. Let $\tilde{\lambda}_{1,p}(u, t)$ be a smooth function defined by (2.6). As is shown in the proof of Theorem 4.5, the evolving hypersurface M_t satisfies the pinching condition (4.1) for some constants $0 < \varepsilon \leq \frac{1}{n} \leq \beta < 1$, then $0 < \varepsilon H g^{ij} \leq h^{ij} \leq \beta H g^{ij}$, so at time $t = t_0$, by (4.5), (4.16), if $\frac{1}{p} \leq \varepsilon \leq \frac{1}{n}$ ($p \geq n$), we have

$$(p\varepsilon - 1)\sigma^2(t)\tilde{\lambda}_{1,p}(t) \leq \delta(t) \leq (p\beta - 1)\psi^2(t)\tilde{\lambda}_{1,p}(t). \quad (4.22)$$

Inputting (4.22) into (3.4), by (4.5), at time $t = t_0$, we have

$$\frac{d}{dt}\tilde{\lambda}_{1,p}(u, t) \leq \tilde{\lambda}_{1,p}(t)(-p\kappa(t) + p\beta\psi^2(t)) \quad (4.23)$$

and

$$\frac{d}{dt}\tilde{\lambda}_{1,p}(u, t) \geq \tilde{\lambda}_{1,p}(t)(-p\kappa(t) + p\varepsilon\sigma^2(t)). \quad (4.24)$$

If $0 < \varepsilon \leq \frac{1}{p}$, by (4.5), (4.16), we have

$$\frac{d}{dt}\tilde{\lambda}_{1,p}(u, t) \geq \tilde{\lambda}_{1,p}(t)(-p\kappa(t) + \sigma(t) + (p\varepsilon - 1)\psi^2(t)), \quad (4.25)$$

then by (4.23), (4.24), following similar arguments of the proof of Theorem 4.5, we conclude that if $\varepsilon \geq \frac{1}{p}$, $\tilde{\lambda}_{1,p}(t)$ is nonincreasing and differentiable almost everywhere under the forced MCF (2.2) provided $\kappa(t) \geq \beta\psi^2(t)$ for $t \in [0, T)$ and nondecreasing and differentiable almost everywhere under the forced MCF (2.2) provided $\kappa(t) \leq \varepsilon\sigma^2(t)$ for $t \in [0, T)$. If $0 < \varepsilon \leq \frac{1}{p}$, then by (4.25), we conclude that $\tilde{\lambda}_{1,p}(t)$ is nondecreasing and differentiable almost everywhere under the forced MCF (2.2) provided $\kappa(t) \leq \frac{\sigma(t) + (p\varepsilon - 1)\psi^2(t)}{p}$ for $t \in [0, T)$. $\square$

Remark 4.10. It should be pointed out that when $\kappa(t) \leq 0$ for $t \in [0, T)$ and $\varepsilon \geq \frac{1}{p}$ in Theorem 4.9, obviously condition (ii) in Theorem 4.9 holds. Particularly, when $\kappa(t) \equiv 0$, for $t \in [0, T)$, the forced MCF (2.2) reduces to the classical MCF, then from the proof of Theorem 4.9, we conclude that the first eigenvalue of the p -Laplace operator is strictly increasing under the classical MCF if the initial closed, strictly convex hypersurface satisfies the pinching condition (4.1) and $\varepsilon \geq \frac{1}{p}$.

In the following, we give an example which satisfies the pinching condition (4.1).

Example 4.11. We consider the rotation ellipsoid surface $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ in $\mathbb{R}^3$ parametrized by equation $\vec{r}(\varphi, \theta) = (a \cos \varphi \cos \theta, a \cos \varphi \sin \theta, b \sin \varphi)$, where a and b are constants satisfying $a \geq b > 0$, and θ and φ are parameters satisfying $\theta \in [0, 2\pi)$, $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then we calculate the first fundamental form (metric) g and the second fundamental form h as follows:

