Research Article

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Blow-up analyses in nonlocal reaction diffusion equations with time-dependent coefficients under Neumann boundary conditions

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Abstract: In this paper, the blow-up analyses in nonlocal reaction diffusion equations with time-dependent coefficients are investigated under Neumann boundary conditions. By constructing some suitable auxiliary functions and using differential inequality techniques, we show some sufficient conditions to ensure that the solution $u(x, t)$ blows up at a finite time under appropriate measure sense. Furthermore, an upper and a lower bound on blow-up time are derived under some appropriate assumptions. At last, two examples are presented to illustrate the application of our main results.

Keywords: reaction diffusion equations, blow up, lower and upper bounds

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1 Introduction

This paper is concerned with the following nonlocal reaction diffusion equations with time-dependent coefficients:

$$\begin{cases} (g(u))_t = \nabla \cdot (\rho(|\nabla u|^p) |\nabla u|^{p-2} \nabla u) + k(t)f(x, u), & (x, t) \in \Omega \times (0, t^*), \\ \frac{\partial u}{\partial n} = 0, & (x, t) \in \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) \geq 0, & x \in \bar{\Omega}, \end{cases} \quad (1)$$

where $p \geq 2$ and $\Omega \subset \mathbb{R}^N$ ($N > 2$) is a bounded convex domain with smooth boundary $\partial\Omega$. $\frac{\partial u}{\partial n}$ represents the outward normal derivative on $\partial\Omega$, $u_0(x)$ is the initial value, t^* is the maximal existence time of u , and $\bar{\Omega}$ is the closure of Ω . Set $R^+ = (0, +\infty)$, g is a $C^2(\bar{R}^+)$ function with $g'(s) > 0$ for all $s \geq 0$, ρ is a positive $C^1(\bar{R}^+)$ function, k is a positive $C^1(\bar{R}^+)$ function, and u_0 is a nonnegative $C^1(\bar{\Omega})$ function. The nonlinearity f is a non-negative $C(\bar{R}^+)$ function and satisfies the following nonlocal conditions:

$$f(x, s(x, t)) \leq c_1 s(x, t)^{r_1} + c_2 s(x, t)^{r_2} \left(\int_{\Omega} s(x, t)^{\delta+1} dx \right)^l, \quad s(x, t) \geq 0, \quad x \in \bar{\Omega}, \quad (2)$$

where c_1, r_1 are nonnegative constants and c_2, r_2, δ, l are positive constants.

Many physical phenomena and biological species theories have been formulated as reaction diffusion equations, see [1–3]. There have been a lot of interesting results about reaction diffusion equations, such as existence of global solution, blow-up solution, estimates of the bounds for the blow-up time, blow-up rate, blow-up set, and asymptotic behavior of the solutions (see [4–9]). For example, Philippin and Proytcheva in

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[8] dealt with a class of semilinear parabolic problems. They established sufficient conditions on the data forcing the solution to blow up at finite time and derived an upper bound for the blow-up time.

In recent years, the blow-up and global solutions for local reaction diffusion equations have been discussed by many authors (see [10–13]). Some special cases of (1) have been studied, see [12–14]. For example, Payne and Philippin in [12] considered an initial boundary value problem for the semilinear parabolic equation with time-dependent coefficients and inner source terms:

$$\begin{cases} u_t = \Delta u + k(t)f(u), & (x, t) \in \Omega \times (0, t^*), \\ \frac{\partial u}{\partial n} = 0, & (x, t) \in \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) \geq 0, & x \in \bar{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a sufficient smooth bounded domain. And under some conditions on $k(t)$ and $f(u)$, they obtained some sufficient conditions for the existence of global solution and an upper bound for the blow-up time. And they derived a lower bound on blow-up time if blow up occurs. Recently, Ding and Hu [13] studied the following nonlinear reaction diffusion equations with time-dependent coefficients under Neumann boundary conditions:

$$\begin{cases} (g(u))_t = \nabla \cdot (\rho(|\nabla u|^2) \nabla u) + k(t)f(u), & (x, t) \in \Omega \times (0, t^*), \\ \frac{\partial u}{\partial n} = 0, & (x, t) \in \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) \geq 0, & x \in \bar{\Omega}, \end{cases}$$

where Ω is a bounded convex domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. By means of a first-order differential inequality technique, the authors showed the conditions to ensure that the blow-up solution occurs. Moreover, a lower bound on blow-up time was derived when blow up occurs.

