Research Article

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Noetherian properties in composite generalized power series rings

<https://doi.org/10.1515/math-2020-0103>

received March 7, 2020; accepted October 20, 2020

Abstract: Let $(\Gamma, \leq)$ be a strictly ordered monoid, and let $\Gamma^* = \Gamma \setminus \{0\}$. Let $D \subseteq E$ be an extension of commutative rings with identity, and let I be a nonzero proper ideal of D . Set

$$D + \llbracket E^{\Gamma^*, \leq} \rrbracket := \{f \in \llbracket E^{\Gamma^*, \leq} \rrbracket \mid f(0) \in D\} \text{ and} \\ D + \llbracket I^{\Gamma^*, \leq} \rrbracket := \{f \in \llbracket D^{\Gamma^*, \leq} \rrbracket \mid f(\alpha) \in I, \text{ for all } \alpha \in \Gamma^*\}.$$

In this paper, we give necessary conditions for the rings $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ to be Noetherian when $(\Gamma, \leq)$ is positively ordered, and sufficient conditions for the rings $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ to be Noetherian when $(\Gamma, \leq)$ is positively totally ordered. Moreover, we give a necessary and sufficient condition for the ring $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ to be Noetherian when $(\Gamma, \leq)$ is positively totally ordered. As corollaries, we give equivalent conditions for the rings $D + (X_1, \dots, X_n)E[X_1, \dots, X_n]$ and $D + (X_1, \dots, X_n)I[X_1, \dots, X_n]$ to be Noetherian.

Keywords: $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$, $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$, generalized power series ring, Noetherian ring

MSC 2020: 13A02, 13A15, 13B35, 13E05

1 Introduction

Throughout this paper, a *monoid* means a commutative semigroup with identity element. The operation is written additively and the identity element is denoted by 0, unless otherwise stated.

A monoid Γ is said to be *cancellative* if every element in Γ is cancellative, i.e., for every $\alpha, \beta, \gamma \in \Gamma$, $\alpha + \beta = \alpha + \gamma$ implies $\beta = \gamma$. We denote by $G(\Gamma)$ the largest subgroup of Γ , i.e., $G(\Gamma) := \{\alpha \in \Gamma \mid \alpha + \beta = 0, \text{ for some } \beta \in \Gamma\}$. A monoid Γ is *torsion-free* if for an $\alpha \in \Gamma$ and a positive integer n , $n\alpha = 0$ implies $\alpha = 0$. If $\alpha_1, \dots, \alpha_n \in \Gamma$, then we denote by $\langle \alpha_1, \dots, \alpha_n \rangle$ the set of all elements $\sum_{i=1}^n k_i \alpha_i$ with nonnegative integers k_i . A monoid Γ is *finitely generated* if there exists a finite subset $\{\alpha_1, \dots, \alpha_n\}$ of Γ such that $\Gamma = \langle \alpha_1, \dots, \alpha_n \rangle$. An *ideal* of Γ is a nonempty subset I of Γ such that $I \supseteq \alpha + I := \{\alpha + \gamma \mid \gamma \in I\}$ for each $\alpha \in \Gamma$.

An *ordered* monoid is a monoid $(\Gamma, +)$ together with a partial order $\leq$ that is compatible with the monoid operation, meaning that for every $\alpha_1, \alpha_2, \beta \in \Gamma$, $\alpha_1 \leq \alpha_2$ implies $\alpha_1 + \beta \leq \alpha_2 + \beta$. An ordered monoid $(\Gamma, \leq)$ is *positively* ordered if $0 \leq \alpha$ for all $\alpha \in \Gamma$. Note that if $(\Gamma, \leq)$ is positively ordered, then $G(\Gamma) = \{0\}$. An ordered monoid $(\Gamma, \leq)$ is a *strictly* ordered monoid if for every $\alpha_1, \alpha_2, \beta \in \Gamma$, $\alpha_1 < \alpha_2$ implies $\alpha_1 + \beta < \alpha_2 + \beta$. We say that an ordered monoid $(\Gamma, \leq)$ is *artinian* if every decreasing sequence of elements of Γ is finite; and $(\Gamma, \leq)$ is *narrow* if every subset of pairwise order-incomparable elements of Γ is finite.

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Let R be a commutative ring with identity and $(\Gamma, \leq)$ a strictly ordered monoid. We denote by $\llbracket R^{\Gamma, \leq} \rrbracket$ the set of all mappings $f: \Gamma \rightarrow R$ such that $\text{supp}(f) := \{\alpha \in \Gamma \mid f(\alpha) \neq 0\}$ is an artinian and narrow subset of Γ . (The set $\text{supp}(f)$ is called the *support* of f .) With pointwise addition, $\llbracket R^{\Gamma, \leq} \rrbracket$ is an (additive) abelian group. Moreover, for every $\alpha \in \Gamma$ and $f, g \in \llbracket R^{\Gamma, \leq} \rrbracket$, the set $X_\alpha(f, g) := \{(\beta, \gamma) \in \Gamma \times \Gamma \mid \alpha = \beta + \gamma, f(\beta) \neq 0, \text{ and } g(\gamma) \neq 0\}$ is finite [1, 1.16]; so this allows us to define the operation of *convolution* $*$ on $\llbracket R^{\Gamma, \leq} \rrbracket$: For every $f, g \in \llbracket R^{\Gamma, \leq} \rrbracket$,

$$(f * g)(\alpha) = \sum_{(\beta, \gamma) \in X_\alpha(f, g)} f(\beta)g(\gamma).$$

It is easy to see that $\llbracket R^{\Gamma, \leq} \rrbracket$ is a commutative ring (under these operations) with identity e , namely, $e(0) = 1$ and $e(\alpha) = 0$ for every $\alpha \in \Gamma \setminus \{0\}$, which is called the *ring of generalized power series* of Γ over R . The elements of $\llbracket R^{\Gamma, \leq} \rrbracket$ are called *generalized power series with coefficients in R and exponents in Γ* . It is well known that R is canonically embedded as a subring of $\llbracket R^{\Gamma, \leq} \rrbracket$ and Γ is canonically embedded as a submonoid of $\llbracket R^{\Gamma, \leq} \rrbracket \setminus \{0\}$ by the mapping $\alpha \in \Gamma \mapsto e_\alpha \in \llbracket R^{\Gamma, \leq} \rrbracket$, where $e_\alpha(\alpha) = 1$ and $e_\alpha(\gamma) = 0$ for every $\gamma \in \Gamma \setminus \{\alpha\}$. The elements of $\llbracket R^{\Gamma, \leq} \rrbracket$ are written in the form $\sum_{\alpha \in \text{supp}(f)} f(\alpha)X^\alpha$, with addition and multiplication defined as for formal power series. Thus, a mapping $e_\alpha \in \llbracket R^{\Gamma, \leq} \rrbracket$ is written in the form X^α . Henceforth, we use the notation $\sum_{\alpha \in \text{supp}(f)} f(\alpha)X^\alpha$ for the elements of $\llbracket R^{\Gamma, \leq} \rrbracket$.

Let $\mathbb{N}$ be the additive monoid of nonnegative integers with the usual order $\leq$. Then every subset of $\mathbb{N}$ is artinian and narrow; thus $\llbracket R^{\mathbb{N}, \leq} \rrbracket \cong R[X]$, ring of formal power series over R in one indeterminate X [2, Example 2]. If $\mathbb{N}$ is the usual ordered monoid, and $\Gamma := \mathbb{N}^n = \mathbb{N} \times \cdots \times \mathbb{N}$ (n times) with the order $\leq$, where $\leq$ is the product order, or the lexicographic order, or the reverse lexicographic order, then $\llbracket R^{\Gamma, \leq} \rrbracket \cong R[X_1, \dots, X_n]$, ring of formal power series over R with n indeterminates [2, Example 3]. For more details on the ring of generalized power series, the readers can refer to [1–3].