$$\begin{aligned}\vec{r}_\varphi &= (-a \sin \varphi \cos \theta, -a \sin \varphi \sin \theta, b \cos \varphi), & \vec{r}_\theta &= (-a \cos \varphi \sin \theta, a \cos \varphi \cos \theta, 0), \\ \vec{r}_{\varphi\varphi} &= (-a \cos \varphi \cos \theta, -a \cos \varphi \sin \theta, -b \sin \varphi), & \vec{r}_{\varphi\theta} &= (a \sin \varphi \sin \theta, -a \sin \varphi \cos \theta, 0), \\ \vec{r}_{\theta\theta} &= (-a \cos \varphi \cos \theta, -a \cos \varphi \sin \theta, 0), & E &= \vec{r}_\varphi \cdot \vec{r}_\varphi = a^2 \sin^2 \varphi + b^2 \cos^2 \varphi, \\ F &= \vec{r}_\varphi \cdot \vec{r}_\theta = 0, & G &= \vec{r}_\theta \cdot \vec{r}_\theta = a^2 \cos^2 \varphi,\end{aligned}$$

then the unit outer normal vector

$$\begin{aligned}\vec{n} &= \frac{(b \cos \varphi \cos \theta, b \cos \varphi \sin \theta, a \sin \varphi)}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}, \\ L = \vec{r}_{\varphi\varphi} \cdot \vec{n} &= \frac{-ab}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}, & M = \vec{r}_{\varphi\theta} \cdot \vec{n} &= 0, \\ N = \vec{r}_{\theta\theta} \cdot \vec{n} &= \frac{-ab \cos^2 \varphi}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}.\end{aligned}$$

So

$$\begin{aligned}g &= Ed\varphi^2 + 2Fd\varphi d\theta + Gd\theta^2 = (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)d\varphi^2 + a^2 \cos^2 \varphi d\theta^2, \\ h &= -(Ld\varphi^2 + 2Md\varphi d\theta + Nd\theta^2) = \frac{ab}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}d\varphi^2 + \frac{ab \cos^2 \varphi}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}d\theta^2.\end{aligned}$$

Since $F = 0$, $M = 0$, the principal curvatures are as follows:

$$\mu_1 = -\frac{L}{E} = \frac{ab}{(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)^{\frac{3}{2}}}, \quad \mu_2 = -\frac{N}{G} = \frac{\frac{b}{a}}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}},$$

and the mean curvature is as follows:

$$H = \mu_1 + \mu_2 = \frac{ab}{(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)^{\frac{3}{2}}} + \frac{\frac{b}{a}}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}.$$

Since $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and a, b are constants satisfying $a \geq b > 0$, we have

$$H_{\min} = \frac{2b}{a^2}, \quad H_{\max} = \frac{a^2 + b^2}{ab^2},$$

and also

$$\frac{b^2}{a^2 + b^2} \leq \frac{\mu_2}{H} \leq \frac{1}{2} \leq \frac{\mu_1}{H} \leq \frac{a^2}{a^2 + b^2},$$

that is,

$$\frac{b^2}{a^2 + b^2}H \leq \mu_1, \mu_2 \leq \frac{a^2}{a^2 + b^2}H.$$

If we consider the compact strictly convex hypersurface M_0 without boundary to be the rotation ellipsoid surface in $\mathbb{R}^3$ above, then the rotation ellipsoid surface satisfies the pinching condition (4.1) with $\varepsilon = \frac{b^2}{a^2 + b^2}$, $\beta = \frac{a^2}{a^2 + b^2}$ by the calculation above. When $a = b$, we know the rotation ellipsoid surface in $\mathbb{R}^3$ reduces to a round sphere with radius a . Therefore, we can consider the monotonicity of the first eigenvalue

of the Laplace operator and the p -Laplace operator under the forced MCF (2.2), especially the classical MCF when the initial hypersurface is the rotation ellipsoid surface $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ in $\mathbb{R}^3$ for some special positive constants a, b and the coefficient function of the forcing term of the forced MCF by applying some of the conclusions obtained above. In fact, we can consider the compact strictly convex hypersurface M_0 without boundary to be even more general n -dimensional ellipsoid $\sum_{i=1}^{n+1} \left(\frac{x_i}{a_i}\right)^2 = 1$ in $\mathbb{R}^{n+1}$, where $a_i, i = 1, \dots, n+1$ are positive constants satisfying $a_1 \geq a_2 \geq \dots \geq a_{n+1}$ since they also satisfy the pinching condition (4.1).

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