On the other hand, there are many papers about the nonlocal models. Zhang et al. [15] introduced the qualitative properties of the solutions to nonlocal reaction diffusion systems in detail. Evidently, the nonlocal models are applied more accurately to practical problems than the local models in a sense. However, they are more challenging and difficult. Some theorems and methods applying the local models do not work in the nonlocal models. Some authors have applied various methods to investigate the nonlocal problems (see [16–22]). For example, Liu and Fang [17] focused on the blow-up phenomena to the following equations with time-dependent coefficients:

$$u_t = \nabla \cdot (h(|\nabla u|^2) \nabla u) - k(t)f(u), \quad (x, t) \in \Omega \times (0, t^*),$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded star-shaped region domain with smooth boundary $\partial\Omega$. For the initial boundary value problem under nonlinear boundary conditions, using the auxiliary functions and modified differential inequality technique, they established some conditions on time-dependent coefficients and nonlinear data to ensure that the solution $u(x, t)$ exists globally and blows up at some finite time. Moreover, when blow up does occur, the upper and lower bounds for the blow-up time were obtained under suitable measure in high-dimensional spaces.

In [19], the authors dealt with the blow-up phenomena of the following quasilinear reaction diffusion equations with weighted nonlocal source under Robin boundary conditions:

$$(g(u))_t = \nabla \cdot (\rho(|\nabla u|^2) \nabla u) + a(x)f(u), \quad (x, t) \in \Omega \times (0, t^*),$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded convex domain, weighted nonlocal source satisfies $a(x)f(u(x, t)) \leq a_1 + a_2(u(x, t))^p \int_{\Omega} (u(x, t))^l dx^m$, and a_1, a_2, p, l , and m are positive constants. By utilizing a differential inequality technique and maximum principles, they established conditions to guarantee that the solution remains global or blows up in a finite time. Furthermore, an upper and a lower bound for blow-up time were derived.

Inspired by the aforementioned studies, we investigate the more complicated and general case than the ones in the aforementioned papers. Our paper's objective is not only to obtain the blow-up solution of (1)

but also to derive the bounds of the blow-up time when the blow up occurs in finite time. Compared with (1), Ding and Hu [13] considered the local model, Liu and Fang [17] and Ding and Shen [19] considered the models with various boundaries. Hence, the auxiliary functions and some techniques in [13,17,19] are no longer applicable to (1). Therefore, we need to construct new and appropriate auxiliary functions to achieve our purpose. We use differential inequalities to prove that the solution $u(x, t)$ blows up at some finite time. And an upper bound of the blow-up time is obtained. Moreover, when blow up occurs, a lower bound for the blow-up time is shown. And our main results are extended for [13,17,19] in a way.

The rest of the paper is constructed as follows. In Section 2, we prove that the solution $u(x, t)$ blows up at some finite time and give an upper bound for the blow-up time. In Section 3, we investigate a lower bound for the blow-up time when blow up occurs. Section 4 mainly presents two examples to illustrate our main results.

2 Blow-up solution and an upper bound for blow-up time

In this section, we introduce the following auxiliary functions:

$$F(x, s) := \int_0^s f(x, y) dy, \quad G(s) := 2p \int_0^s yg'(y) dy, \quad P(s) := \int_0^s \rho(y) dy, \quad s \geq 0, \quad (3)$$

$$\Phi(t) := \int_{\Omega} G(u(x, t)) dx, \quad \Psi(t) := -p \int_{\Omega} P(|\nabla u|^p) dx + pk(t) \int_{\Omega} F(x, u) dx, \quad t \geq 0. \quad (4)$$

Now, we give the main result Theorem 1.