In [2], Ribenboim determined when a generalized power series ring $\llbracket R^{\Gamma, \leq} \rrbracket$ is a Noetherian ring, where $(\Gamma, \leq)$ is a strictly ordered monoid. He showed that if $\llbracket R^{\Gamma, \leq} \rrbracket$ is Noetherian, then the following three conditions hold: (1) R is Noetherian; (2) if Γ is cancellative, then there exist $\alpha_1, \dots, \alpha_n \in \Gamma \setminus G(\Gamma)$ such that $\Gamma \subseteq \langle \alpha_1, \dots, \alpha_n \rangle + G(\Gamma)$; and (3) if $0 \leq \alpha$ for every $\alpha \in \Gamma$, then $(\Gamma, \leq)$ is narrow [2, 5.2]. He also proved the converse under an additional hypothesis which states that $\llbracket R^{\Gamma, \leq} \rrbracket$ is Noetherian if R and Γ satisfy the following: (1) $(\Gamma, \leq)$ is narrow and Γ is torsion-free cancellative; (2) there exist $\alpha_1, \dots, \alpha_n \in \Gamma \setminus G(\Gamma)$ such that $\Gamma = \langle \alpha_1, \dots, \alpha_n \rangle + G(\Gamma)$; and (3) R is Noetherian [2, 5.5]. In [4, Theorem 4], Hizem and Benhissi showed that if $D \subseteq E$ is an extension of commutative rings, then $D + XE[X]$ is a Noetherian ring if and only if D is a Noetherian ring and E is a finitely generated D -module. In [5, Proposition 2.4], Hizem proved that $D + XI[X]$ is a Noetherian ring if and only if D is a Noetherian ring and $I = I^2$.

Let $(\Gamma, \leq)$ be a strictly ordered monoid, and let $\Gamma^* := \Gamma \setminus \{0\}$. Let $D \subseteq E$ be an extension of commutative rings with identity, and let I be a nonzero proper ideal of D . Set

$$\begin{aligned} D + \llbracket E^{\Gamma^*, \leq} \rrbracket &:= \{f \in \llbracket E^{\Gamma, \leq} \rrbracket \mid f(0) \in D\} \quad \text{and} \\ D + \llbracket I^{\Gamma^*, \leq} \rrbracket &:= \{f \in \llbracket D^{\Gamma, \leq} \rrbracket \mid f(\alpha) \in I, \text{ for all } \alpha \in \Gamma^*\}. \end{aligned}$$

It is easy to see that $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ and $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ are commutative rings with identity, which are called the *composite generalized power series rings*. Then $D \subsetneq D + \llbracket I^{\Gamma^*, \leq} \rrbracket \subsetneq \llbracket D^{\Gamma, \leq} \rrbracket \subseteq D + \llbracket E^{\Gamma^*, \leq} \rrbracket \subseteq \llbracket E^{\Gamma, \leq} \rrbracket$. Note that if $G(\Gamma) \neq \{0\}$, then for $0 \neq \alpha \in G(\Gamma)$, there exists $\beta \in \Gamma$ such that $\alpha + \beta = 0$. For $a \in E$, $a = aX^\alpha X^\beta \in D + \llbracket E^{\Gamma^*, \leq} \rrbracket$. Thus, $D + \llbracket E^{\Gamma^*, \leq} \rrbracket = \llbracket E^{\Gamma, \leq} \rrbracket$. Hence, if E properly contains D and $(\Gamma, \leq)$ is positively ordered, then $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is properly between $\llbracket D^{\Gamma, \leq} \rrbracket$ and $\llbracket E^{\Gamma, \leq} \rrbracket$. We note that if $\Gamma = \mathbb{N}^n$ with the product order, or the lexicographic order, or the reverse lexicographic order, then $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ and $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ are isomorphic to $D + (X_1, \dots, X_n)E[X_1, \dots, X_n]$ and $D + (X_1, \dots, X_n)I[X_1, \dots, X_n]$, respectively.

The composite generalized power series rings are also appropriate examples of $D + M$ constructions. Also, $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ guarantees some algebraic properties of intermediate rings between $\llbracket D^{\Gamma, \leq} \rrbracket$ and $\llbracket E^{\Gamma, \leq} \rrbracket$, and $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ provides us some information of generalized power series rings which are contained in the usual generalized power series ring $\llbracket D^{\Gamma, \leq} \rrbracket$.

Let $(\Gamma, \leq)$ be a strictly ordered monoid, and let $\Gamma^* := \Gamma \setminus \{0\}$. Let $D \subseteq E$ be an extension of commutative rings with identity, and let I be a nonzero proper ideal of D . In this article, we study Noetherian properties

on the composite generalized power series rings $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ and $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$. In Section 2, we determine when the ring $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring. More precisely, we give necessary conditions for the ring $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ to be Noetherian when $(\Gamma, \leq)$ is positively ordered, and sufficient conditions for the ring $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ to be Noetherian when $(\Gamma, \leq)$ is positively totally ordered. As a corollary, if $(\Gamma, \leq)$ is positively totally ordered, then $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring if and only if D is a Noetherian ring, E is a finitely generated D -module, and Γ is finitely generated. Thus, we recover [4, Theorem 4] that $D + XE[X]$ is a Noetherian ring if and only if D is a Noetherian ring and E is a finitely generated D -module. In Section 3, when $(\Gamma, \leq)$ is positively totally ordered, we show that $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring if and only if D is Noetherian, $I = I^2$, and Γ is finitely generated. Thus, we recover [5, Proposition 2.4] that $D + XI[X]$ is a Noetherian ring if and only if D is a Noetherian ring and $I = I^2$.

2 Composite generalized power series ring of the form $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$

Throughout this section, an ordered monoid $(\Gamma, \leq)$ means a nonzero *strictly* ordered monoid.

Let $(\Gamma, \leq)$ be a positively ordered monoid, and $\Gamma^* = \Gamma \setminus \{0\}$. Let $D \subseteq E$ be an extension of commutative rings with identity. In this section, we determine when the ring $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring.

Let A be either $\llbracket E^{\Gamma^*, \leq} \rrbracket$ or $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$. For simplicity, the ideal of A generated by $f_1, \dots, f_n$ of A is denoted by $(f_1, \dots, f_n)$ or $(f_1, \dots, f_n)A$ instead of $(f_1, \dots, f_n) \star A$, and the principal ideal of A generated by $f \in A$ is denoted by fA instead of $f \star A$. Note that for every $f, g \in A$,

$$\text{supp}(f + g) \subseteq \text{supp}(f) \cup \text{supp}(g) \quad \text{and} \quad \text{supp}(f \star g) \subseteq \text{supp}(f) + \text{supp}(g).$$

For $0 \neq f \in A$, we denote by $\pi(f)$ the set of minimal elements in $\text{supp}(f)$. Then $\pi(f)$ is a nonempty finite set consisting of pairwise order incomparable elements. If $\pi(f)$ consists of only one element α , then we write $\pi(f) = \alpha$ and call it the *order* of f . If $(\Gamma, \leq)$ is totally ordered and $0 \neq f \in A$, then $\text{supp}(f)$ is a nonempty well-ordered subset of Γ ; so $\pi(f)$ always consists only one element.

We now give necessary conditions for the ring $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ to be Noetherian. For the most part of the proof, we follow the proof of [2, 5.2].

Theorem 2.1. *Let $D \subseteq E$ be an extension of commutative rings with identity and $(\Gamma, \leq)$ a positively ordered monoid. If $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring, then the following statements hold.*

- (1) *D is a Noetherian ring.*
- (2) *If Γ is cancellative, then E is a finitely generated D -module and Γ is finitely generated.*
- (3) *$(\Gamma, \leq)$ is narrow.*

Proof.

- (1) Note that $\llbracket E^{\Gamma^*, \leq} \rrbracket$ is an ideal of $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$, and $D \cong (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) / \llbracket E^{\Gamma^*, \leq} \rrbracket$; so D is a Noetherian ring.
- (2) Let Γ be cancellative. We first show that E is finitely generated as a D -module. Let $\alpha \in \Gamma^*$. Consider the ideal $X^\alpha \llbracket E^{\Gamma^*, \leq} \rrbracket$ of $\llbracket E^{\Gamma^*, \leq} \rrbracket$ generated by X^α :

$$X^\alpha \llbracket E^{\Gamma^*, \leq} \rrbracket := \{X^\alpha \star g \mid g \in \llbracket E^{\Gamma^*, \leq} \rrbracket\}.$$

Note that $\alpha \in \Gamma^*$, so for every $g \in \llbracket E^{\Gamma^*, \leq} \rrbracket$, $X^\alpha \star g \in D + \llbracket E^{\Gamma^*, \leq} \rrbracket$. Thus, $X^\alpha \llbracket E^{\Gamma^*, \leq} \rrbracket$ is an ideal of $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$. Since $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is Noetherian, $X^\alpha \llbracket E^{\Gamma^*, \leq} \rrbracket = (g_1, \dots, g_n)(D + \llbracket E^{\Gamma^*, \leq} \rrbracket)$, for some $g_1, \dots, g_n \in X^\alpha \llbracket E^{\Gamma^*, \leq} \rrbracket$. Then, for some $f_i \in \llbracket E^{\Gamma^*, \leq} \rrbracket$, each $g_i := X^\alpha \star f_i$. Hence, we have

$$X^\alpha \llbracket E^{\Gamma^*, \leq} \rrbracket = X^\alpha \star f_1 (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) + \dots + X^\alpha \star f_n (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) = X^\alpha \star (f_1 (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) + \dots + f_n (D + \llbracket E^{\Gamma^*, \leq} \rrbracket)).$$

Since Γ is cancellative, $E = f_1(0)D + \dots + f_n(0)D$. Thus, E is a finitely generated D -module.