Theorem 1. *Let u be a nonnegative classical solution of problem (1). Assume that ρ, g, f, k, F, P satisfy the following conditions:*

$$\lim_{s \rightarrow 0^+} s^2 g'(s) = 0. \quad (5)$$

$$k(0) > 0, \quad k'(t) \geq 0, \quad t \geq 0, \quad g'(s) > 0, \quad g''(s) \leq 0, \quad s \geq 0. \quad (6)$$

$$0 < sp(s) \leq (1 + \alpha)P(s), \quad sf(x, s) \geq (1 + \beta)F(x, s), \quad s \geq 0. \quad (7)$$

Here α, β are positive constants and they satisfy $\beta \geq \alpha > 1$.

$$\Psi(0) := -p \int_{\Omega} P(|\nabla u_0|^p) dx + pk(0) \int_{\Omega} F(x, u_0) dx > 0, \quad (8)$$

where u_0 is the initial value. Then when $p \geq 2$, u blows up at some finite time $t^* < T^*$ in the measure $\Phi(t)$ and

$$T^* = \frac{\Phi(0)}{(1 + \alpha)(\alpha - 1)\Psi(0)}.$$

Proof. From (3)–(4) and the divergence theorem, we get

$$\begin{aligned} \Phi'(t) &= 2p \int_{\Omega} u(\nabla \cdot (\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + k(t)f(x, u)) dx \\ &= 2p \int_{\Omega} \nabla \cdot (u\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) dx - 2p \int_{\Omega} \rho(|\nabla u|^p)|\nabla u|^p dx + 2p \int_{\Omega} uk(t)f(x, u) dx \\ &= 2p \int_{\Omega} u\rho(|\nabla u|^p)|\nabla u|^{p-2} \frac{\partial u}{\partial n} dS - 2p \int_{\Omega} \rho(|\nabla u|^p)|\nabla u|^p dx + 2p \int_{\Omega} uk(t)f(x, u) dx. \end{aligned}$$

The Neumann boundary conditions in (1) and (7) imply

$$\Phi'(t) \geq -2(1 + \alpha)p \int_{\Omega} P(|\nabla u|^p) dx + 2p(1 + \beta)k(t) \int_{\Omega} F(x, u) dx \geq 2(1 + \alpha)\Psi(t). \quad (9)$$

On the other hand, in virtue of the divergence theorem and (6), we have

$$\begin{aligned} \Psi'(t) &= -p^2 \int_{\Omega} \rho(|\nabla u|^p) |\nabla u|^{p-2} (\nabla u \cdot \nabla u_t) dx + pk'(t) \int_{\Omega} F(x, u) dx + pk(t) \int_{\Omega} f(x, u) u_t dx \\ &\geq -p^2 \int_{\Omega} \rho(|\nabla u|^p) |\nabla u|^{p-2} (\nabla u \cdot \nabla u_t) dx + pk(t) \int_{\Omega} f(x, u) u_t dx \\ &= p^2 \int_{\Omega} \nabla \cdot (\rho(|\nabla u|^p) |\nabla u|^{p-2} \nabla u) u_t dx + pk(t) \int_{\Omega} f(x, u) u_t dx \\ &\geq p \int_{\Omega} \nabla \cdot (\rho(|\nabla u|^p) |\nabla u|^{p-2} \nabla u) u_t dx + pk(t) \int_{\Omega} f(x, u) u_t dx \\ &= p \int_{\Omega} g'(u) u_t^2 dx \geq 0. \end{aligned} \quad (10)$$

Consequently, $\Psi(t)$ is a nondecreasing function in t . It follows from (8) that

$$\Psi(t) \geq \Psi(0) > 0, \quad t \in (0, T).$$

Therefore, by (9), we obtain

$$\Phi'(t) \geq 0. \quad (11)$$

Then (9) and (11) yield

$$2(1 + \alpha)\Psi(t)\Phi'(t) \leq (\Phi'(t))^2 = \left(2p \int_{\Omega} ug'(u) u_t dx \right)^2 \leq 4p^2 \int_{\Omega} g'(u) u^2 dx \int_{\Omega} g'(u) u_t^2 dx, \quad (12)$$

where the Schwarz inequality is used. Integrating by parts and (6), we obtain

$$G(u) = 2p \int_0^u yg'(y) dy = p \int_0^u g'(y) dy^2 = pg'(u)u^2 - p \int_0^u y^2 g''(y) dy \geq pg'(u)u^2. \quad (13)$$

Combining (12) and (13), we obtain

$$(1 + \alpha)\Psi(t)\Phi'(t) \leq 2p \int_{\Omega} G(u) dx \int_{\Omega} g'(u) u_t^2 dx \leq 2\Phi(t)\Psi'(t).$$