Next, we prove that Γ is finitely generated. Suppose that $\Gamma \neq \langle \alpha_1, \dots, \alpha_n \rangle$ for any finite subset $\{\alpha_1, \dots, \alpha_n\}$ of Γ^* . For each $n \geq 1$, set

$$I_n := X^{\alpha_1} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) + \dots + X^{\alpha_n} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket).$$

Claim: There exists an element $\alpha_{n+1} \in \Gamma \setminus \langle \alpha_1, \dots, \alpha_n \rangle$ such that $X^{\alpha_{n+1}} \notin I_n$.

Proof of Claim. Suppose, by way of contradiction, that there is no such element α_{n+1} . Let $\beta \in \Gamma \setminus \langle \alpha_1, \dots, \alpha_n \rangle$. If $X^\beta \notin I_n$, then $\alpha_{n+1} := \beta$ contradicts our assumption and so the claim holds. Thus, assume $X^\beta \in I_n$. Then there exist $f_1, \dots, f_n \in D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ such that $X^\beta = \sum_{i=1}^n X^{\alpha_i} * f_i$. Hence, we have

$$\beta \in \text{supp} \left(\sum_{i=1}^n X^{\alpha_i} * f_i \right) \subseteq \bigcup_{i=1}^n \text{supp}(X^{\alpha_i} * f_i) = \bigcup_{i=1}^n (\alpha_i + \text{supp}(f_i));$$

so $\beta = \alpha_{i_1} + \gamma_1$ for some $i_1 \in \{1, \dots, n\}$ and $\gamma_1 \in \text{supp}(f_{i_1})$. Note that $\gamma_1 \notin \langle \alpha_1, \dots, \alpha_n \rangle$ because $\beta \notin \langle \alpha_1, \dots, \alpha_n \rangle$. If $X^{\gamma_1} \notin I_n$, then setting $\alpha_{n+1} := \gamma_1$ again contradicts our assumption because $\gamma_1 \notin \langle \alpha_1, \dots, \alpha_n \rangle$. Thus, assume $X^{\gamma_1} \in I_n$. We proceed by induction. Assume $m \geq 1$ and that $\gamma_0 = \beta, \gamma_1, \dots, \gamma_m \in \Gamma$ and $i_1, \dots, i_m \in \{1, \dots, n\}$ have been chosen so that, for each $1 \leq j \leq m$,

$$\gamma_j \notin \langle \alpha_1, \dots, \alpha_n \rangle, \quad X^{\gamma_j} \in I_n, \quad \gamma_{j-1} = \alpha_{i_j} + \gamma_j, \quad \gamma_j \in \text{supp}(f_{i_j}).$$

Note that for $\beta = \gamma_0, \gamma_0 \notin \langle \alpha_1, \dots, \alpha_n \rangle$ and $X^{\gamma_0} \in I_n$. Then $X^{\gamma_m} \in I_n$ implies, as in the argument above, that $\gamma_m = \alpha_{i_{m+1}} + \gamma_{m+1}$, for some $i_{m+1} \in \{1, 2, \dots, n\}$ and some $\gamma_{m+1} \in \text{supp}(f_{i_{m+1}}) \setminus \langle \alpha_1, \dots, \alpha_n \rangle$. By the contradiction hypothesis, $X^{\gamma_{m+1}} \in I_n$. It follows that we obtain, for every $m \geq 1$,

$$\beta = \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_m} + \gamma_m = \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_m} + \alpha_{i_{m+1}} + \gamma_{m+1}.$$

Since Γ is cancellative, $\gamma_m = \alpha_{i_{m+1}} + \gamma_{m+1}$, for every $m \geq 1$. Hence, for every $m \geq 1$,

$$X^{\gamma_m} = X^{\alpha_{i_{m+1}}} * X^{\gamma_{m+1}},$$

and so we have an infinite nondecreasing chain of ideals in $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$:

$$X^{\gamma_1} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) \subseteq \dots \subseteq X^{\gamma_m} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) \subseteq X^{\gamma_{m+1}} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) \subseteq \dots.$$

Since $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is Noetherian, there exists an integer $N \geq 1$ such that, for every $k \geq N$,

$$X^{\gamma_k} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) = X^{\gamma_N} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket).$$

Therefore, $X^{\gamma_{N+1}} = X^{\gamma_N} * f$, for some $f \in D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ and thus $\gamma_{N+1} = \gamma_N + \delta$, for some $\delta \in \text{supp}(f)$. Since $\gamma_N = \alpha_{i_{N+1}} + \gamma_{N+1}$ and Γ is cancellative, $\alpha_{i_{N+1}} + \delta = 0$. Since $0 \leq \alpha$ for all $\alpha \in \Gamma$, $G(\Gamma) = \{0\}$; so $\alpha_{i_{N+1}} = 0$, which contradicts the fact that $\alpha_i \in \Gamma^*$. This proves the claim.

By the claim, we obtain a strictly infinite chain $I_1 \subsetneq I_2 \subsetneq \dots$ of ideals in $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$, which is a contradiction to the fact that $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring.

(3) Suppose to the contrary that there are infinitely many pairwise incomparable elements $\alpha_1, \alpha_2, \dots$ of Γ .

Consider the chain of ideals in $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$:

$$X^{\alpha_1} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) \subseteq X^{\alpha_1} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) + X^{\alpha_2} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) \subseteq \dots.$$

Since $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring, there exists an integer $N \geq 1$ such that, for every $k \geq N$,

$$X^{\alpha_1} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) + \dots + X^{\alpha_k} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) = X^{\alpha_1} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) + \dots + X^{\alpha_N} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket).$$

Therefore, $X^{\alpha_{N+1}} = \sum_{i=1}^N X^{\alpha_i} * f_i$, for some $f_1, \dots, f_N \in D + \llbracket E^{\Gamma^*, \leq} \rrbracket$. Hence, we obtain

$$\alpha_{n+1} \in \text{supp} \left(\sum_{i=1}^N X^{\alpha_i} * f_i \right) \subseteq \bigcup_{i=1}^N \text{supp}(X^{\alpha_i} * f_i) = \bigcup_{i=1}^N (\alpha_i + \text{supp}(f_i)).$$

Thus, there exist $k \in \{1, \dots, N\}$ and $\beta \in \text{supp}(f_k)$ such that $\alpha_{N+1} = \alpha_k + \beta$, which implies that $\alpha_{N+1} \geq \alpha_k$, a contradiction. $\square$

Note that if $(\Gamma, \leq)$ is positively ordered, then $G(\Gamma) = \{0\}$. By applying Theorem 2.1 to the case of $D = E$, we have the following which is the same as [2, 5.2] under the condition that $(\Gamma, \leq)$ is positively ordered.

Corollary 2.2. (cf. [2, 5.2]) *Let D be a commutative ring with identity and $(\Gamma, \leq)$ a positively ordered monoid. If $\llbracket D^{\Gamma, \leq} \rrbracket$ is a Noetherian ring, then the following statements hold.*

- (1) D is a Noetherian ring.
- (2) If Γ is cancellative, then Γ is finitely generated.
- (3) $(\Gamma, \leq)$ is narrow.

We next prove the converse of Theorem 2.1 under some additional conditions on Γ . To do this, we need some lemmas.

Lemma 2.3. *Let $D \subseteq E$ be an extension of commutative rings with identity and $(\Gamma, \leq)$ an ordered monoid. Then an ideal of $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ containing $\llbracket E^{\Gamma^*, \leq} \rrbracket$ is of the form $I + \llbracket E^{\Gamma^*, \leq} \rrbracket$ for some ideal I of D .*

Proof. Let A be an ideal of $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ containing $\llbracket E^{\Gamma^*, \leq} \rrbracket$ and set $I := \{f(0) \mid f \in A\}$. Clearly, I is an ideal of D . Choose $d \in I$. Then there exists an element $f \in A$ such that $d = f(0)$. Since $f - d \in \llbracket E^{\Gamma^*, \leq} \rrbracket$, $d \in A$; so $I \subseteq A$. Hence, $I + \llbracket E^{\Gamma^*, \leq} \rrbracket \subseteq A$. The reverse containment is obvious, which completes the proof. $\square$

Let $(\Gamma, \leq)$ and $(\Gamma, \leq')$ be ordered monoids. We say that $\leq'$ is finer than $\leq$ if for every $\alpha, \beta \in \Gamma$, $\alpha \leq \beta$ implies $\alpha \leq' \beta$.

Lemma 2.4. [2, 3.2, 3.3] *Let $(\Gamma, \leq)$ be an ordered monoid. Then the following statements hold:*

- (1) *If $(\Gamma, \leq)$ is a totally ordered monoid, then Γ is torsion-free and cancellative.*
- (2) *If $(\Gamma, \leq)$ is an ordered monoid, and Γ is torsion-free and cancellative, then there exists a compatible strict total order on Γ , which is finer than $\leq$.*

Lemma 2.5. *Let $D \subseteq E$ be an extension of commutative rings with identity, and I be a finitely generated ideal of D . Let $(\Gamma, \leq)$ be a positively totally ordered monoid. If E is a finitely generated D -module and Γ is finitely generated, then $I + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is finitely generated.*

Proof. Note that if $(\Gamma, \leq)$ is a totally ordered monoid, then by Lemma 2.4, Γ is torsion-free and cancellative. Let $E = e_1 D + \dots + e_m D$ for some $e_1, \dots, e_m \in E$, and $\Gamma = \langle \alpha_1, \dots, \alpha_n \rangle$ with $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n$. We first claim that $\llbracket E^{\Gamma^*, \leq} \rrbracket$ is a finitely generated ideal of $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$. Let $f \in \llbracket E^{\Gamma^*, \leq} \rrbracket$. Define

$$S_{\alpha_1} := \left\{ \alpha \in \text{supp}(f) \mid \alpha = \sum_{i=1}^n k_i \alpha_i \text{ and } k_1 \neq 0 \right\},$$

and for $i = 1, \dots, n-1$, set

$$S_{\alpha_{i+1}} := \left\{ \alpha \in \text{supp}(f) \setminus \bigcup_{j=1}^i S_{\alpha_j} \mid \alpha = \sum_{j=i+1}^n k_j \alpha_j \text{ and } k_{i+1} \neq 0 \right\}.$$

Then $\text{supp}(f) = \bigcup_{i=1}^n S_{\alpha_i}$ and hence we write f as follows:

For $1 \leq t \leq n$,

$$f = \sum_{t=1}^n f_t, \quad \text{where } f_t = \sum_{\alpha \in S_{\alpha_t}} a_{\alpha} X^{\alpha} \in \llbracket E^{\Gamma^*, \leq} \rrbracket.$$

Note that for each $\alpha \in S_{\alpha_t}$,

$$\begin{cases} a_\alpha = e_1 d_{\alpha_1} + \cdots + e_m d_{\alpha_m}, \text{ for some } d_{\alpha_1}, \dots, d_{\alpha_m} \in D, \\ \alpha = \sum_{i=t}^n k_i \alpha_i, \text{ for some nonnegative integers } k_t, \dots, k_n \text{ with } k_t \geq 1. \end{cases}$$

So for each $\alpha \in S_{\alpha_t}$, we have

$$a_\alpha X^\alpha = (e_1 d_{\alpha_1} + \cdots + e_m d_{\alpha_m}) X^{\sum_{i=t}^n k_i \alpha_i} = (e_1 X^{\alpha_t}) \star (d_{\alpha_1} X^{\alpha - \alpha_t}) + \cdots + (e_m X^{\alpha_t}) \star (d_{\alpha_m} X^{\alpha - \alpha_t}).$$

Note that if $\alpha \in S_{\alpha_t}$, then $\alpha - \alpha_t \in \Gamma$. Since $(\Gamma, \leq)$ is compatible, we have that if $\alpha, \beta \in S_{\alpha_t}$ with $\alpha \neq \beta$, $\alpha - \alpha_t \neq \beta - \alpha_t$. We also note that $\{\alpha - \alpha_t \mid \alpha \in S_{\alpha_t}\}$ is artinian because $\text{supp}(f)$ is artinian, which means that $\sum_{\alpha \in S_{\alpha_t}} X^{\alpha - \alpha_t} \in D + \llbracket E^{\Gamma^*, \leq} \rrbracket$.

Therefore, we have

$$\begin{aligned} f_t &= \sum_{\alpha \in S_{\alpha_t}} a_\alpha X^\alpha \\ &= \sum_{\alpha \in S_{\alpha_t}} (e_1 d_{\alpha_1} + \cdots + e_m d_{\alpha_m}) X^{\sum_{i=t}^n k_i \alpha_i} \\ &= e_1 X^{\alpha_t} \star \sum_{\alpha \in S_{\alpha_t}} d_{\alpha_1} X^{\alpha - \alpha_t} + \cdots + e_m X^{\alpha_t} \star \sum_{\alpha \in S_{\alpha_t}} d_{\alpha_m} X^{\alpha - \alpha_t} \\ &\in e_1 X^{\alpha_t} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket) + \cdots + e_m X^{\alpha_t} (D + \llbracket E^{\Gamma^*, \leq} \rrbracket), \end{aligned}$$

and hence $f = \sum_{t=1}^n f_t \in (\{e_i X^{\alpha_t} \mid i = 1, \dots, m \text{ and } t = 1, \dots, n\})$. Thus, we obtain

$$\llbracket E^{\Gamma^*, \leq} \rrbracket = (\{e_i X^{\alpha_t} \mid i = 1, \dots, m \text{ and } t = 1, \dots, n\}).$$

If $I = (b_1, \dots, b_q)$, then we have

$$I + \llbracket E^{\Gamma^*, \leq} \rrbracket = (\{b_i, e_j X^{\alpha_k} \mid i = 1, \dots, q, j = 1, \dots, m, \text{ and } k = 1, \dots, n\}),$$

and thus $I + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is finitely generated as desired. $\square$

Recall that an additive semigroup Γ is *Archimedean* if $\bigcap_{n \geq 1} (n\alpha + \Gamma) = \emptyset$ for each $\alpha \in \Gamma^*$. A simple example of an Archimedean semigroup is $\mathbb{N}^m$, where $\mathbb{N}$ is the monoid of nonnegative integers and m is a positive integer. (To see this, for any $(a_1, \dots, a_m) \in \mathbb{N}^{m*}$, if $(b_1, \dots, b_m) \in \bigcap_{n \geq 1} (n(a_1, \dots, a_m) + \mathbb{N}^m)$, then $b_i \geq na_i$ for all $i = 1, \dots, m$ and all $n \geq 1$. However, this is impossible, because at least one of a_i is nonzero. Thus, $\bigcap_{n \geq 1} (n(a_1, \dots, a_m) + \mathbb{N}^m) = \emptyset$.) Clearly, every submonoid of an Archimedean monoid is also Archimedean; so each submonoid of $\mathbb{N}^m$ is Archimedean.

Lemma 2.6. *Let $(\Gamma, \leq)$ be a positively totally ordered monoid. If Γ is finitely generated, then Γ is Archimedean.*

Proof. This is an immediate consequence of Lemma 2.4 and the facts that (1) S is a torsion-free, cancellative, finitely generated monoid with $G(S) = \{0\}$ if and only if S is isomorphic to a submonoid of $\mathbb{N}^m$ for some integer $m \geq 1$ [6, Theorem 3.11] (or [7, II. 7]); and (2) every submonoid of $\mathbb{N}^m$ is Archimedean. $\square$

Let R be a commutative ring with identity and I a nonzero proper ideal of R . It is well known that by taking $\{I^n\}_{n \geq 1}$ to be a fundamental system of neighborhoods of 0 in R , R becomes a topological ring and the topology of R is called the *I-adic topology* on R . Note that the *I-adic topology* on R is Hausdorff if and only if $\bigcap_{n \geq 1} I^n = (0)$.