Namely,

$$\frac{\Psi'(t)}{\Psi(t)} \geq \frac{(1 + \alpha)\Phi'(t)}{2\Phi(t)}. \quad (14)$$

Integrating (14) from 0 to t and making use of (8), we obtain

$$\Psi(t) \geq \Psi(0)\Phi(0)^{-\frac{1+\alpha}{2}}\Phi(t)^{\frac{1+\alpha}{2}}. \quad (15)$$

Again using (9), we obtain

$$\Phi'(t) \geq 2(1 + \alpha)\Psi(t) \geq 2(1 + \alpha)\Psi(0)\Phi(0)^{-\frac{1+\alpha}{2}}\Phi(t)^{\frac{1+\alpha}{2}}.$$

Hence,

$$\Phi'(t)\Phi(t)^{-\frac{1+\alpha}{2}} \geq 2(1 + \alpha)\Psi(0)\Phi(0)^{-\frac{1+\alpha}{2}}. \quad (16)$$

An integration of (16) from 0 to t implies

$$\Phi(t)^{-\frac{\alpha-1}{2}} \leq -(\alpha-1)(\alpha+1)\Psi(0)\Phi(0)^{-\frac{1+\alpha}{2}}t + \Phi(0)^{-\frac{\alpha-1}{2}}. \quad (17)$$

From (17), we can obtain that the solution u of (1) blows up at some finite time $t^* < T^*$ in the measure $\Phi(t)$ and

$$t^* < T^* = \frac{\Phi(0)}{(\alpha-1)(\alpha+1)\Psi(0)}. \quad \square$$

3 A lower bound for blow-up time

In this section, we define $\Omega \subset R^3$ and show a lower bound for the blow-up time t^* . Now we define the new auxiliary functions:

$$A(t) = \int_{\Omega} B(u) dx, \quad t \geq 0, \quad B(s) = (\delta+1) \int_0^s g'(y)y^\delta dy, \quad s \geq 0, \quad (18)$$

where δ is a positive constant.

Theorem 2. Let u be a nonnegative classical solution of problem (1). Assume that function f satisfies condition (2) and the following assumptions hold:

$$sp(s) \geq b_1 s^{2q} + b_2, \quad s \geq 0, \quad (19)$$

$$k(t) \leq \eta, \quad t \geq 0, \quad (20)$$

$$g'(s) \geq m_0, \quad s \geq 0, \quad (21)$$

where q, b_1, b_2, m_0, η are positive constants. And they satisfy

$$r_1 < 1, \quad pq > 1, \quad 0 < r_2 < \frac{\delta+3}{2}.$$

u becomes unbounded in the measure $A(t)$ at some finite time t^* . Then the blow-up time t^* is bounded from below, as follows:

$$t^* \geq \int_{A_0}^{\infty} \frac{d\tau}{C_1 \tau^{\frac{\delta+r_1}{\delta+1}} + C_2 \tau^{l+\frac{\delta+r_2}{\delta+1}} + C_3 \tau^{\frac{4pq-3}{4pq-3} + \frac{2pq(2\delta+2r_2-3)-3\delta+3}{(4pq-3)(\delta+1)}}}.$$

Here

$$C_1 = \eta c_1 (\delta+1) m_0^{-\frac{\delta+r_1}{\delta+1}} |\Omega|^{\frac{1-r_1}{\delta+1}}, \quad C_2 = \frac{3\sqrt{3}}{2} \eta c_2 (\delta+1) \left(\frac{1}{\rho_0} \right)^{\frac{3}{2}} m_0^{-\frac{\delta+r_2}{\delta+1}} |\Omega|^{\frac{\delta+3-2r_2}{2(\delta+1)}}, \quad (22)$$

$$C_3 = \frac{\sqrt{6} \eta c_2 (4pq-3)}{36pq} (\delta+1) (\delta+r_2)^{\frac{3}{2}} \left(1 + \frac{d}{\rho_0} \right)^{\frac{3}{2}} \varepsilon^{-\frac{3}{4pq-3}} m_0^{-l - \frac{2pq(2\delta+2r_2-3)-3\delta+3}{4pq(\delta+1)}} |\Omega|^{\frac{pq(\delta-2r_2+6)-3}{2pq(\delta+1)}}, \quad (23)$$

where

$$\varepsilon = \frac{2\sqrt{6} \delta b_1 pq}{\eta c_2} (\delta+r_2)^{-\frac{3}{2}} \left(1 + \frac{d}{\rho_0} \right)^{-\frac{3}{2}} m_0^{l + \frac{2pq(2\delta+2r_2-3)-3\delta+3}{4pq(\delta+1)}} |\Omega|^{-\frac{pq(\delta-2r_2+6)-3}{2pq(\delta+1)}}, \quad (24)$$

and $|\Omega|$ is the volume of Ω , $\rho_0 = \min_{\partial\Omega} (x \cdot n)$, $d = \max_{\bar{\Omega}} |x|$.

Proof. By the divergence theorem and the Neumann boundary conditions, we obtain

$$\begin{aligned}
 A'(t) &= (\delta + 1) \int_{\Omega} g'(u) u^{\delta} u_t \, dx \\
 &= (\delta + 1) \int_{\Omega} u^{\delta} (\nabla \cdot (\rho(|\nabla u|^p) |\nabla u|^{p-2} \nabla u) + k(t) f(x, u)) \, dx \\
 &= (\delta + 1) \int_{\Omega} \nabla \cdot (u^{\delta} \rho(|\nabla u|^p) |\nabla u|^{p-2} \nabla u) \, dx \\
 &\quad - \delta(\delta + 1) \int_{\Omega} u^{\delta-1} \rho(|\nabla u|^p) |\nabla u|^p \, dx + (\delta + 1) k(t) \int_{\Omega} u^{\delta} f(x, u) \, dx \\
 &= -\delta(\delta + 1) \int_{\Omega} u^{\delta-1} \rho(|\nabla u|^p) |\nabla u|^p \, dx + (\delta + 1) k(t) \int_{\Omega} u^{\delta} f(x, u) \, dx.
 \end{aligned}$$

Then we use conditions (19), (20), and (2) to have

$$\begin{aligned}
 A'(t) &\leq -\delta(\delta + 1) b_1 \int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} \, dx - \delta(\delta + 1) b_2 \int_{\Omega} u^{\delta-1} \, dx + (\delta + 1) c_1 k(t) \int_{\Omega} u^{\delta+r_1} \, dx \\
 &\quad + c_2 (\delta + 1) k(t) \int_{\Omega} u^{\delta+r_2} \, dx \cdot \left(\int_{\Omega} u^{\delta+1} \, dx \right)^l \\
 &\leq -\delta(\delta + 1) b_1 \int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} \, dx + \eta(\delta + 1) c_1 \int_{\Omega} u^{\delta+r_1} \, dx + \eta(\delta + 1) c_2 \int_{\Omega} u^{\delta+r_2} \, dx \left(\int_{\Omega} u^{\delta+1} \, dx \right)^l.
 \end{aligned} \tag{25}$$

The Hölder inequality implies

$$\int_{\Omega} u^{\delta+r_1} \, dx \leq \left(\int_{\Omega} u^{\delta+1} \, dx \right)^{\frac{\delta+r_1}{\delta+1}} |\Omega|^{\frac{\delta+r_1}{\delta+1}}. \tag{26}$$

Replacing (26) in (25), we achieve

$$\begin{aligned}
 A'(t) &\leq -\delta(\delta + 1) b_1 \int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} \, dx + \eta(\delta + 1) c_1 \left(\int_{\Omega} u^{\delta+1} \, dx \right)^{\frac{\delta+r_1}{\delta+1}} |\Omega|^{\frac{1-r_1}{\delta+1}} \\
 &\quad + \eta(\delta + 1) c_2 \int_{\Omega} u^{\delta+r_2} \, dx \left(\int_{\Omega} u^{\delta+1} \, dx \right)^l.
 \end{aligned} \tag{27}$$

In the sequel, we estimate the last term in the right-hand side of (27). Using Lemma A.2 in [22], we have

$$\int_{\Omega} u^{\delta+r_2} \, dx \leq \left\{ \frac{3}{2\rho_0} \int_{\Omega} u^{\frac{2}{3}(\delta+r_2)} \, dx + \frac{\delta+r_2}{3} \left(1 + \frac{d}{\rho_0} \right) \int_{\Omega} u^{\frac{2}{3}(\delta+r_2)-1} |\nabla u| \, dx \right\}^{\frac{3}{2}}. \tag{28}$$