Lemma 2.7. *Let $D \subseteq E$ be an extension of commutative rings with identity and $(\Gamma, \leq)$ a positively totally ordered monoid. If Γ is generated by $\alpha_1, \dots, \alpha_n$, then the $(X^{\alpha_1} \llbracket E^{\Gamma^*, \leq} \rrbracket + \cdots + X^{\alpha_n} \llbracket E^{\Gamma^*, \leq} \rrbracket)$ -adic topology on $D + \llbracket E^{\Gamma^*, \leq} \rrbracket$ is Hausdorff.*

Proof. Let $0 \neq g \in \bigcap_{m \geq 1} (X^{\alpha_1} \llbracket E^{\Gamma, \leq} \rrbracket + \cdots + X^{\alpha_n} \llbracket E^{\Gamma, \leq} \rrbracket)^m$. Then for each $m \geq 1$, g is a finite sum of generalized power series of the form $X^{k_{m1}\alpha_1 + \cdots + k_{mn}\alpha_n} \star g_m$, where $g_m \in \llbracket E^{\Gamma, \leq} \rrbracket$ and $k_{m1}, \dots, k_{mn}$ are nonnegative integers with $k_{m1} + \cdots + k_{mn} = m$. Recall that $\pi(g)$ is the minimal element in $\text{supp}(g)$. Therefore, we have

$$\pi(g) \in \text{supp} \left(\sum_{\text{finite}} X^{k_{m1}\alpha_1 + \cdots + k_{mn}\alpha_n} \star g_m \right) \subseteq \bigcup_{\text{finite}} (k_{m1}\alpha_1 + \cdots + k_{mn}\alpha_n + \text{supp}(g_m)),$$

that is, $\pi(g) = k_{m1}\alpha_1 + \cdots + k_{mn}\alpha_n + \pi(g_m)$, where k_{mi} are nonnegative integers such that $k_{m1} + \cdots + k_{mn} = m$ and $g_m \in \llbracket E^{\Gamma, \leq} \rrbracket$. Note that as m approaches ∞ , at least one of the k_{mi} , say k_{mr} , for some r with $1 \leq r \leq n$, gets arbitrarily large. Thus, by replacing the sequence $\{k_{mr}\}_{m=1}^{\infty}$ with a subsequence if necessary, we may assume that for each $m \geq 1$, we have $k_{mr} < k_{m+1r}$ and $k_{mr} \geq m$. Now

$$\begin{aligned} \pi(g) &= k_{m1}\alpha_1 + \cdots + k_{mr}\alpha_r + \cdots + k_{mn}\alpha_n + \pi(g_m) \in \Gamma \Rightarrow \\ \pi(g) &= k_{mr}\alpha_r + \gamma_m, \quad \text{for } \gamma_m \in \Gamma \Rightarrow \\ \pi(g) &\in m\alpha_r + \Gamma. \end{aligned}$$

Thus, $\pi(g) \in \bigcap_{m \geq 1} (m\alpha_r + \Gamma)$. By Lemma 2.6, $\pi(g) = 0$, a contradiction to the definition of g . Therefore, $\bigcap_{m \geq 1} (X^{\alpha_1} \llbracket E^{\Gamma, \leq} \rrbracket + \cdots + X^{\alpha_n} \llbracket E^{\Gamma, \leq} \rrbracket)^m = (0)$. $\square$

Let M be a monoid. The *algebraic preorder* (or *natural preorder*) on a monoid M is the relation $\preceq$ defined as follows:

For every $a, b \in M$,

$$a \preceq b \text{ if and only if } a + c = b, \text{ for some } c \in M.$$

In general, $a \preceq b \preceq a$ does not imply $a = b$; so $\preceq$ is not always a partial order on M . We first introduce some terminology in [8]. We say that a monoid M is *strict* if, for every $a, b, c \in M$, $a + b + c = c$ implies $a = b = 0$. For a partially ordered set $(M, \leq)$, a *lower set* of M is a subset I of M such that, for all $x, y \in M$, $x \leq y$, $y \in I$ implies $x \in I$. We write $\Downarrow(M, \leq)$ for the set of lower sets of M ordered by inclusion. Recall that an ordered monoid means a strictly ordered monoid. We collect some known results on a monoid M with algebraic preorder $\preceq$ in [8].

Lemma 2.8. *Let M be a nonzero monoid. Then the following statements hold:*

- (1) [8, Lemma 3.1] *If $(M, \leq)$ is a positively ordered monoid, then M is strict.*
- (2) [8, Lemma 3.2] *M is strict if and only if $(M, \preceq)$ is an ordered monoid.*
- (3) [8, Lemma 3.3] *Let M be a strict monoid. Then M is finitely generated if and only if $\Downarrow(M, \preceq)$ is artinian.*
- (4) [8, Lemma 2.2] *Let $(M, \leq)$ be a partially ordered set. Then $\Downarrow(M, \leq)$ is artinian if and only if $(M, \leq)$ is artinian and narrow.*
- (5) [8, Lemma 2.1 (2)] *Let $\sigma : M \rightarrow N$ be an increasing map between partially ordered sets. If σ is surjective and $\Downarrow M$ is artinian, then $\Downarrow N$ is artinian.*

Lemma 2.9. *Let $(\Gamma, \leq)$ be a positively totally ordered monoid. If Γ is finitely generated, then $(\Gamma, \leq)$ is artinian.*

Proof. Let $(\Gamma, \leq)$ be a positively totally ordered monoid that is finitely generated. By Lemma 2.8(1), (2), and (3), $(\Gamma, \preceq)$ is an ordered monoid and $\Downarrow(\Gamma, \preceq)$ is artinian. Since $\leq$ is a positive order, we obtain

$$\alpha \preceq \beta \Rightarrow \alpha \leq \beta, \text{ for all } \alpha, \beta \in \Gamma.$$

Thus, the identity map from $(\Gamma, \preceq)$ to $(\Gamma, \leq)$ is a monoid surjection. Since $\Downarrow(\Gamma, \preceq)$ is artinian, $\Downarrow(\Gamma, \leq)$ is artinian by Lemma 2.8(5). Hence, $(\Gamma, \leq)$ is artinian by Lemma 2.8(4). $\square$

Now we are ready to give sufficient conditions for $D + \llbracket E^{\Gamma, \leq} \rrbracket$ to be Noetherian. For simplicity, we use the notation $f^n = f \star \cdots \star f$ (n times) for $f \in D + \llbracket E^{\Gamma, \leq} \rrbracket$.

Theorem 2.10. *If E is a finitely generated D -module over a Noetherian ring D and $(\Gamma, \leq)$ is a positively totally ordered monoid that is finitely generated, then $D + \llbracket E^{\Gamma, \leq} \rrbracket$ is a Noetherian ring.*

Proof. By the hypothesis, there exist $e_1, \dots, e_n \in E$ and $\alpha_1, \dots, \alpha_m \in \Gamma^*$ such that

$$E = e_1 D + \dots + e_n D \quad \text{and} \quad \Gamma = \langle \alpha_1, \dots, \alpha_m \rangle.$$

Suppose to the contrary that $D + \llbracket E^{\Gamma, \leq} \rrbracket$ is not a Noetherian ring. Let $\mathcal{A}$ be the set of non-finitely generated ideals of $D + \llbracket E^{\Gamma, \leq} \rrbracket$. Then $\mathcal{A}$ is nonempty. Let $\{B_k\}_{k \in \Lambda}$ be a chain of members of $\mathcal{A}$, where Λ is an indexed set. Clearly, $\bigcup_{k \in \Lambda} B_k$ is an upper bound of $\{B_k\}_{k \in \Lambda}$ and is non-finitely generated. By Zorn's lemma, there exists an ideal I of $D + \llbracket E^{\Gamma, \leq} \rrbracket$ that is maximal among non-finitely generated ideals. Note that I is a prime ideal of $D + \llbracket E^{\Gamma, \leq} \rrbracket$ [9, Theorem 7]. If $\llbracket E^{\Gamma, \leq} \rrbracket \subseteq I$, then $I = I(0) + \llbracket E^{\Gamma, \leq} \rrbracket$, where $I(0) = \{f(0) \mid f \in I\}$ by Lemma 2.3. Since D is a Noetherian ring, $I(0)$ is finitely generated; so I is also finitely generated by Lemma 2.5, which contradicts the choice of I . Therefore, $\llbracket E^{\Gamma, \leq} \rrbracket \not\subseteq I$. Hence, $e_i X^{\alpha_i} \notin I$ for some e_i and α_i . Without loss of generality, we may assume that $e_1 X^{\alpha_1} \notin I$; so $I \subsetneq I + e_1 X^{\alpha_1} (D + \llbracket E^{\Gamma, \leq} \rrbracket)$. By the maximality of I , $I + e_1 X^{\alpha_1} (D + \llbracket E^{\Gamma, \leq} \rrbracket)$ is a finitely generated ideal of $D + \llbracket E^{\Gamma, \leq} \rrbracket$; so there exists a finitely generated subideal $J := (f_1, \dots, f_q)$ of I such that $I + e_1 X^{\alpha_1} (D + \llbracket E^{\Gamma, \leq} \rrbracket) = J + e_1 X^{\alpha_1} (D + \llbracket E^{\Gamma, \leq} \rrbracket)$.

We first show that $I = J + e_1 X^{\alpha_1} I$. Let $f \in I$. Then $f \in J + e_1 X^{\alpha_1} (D + \llbracket E^{\Gamma, \leq} \rrbracket)$; so $f = g + e_1 X^{\alpha_1} * h$ for some $g \in J$ and $h \in D + \llbracket E^{\Gamma, \leq} \rrbracket$. Hence, $e_1 X^{\alpha_1} * h = f - g \in I$. Since I is a prime ideal of $D + \llbracket E^{\Gamma, \leq} \rrbracket$ and $e_1 X^{\alpha_1} \notin I$, we have $h \in I$. Therefore, $f \in J + e_1 X^{\alpha_1} I$. The reverse containment is obvious.

Next, we claim that $I = J$, which is a contradiction. Let $g \in I$. Since $I = J + e_1 X^{\alpha_1} I$, for each $n \geq 1$, we write g as follows:

$$\begin{cases} g = g_0 + (e_1 X^{\alpha_1}) * g_1 + (e_1 X^{\alpha_1})^2 * g_2 + \dots + (e_1 X^{\alpha_1})^{n-1} * g_{n-1} + (e_1 X^{\alpha_1})^n * h_n, \\ h_n = g_n + (e_1 X^{\alpha_1}) * h_{n+1}, \end{cases}$$

where $g_0, \dots, g_n \in J$ and $h_n, h_{n+1} \in I$. Note that if $e_1^i = 0$ for some $i > 1$, then $(e_1 X^{\alpha_1})^j = 0$ for all $j \geq i$; so $g = g_0 + (e_1 X^{\alpha_1}) * g_1 + \dots + (e_1 X^{\alpha_1})^{i-1} * g_{i-1} \in J$. Hence, we assume that e_1 is not a nilpotent element. Put $t_r := \sum_{i=0}^r (e_1 X^{\alpha_1})^i * g_i$. Then for each $N \geq 1$, we can find a nonnegative integer r such that

$$g - t_r = (e_1 X^{\alpha_1})^{r+1} * h_{r+1} \in (e_1 X^{\alpha_1})^{r+1} I \subseteq (X^{\alpha_1} \llbracket E^{\Gamma, \leq} \rrbracket + \dots + X^{\alpha_m} \llbracket E^{\Gamma, \leq} \rrbracket)^N$$

for all $r \geq N - 1$. Hence by Lemma 2.7, $\lim_{r \rightarrow \infty} t_r = g$.

We now consider the sequence $(g_n)_{n \geq 0}$ in J . Since $J = (f_1, \dots, f_q)$, each g_k is written as $\sum_{i=1}^q z_{ki} * f_i$ for some $z_{ki} \in D + \llbracket E^{\Gamma, \leq} \rrbracket$. For each $r \geq 0$, we obtain

$$\sum_{i=0}^r (e_1 X^{\alpha_1})^i * g_i = \sum_{i=0}^r \left((e_1 X^{\alpha_1})^i * \sum_{j=1}^q (z_{ij} * f_j) \right) = f_1 * \left(\sum_{i=0}^r z_{i1} * (e_1 X^{\alpha_1})^i \right) + \dots + f_q * \left(\sum_{i=0}^r z_{iq} * (e_1 X^{\alpha_1})^i \right).$$

For each $k = 1, \dots, q$, put

$$u_{rk} := \sum_{i=0}^r z_{ik} * (e_1 X^{\alpha_1})^i \quad \text{and} \quad u_k := \sum_{i=0}^{\infty} z_{ik} * (e_1 X^{\alpha_1})^i.$$

We first show that $u_k \in D + \llbracket E^{\Gamma, \leq} \rrbracket$ for each $k = 1, \dots, q$. Fix an element $k \in \{1, \dots, q\}$. Note that 0 does not appear in $\text{supp}(z_{ik} * (e_1 X^{\alpha_1})^i)$ for all $i \geq 1$, because $G(\Gamma) = \{0\}$. Let $0 \neq \beta \in \text{supp}(u_k)$. Note that if $\beta \in \text{supp}(z_{ik} * (e_1 X^{\alpha_1})^i)$, then $\beta = i\alpha_1 + \gamma_{ik}$ for some $\gamma_{ik} \in \text{supp}(z_{ik})$. Therefore, if β belongs to infinitely many $\text{supp}(z_{ik} * (e_1 X^{\alpha_1})^i)$, then $\beta \in \bigcap_{i \geq 1} (i\alpha_1 + \Gamma)$, which contradicts Lemma 2.6. Hence, β occurs in only finitely many $z_{ik} * (e_1 X^{\alpha_1})^i$. Thus, for every $\beta \in \text{supp}(u_k)$, the coefficient of X^β in u_k is the sum of coefficients of X^β in only finitely many suitable $z_{ik} * (e_1 X^{\alpha_1})^i$. By Lemma 2.9, $(\Gamma, \leq)$ is artinian; so $\text{supp}(u_k)$ is artinian. Thus, $u_k \in D + \llbracket E^{\Gamma, \leq} \rrbracket$ for each $k = 1, \dots, q$. Next, we show that for each $k = 1, \dots, q$, the sequence $(u_{rk})_{r \geq 1}$ converges to u_k as r goes to ∞ . For each $k = 1, \dots, q$ and each $N \geq 1$, there exists a nonnegative integer t such that

$$u_k - u_{kt} = \sum_{i \geq t+1} z_{ik} \star (e_1 X^{\alpha_i})^i \in (X^{\alpha_1} \llbracket E^{\Gamma, \leq} \rrbracket + \cdots + X^{\alpha_m} \llbracket E^{\Gamma, \leq} \rrbracket)^N$$

for all $t \geq N - 1$. By Lemma 2.7, u_{rk} converges to u_k ; so

$$\lim_{r \rightarrow \infty} \sum_{i=0}^r z_{ik} \star (e_1 X^{\alpha_i})^i = \sum_{i=0}^{\infty} z_{ik} \star (e_1 X^{\alpha_i})^i.$$

Therefore,

$$\begin{aligned} g &= \lim_{r \rightarrow \infty} \left(f_1 \star \sum_{j=0}^r (z_{j1} \star (e_1 X^{\alpha_j})^j) + \cdots + f_q \star \sum_{j=0}^r (z_{jq} \star (e_1 X^{\alpha_j})^j) \right) \\ &= f_1 \star \sum_{j=0}^{\infty} (z_{j1} \star (e_1 X^{\alpha_j})^j) + \cdots + f_q \star \sum_{j=0}^{\infty} (z_{jq} \star (e_1 X^{\alpha_j})^j) \\ &\in J, \end{aligned}$$

which proves the claim. Thus, $D + \llbracket E^{\Gamma, \leq} \rrbracket$ is a Noetherian ring. $\square$

By applying Theorem 2.10 to the case of $D = E$, we have the following which is the same as [2, 5.5] under the additional condition that $(\Gamma, \leq)$ is positively ordered.

Corollary 2.11. (cf. [2, 5.5]) *Let Γ be cancellative and torsion-free and $(\Gamma, \leq)$ be a positively narrow ordered monoid. If D is a Noetherian ring and Γ is finitely generated, then $[D^{\Gamma, \leq}]$ is a Noetherian ring.*

Proof. By Lemma 2.4, there exists a compatible strict total order $\leq'$ on Γ , which is finer than $\leq$. Note that $0 \leq' \alpha$ for all $\alpha \in \Gamma$. Since $\leq'$ is finer than $\leq$ and $(\Gamma, \leq)$ is narrow, $\llbracket D^{\Gamma, \leq} \rrbracket = \llbracket D^{\Gamma, \leq'} \rrbracket$. By applying Theorem 2.10 to the case of $D = E$, $\llbracket D^{\Gamma, \leq} \rrbracket$ is a Noetherian ring. $\square$

Recall that for the usual ordered monoid $\mathbb{N}$, if $(\Gamma = \mathbb{N}^n, \leq)$ with the lexicographic order, then the ring $D + \llbracket E^{\Gamma, \leq} \rrbracket$ is isomorphic to $D + (X_1, \dots, X_n)E[X_1, \dots, X_n]$. By applying Theorems 2.1 and 2.10 to the case when $(\Gamma, \leq)$ is the monoid $\mathbb{N}^n$ with the lexicographic order, we recover

Corollary 2.12. [4, Theorem 4] *Let $D \subseteq E$ be an extension of commutative rings with identity. Then D is a Noetherian ring and E is a finitely generated D -module if and only if $D + (X_1, \dots, X_n)E[X_1, \dots, X_n]$ is a Noetherian ring. In particular, D is Noetherian if and only if $D[X_1, \dots, X_n]$ is Noetherian.*

3 Composite generalized power series ring of the form $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$

Throughout this section, an ordered monoid $(\Gamma, \leq)$ means a nonzero strictly ordered monoid.

Let D be a commutative ring with identity and let I be a nonzero proper ideal of D . In this section, we give an equivalent condition for the ring $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ to be Noetherian when $(\Gamma, \leq)$ is positively totally ordered. We also give some applications of composite generalized power series rings.

Theorem 3.1. *Let D be a commutative ring with identity and let I be a nonzero proper ideal of D . Let $(\Gamma, \leq)$ be a positively totally ordered monoid. Then $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring if and only if D is a Noetherian ring, $I^2 = I$, and Γ is finitely generated.*

Proof. $(\Rightarrow)$ Let $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ be a Noetherian ring. Then $D \cong (D + \llbracket I^{\Gamma^*, \leq} \rrbracket) / \llbracket I^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring. Let $0 \neq a \in I$ and $\alpha \in \Gamma^*$. Since $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ is Noetherian, the ideal $(aX^\alpha, aX^{2\alpha}, \dots)$ of $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ is finitely generated; so there exists a positive integer m such that $aX^{(m+1)\alpha} \in (aX^\alpha, \dots, aX^{m\alpha})$. Hence, for some $f_1, \dots, f_m \in D + \llbracket I^{\Gamma^*, \leq} \rrbracket$,

$$aX^{(m+1)\alpha} = \sum_{i=1}^m aX^{i\alpha} * f_i.$$

Comparing the coefficients of $X^{(m+1)\alpha}$ from each side of the equality, we conclude that $a = ba$ for some $b \in I$. Hence, $I \subseteq I^2$, and thus $I = I^2$.

Next, we show that Γ is finitely generated by some modification of the proof of Theorem 2.1(2). Suppose that Γ is not finitely generated. Then $\Gamma \neq \langle \alpha_1, \dots, \alpha_n \rangle$ for any finite subset $\{\alpha_1, \dots, \alpha_n\}$ of Γ^* . Since D is a Noetherian ring and $I^2 = I$, I is a principal ideal generated by an idempotent element a [10, Lemma 1]. For each $n \geq 1$, set

$$I_n := aX^{\alpha_1}(D + \llbracket I^{\Gamma^*, \leq} \rrbracket) + \dots + aX^{\alpha_n}(D + \llbracket I^{\Gamma^*, \leq} \rrbracket).$$

Claim: There exists an element $\alpha_{n+1} \in \Gamma \setminus \langle \alpha_1, \dots, \alpha_n \rangle$ such that $aX^{\alpha_{n+1}} \notin I_n$.

Proof of Claim. Suppose, by way of contradiction, that there is no such element α_{n+1} . Let $\beta \in \Gamma \setminus \langle \alpha_1, \dots, \alpha_n \rangle$. If $aX^\beta \notin I_n$, then $\alpha_{n+1} := \beta$ contradicts our assumption and so the claim holds. Thus, assume $aX^\beta \in I_n$. Then $aX^\beta = \sum_{i=1}^n aX^{\alpha_i} * f_i$, for some $f_1, \dots, f_n \in D + \llbracket I^{\Gamma^*, \leq} \rrbracket$. Hence, we have

$$\beta \in \sup p \left(\sum_{i=1}^n (aX^{\alpha_i} * f_i) \right) \subseteq \bigcup_{i=1}^n \sup p(aX^{\alpha_i} * f_i) = \bigcup_{i=1}^n (\alpha_i + \sup p(f_i));$$

so $\beta = \alpha_{i_1} + \gamma_1$ for some $i_1 \in \{1, \dots, n\}$ and $\gamma_1 \in \sup p(f_{i_1})$. Note that $\gamma_1 \notin \langle \alpha_1, \dots, \alpha_n \rangle$ because $\beta \notin \langle \alpha_1, \dots, \alpha_n \rangle$. If $aX^{\gamma_1} \notin I_n$, then setting $\alpha_{n+1} := \gamma_1$ again contradicts our assumption because $\gamma_1 \notin \langle \alpha_1, \dots, \alpha_n \rangle$. Thus, assume $X^{\gamma_1} \in I_n$. We proceed by induction. Assume $m \geq 1$ and that $\gamma_0 = \beta, \gamma_1, \dots, \gamma_m \in \Gamma$ and $i_1, \dots, i_m \in \{1, \dots, n\}$ have been chosen so that, for each $1 \leq j \leq m$,

$$\gamma_j \notin \langle \alpha_1, \dots, \alpha_n \rangle, \quad aX^{\gamma_j} \in I_n, \quad \gamma_{j-1} = \alpha_{i_j} + \gamma_j, \quad \gamma_j \in \sup p(f_{i_j}).$$

Note that for $\beta = \gamma_0$, $\gamma_0 \notin \langle \alpha_1, \dots, \alpha_n \rangle$ and $X^{\gamma_0} \in I_n$. Then $aX^{\gamma_m} \in I_n$ implies, as in the argument above, that $\gamma_m = \alpha_{i_{m+1}} + \gamma_{m+1}$, for some $i_{m+1} \in \{1, 2, \dots, n\}$ and some $\gamma_{m+1} \in \sup p(f_{i_{m+1}}) \setminus \langle \alpha_1, \dots, \alpha_n \rangle$. By the contradiction hypothesis, $aX^{\gamma_{m+1}} \in I_n$. It follows that we obtain, for every $m \geq 1$,

$$\beta = \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_m} + \gamma_m = \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_m} + \alpha_{i_{m+1}} + \gamma_{m+1}.$$

Note that since $(\Gamma, \leq)$ is a (positively) totally ordered monoid, Γ is cancellative by Lemma 2.4. Thus, for every $m \geq 1$, $\gamma_m = \alpha_{i_{m+1}} + \gamma_{m+1}$. Hence, for every $m \geq 1$,

$$aX^{\gamma_m} = X^{\alpha_{i_{m+1}}} * aX^{\gamma_{m+1}},$$

and so we have an infinite nondecreasing chain of ideals in $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$:

$$aX^{\gamma_1}(D + \llbracket I^{\Gamma^*, \leq} \rrbracket) \subseteq \dots \subseteq aX^{\gamma_m}(D + \llbracket I^{\Gamma^*, \leq} \rrbracket) \subseteq aX^{\gamma_{m+1}}(D + \llbracket I^{\Gamma^*, \leq} \rrbracket) \subseteq \dots$$

Since $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ is Noetherian, there exists an integer $N \geq 1$ such that, for every $k \geq N$,

$$aX^{\gamma_k}(D + \llbracket I^{\Gamma^*, \leq} \rrbracket) = aX^{\gamma_N}(D + \llbracket I^{\Gamma^*, \leq} \rrbracket).$$

Therefore, $aX^{\gamma_{N+1}} = aX^{\gamma_N} * f$, for some $f \in D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ and thus $\gamma_{N+1} = \gamma_N + \delta$, for some $\delta \in \sup p(f)$. Since $\gamma_N = \alpha_{i_{N+1}} + \gamma_{N+1}$ and Γ is cancellative, $\alpha_{i_{N+1}} + \delta = 0$. Since $(\Gamma, \leq)$ is a positive strictly ordered monoid, $\alpha_{i_{N+1}} = 0$, which contradicts the fact that $\alpha_i \in \Gamma^*$. This proves the claim. By the claim, we obtain a strictly infinite chain $I_1 \subsetneq I_2 \subsetneq \dots$ of ideals of $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$, which is a contradiction to the fact that $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ is Noetherian.

($\Leftarrow$) Let D be a Noetherian ring, $I^2 = I$, and Γ be finitely generated. Then I is principal and is generated by an idempotent element a [10, Lemma 1]. Then $D \cong I \oplus J$, where $J = \{d - ad \mid d \in D\}$ is an ideal of D . Since D is a Noetherian ring, $I \cong D/J$ and $J \cong D/I$ are Noetherian rings. Since Γ is finitely generated, $\llbracket I^{\Gamma, \leq} \rrbracket$ is Noetherian [2, 5.5]. Note that $D + \llbracket I^{\Gamma^*, \leq} \rrbracket = J \oplus \llbracket I^{\Gamma, \leq} \rrbracket$. Thus, $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring. $\square$

Remark 3.2. Note that an integral domain has only two idempotent elements 0 and 1; so if $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ is a Noetherian domain, then we have either $I = (0)$ or $I = D$ by the proof of Theorem 3.1. Thus, if I is a nonzero proper ideal of an integral domain D , then $D + \llbracket I^{\Gamma^*, \leq} \rrbracket$ is never a Noetherian domain.

By applying Theorem 3.1 to the case when $(\Gamma = \mathbb{N}^n, \leq)$ with the lexicographic order, we recover.

Corollary 3.3. [5, Proposition 2.4] *Let D be a commutative ring with identity and I a nonzero proper ideal of D . Then D is a Noetherian ring and $I = I^2$ if and only if $D + (X_1, \dots, X_n)I[X_1, \dots, X_n]$ is a Noetherian ring.*

Recall that Γ is a *numerical semigroup* if Γ is a subsemigroup of $\mathbb{N}$ containing 0 such that it generates $\mathbb{Z}$ as a group. It is well known that a numerical semigroup is finitely generated. We are closing this paper with some applications of composite generalized power series rings.

Example 3.4.

- (1) Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}[i]$ (resp., $\mathbb{Z}[\omega]$) the ring of Gaussian integers (resp., Eisenstein integers). Then $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ are finitely generated as $\mathbb{Z}$ -modules; so if Γ is a numerical semigroup, then by Theorem 2.10, both $\mathbb{Z} + \llbracket \mathbb{Z}[i]^{\Gamma^*, \leq} \rrbracket$ and $\mathbb{Z} + \llbracket \mathbb{Z}[\omega]^{\Gamma^*, \leq} \rrbracket$ are Noetherian rings. Moreover, if $\Gamma = \mathbb{N} \times \dots \times \mathbb{N}$ (n times), then by Theorem 2.10 (or Corollary 2.12), $\mathbb{Z} + (X_1, \dots, X_n)\mathbb{Z}[i][X_1, \dots, X_n]$ and $\mathbb{Z} + (X_1, \dots, X_n)\mathbb{Z}[\omega][X_1, \dots, X_n]$ are Noetherian rings.
- (2) Let $\mathbb{R}$ (resp., $\mathbb{C}$) be the field of real numbers (resp., complex numbers). Then $\mathbb{C}$ is a finitely generated $\mathbb{R}$ -module; so if Γ is a numerical semigroup, then by Theorem 2.10, $\mathbb{R} + \llbracket \mathbb{C}^{\Gamma^*, \leq} \rrbracket$ is a Noetherian ring. Furthermore, if $\Gamma = \mathbb{N} \times \dots \times \mathbb{N}$ (n times), then by Theorem 2.10 (or Corollary 2.12), $\mathbb{R} + (X_1, \dots, X_n)\mathbb{C}[X_1, \dots, X_n]$ is a Noetherian ring.
- (3) Let R be any Noetherian ring and let n be an integer ≥ 2 . Let

$$\mathcal{V}_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{pmatrix} \mid a_1, \dots, a_n \in R \right\} \text{ and } \mathcal{I}_n(R) = \left\{ \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix} \mid a \in R \right\}.$$

Since $\mathcal{I}_n(R)$ is isomorphic to R , $\mathcal{I}_n(R)$ is a Noetherian ring. Also, $\mathcal{V}_n(R)$ is finitely generated as an $\mathcal{I}_n(R)$ -module. Thus, by Theorem 2.10 (or Corollary 2.12), $\mathcal{I}_n(R) + (X_1, \dots, X_n)\mathcal{V}_n(R)[X_1, \dots, X_n]$ is a Noetherian ring.

- (4) Let R , n , and $\mathcal{V}_n(R)$ be as in (3). Note that $R[Y]/(Y^n)$ is isomorphic to $\mathcal{V}_n(R)$ [11, Section 1]; so by (3), $R + (X_1, \dots, X_n)(R[Y]/(Y^n))[X_1, \dots, X_n]$ is a Noetherian ring.
- (5) Let $\mathbb{Q}$ be the field of rational numbers. Then $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$ -module; so by Theorem 2.1 (or Corollary 2.12), $\mathbb{Z} + X\mathbb{Q}[X]$ is not a Noetherian ring. More precisely, we have a strictly ascending chain of ideals in $\mathbb{Z} + X\mathbb{Q}[X]$:

$$\left(\frac{1}{2}X\right) \subsetneq \left(\frac{1}{4}X\right) \subsetneq \left(\frac{1}{8}X\right) \subsetneq \dots.$$

- (6) Let $D = \mathbb{Z}[Y]/(Y^2 - Y)$ and I the ideal of D generated by $\overline{1 - Y}$. Then D is a Noetherian ring and I is an idempotent ideal of D ; so if Γ is a numerical semigroup, then by Theorem 3.1, $D + [I^{\Gamma^*, \leq}]$ is a Noetherian ring. Moreover, if $\Gamma = \mathbb{N} \times \dots \times \mathbb{N}$ (n times), then by Theorem 3.1 (or Corollary 3.3), $D + (X_1, \dots, X_n)I[X_1, \dots, X_n]$ is a Noetherian ring.
- (7) Let n be an integer ≥ 2 . Then $n\mathbb{Z}$ is not idempotent; so by Theorem 3.1, $\mathbb{Z} + \llbracket (n\mathbb{Z})^{\Gamma^*, \leq} \rrbracket$ is not a Noetherian ring for any strictly totally ordered monoid Γ with $0 \leq \alpha$ for all $\alpha \in \Gamma$. In fact, for $\alpha \in \Gamma^*$, we have a strictly ascending chain of ideals in $\mathbb{Z} + \llbracket (n\mathbb{Z})^{\Gamma^*, \leq} \rrbracket$:

$$(nX^\alpha) \subsetneq (nX^\alpha, nX^{2\alpha}) \subsetneq (nX^\alpha, nX^{2\alpha}, nX^{3\alpha}) \subsetneq \dots.$$

In particular, $\mathbb{Z} + X(n\mathbb{Z})[X]$ is not a Noetherian ring (by Corollary 3.3).

- (8) Let D be a Noetherian ring, E a ring extension of D , I an idempotent ideal of D generated by a , and $\mathbb{Q}_0$ the monoid of nonnegative rational numbers. Then neither $D + \llbracket E^{\mathbb{Q}_0, \leq} \rrbracket$ nor $D + \llbracket I^{\mathbb{Q}_0, \leq} \rrbracket$ is a Noetherian ring. More precisely, we have strictly ascending chains of ideals of $D + \llbracket E^{\mathbb{Q}_0, \leq} \rrbracket$ and $D + \llbracket I^{\mathbb{Q}_0, \leq} \rrbracket$, respectively:

$$\begin{cases} (X^{\frac{1}{2}}) \subsetneq (X^{\frac{1}{4}}) \subsetneq (X^{\frac{1}{8}}) \subsetneq \cdots, \\ (aX^{\frac{1}{2}}) \subsetneq (aX^{\frac{1}{4}}) \subsetneq (aX^{\frac{1}{8}}) \subsetneq \cdots. \end{cases}$$

Also, if $\Gamma = \bigoplus_{n=1}^{\infty} \mathbb{N}$ is the weak direct product of $\mathbb{N}$, then neither $D + \llbracket E^{\Gamma, \leq} \rrbracket$ nor $D + \llbracket I^{\Gamma, \leq} \rrbracket$ is a Noetherian ring, i.e., if $\mathbf{X} = \{X_n \mid n \in \mathbb{N}\}$, then neither $D + \mathbf{X}E[\mathbf{X}]$ nor $D + \mathbf{X}I[\mathbf{X}]$ is a Noetherian ring.

Acknowledgments: The authors would like to thank the referees for their several valuable comments and suggestions. D. Y. Oh was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2018R1D1A1B07041083).

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