And we apply the inequality $(a + b)^{\frac{3}{2}} \leq \sqrt{2} (a^{\frac{3}{2}} + b^{\frac{3}{2}})$ to have

$$\int_{\Omega} u^{\delta+r_2} \, dx \leq \frac{3\sqrt{3}}{2} \left(\frac{1}{\rho_0} \right)^{\frac{3}{2}} \left(\int_{\Omega} u^{\frac{2}{3}(\delta+r_2)} \, dx \right)^{\frac{3}{2}} + \frac{\sqrt{6}}{9} (\delta + r_2)^{\frac{3}{2}} \left(1 + \frac{d}{\rho_0} \right)^{\frac{3}{2}} \left(\int_{\Omega} u^{\frac{2}{3}(\delta+r_2)-1} |\nabla u| \, dx \right)^{\frac{3}{2}}. \tag{29}$$

Thanks to the Hölder inequality, we obtain

$$\int_{\Omega} u^{\frac{2}{3}(\delta+r_2)} dx \leq \left(\int_{\Omega} u^{\delta+1} dx \right)^{\frac{2(\delta+r_2)}{3(\delta+1)}} |\Omega|^{\frac{\delta+3-2r_2}{3(\delta+1)}}. \quad (30)$$

And

$$\begin{aligned} \int_{\Omega} u^{\frac{2}{3}(\delta+r_2)-1} |\nabla u| dx &\leq \left(\int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} dx \right)^{\frac{1}{2pq}} \left(\int_{\Omega} u^{\frac{2pq(2\delta+2r_2-3)-3\delta+3}{3(2pq-1)}} dx \right)^{\frac{2pq-1}{2pq}} \\ &\leq \left(\int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} dx \right)^{\frac{1}{2pq}} \left(\int_{\Omega} u^{\delta+1} dx \right)^{\frac{2pq(2\delta+2r_2-3)-3\delta+3}{6pq(\delta+1)}} \cdot |\Omega|^{\frac{pq(\delta-2r_2+6)-3}{3pq(\delta+1)}}. \end{aligned} \quad (31)$$

By (28)–(31), (27) can be rewritten as:

$$\begin{aligned} A'(t) &\leq -\delta(\delta+1)b_1 \int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} dx + \eta(\delta+1)c_1 \left(\int_{\Omega} u^{\delta+1} dx \right)^{\frac{\delta+r_1}{\delta+1}} |\Omega|^{\frac{1-r_1}{\delta+1}} + \frac{3\sqrt{3}}{2} \eta(\delta+1)c_2 \left(\frac{1}{\rho_0} \right)^{\frac{3}{2}} \\ &\quad \cdot \left(\int_{\Omega} u^{\delta+1} dx \right)^{\frac{\delta+r_2}{\delta+1}+l} |\Omega|^{\frac{\delta+3-2r_2}{2(\delta+1)}} + \frac{\sqrt{6}}{9} \eta c_2(\delta+1)(\delta+r_2)^{\frac{3}{2}} \left(1 + \frac{d}{\rho_0} \right)^{\frac{3}{2}} \left(\int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} dx \right)^{\frac{3}{4pq}} \\ &\quad \cdot \left(\int_{\Omega} u^{\delta+1} dx \right)^{l+\frac{2pq(2\delta+2r_2-3)-3\delta+3}{4pq(\delta+1)}} |\Omega|^{\frac{pq(\delta-2r_2+6)-3}{2pq(\delta+1)}}. \end{aligned} \quad (32)$$

Since $g'(s) \geq m_0$, $s \geq 0$, we have

$$B(u) \geq m_0(\delta+1) \int_0^u y^{\delta} dy = m_0 u^{\delta+1},$$

that is,

$$\int_{\Omega} u^{\delta+1} dx \leq \frac{1}{m_0} \int_{\Omega} B(u) dx = \frac{1}{m_0} A(t). \quad (33)$$

Then

$$\begin{aligned} A'(t) &\leq -\delta(\delta+1)b_1 \int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} dx + \eta c_1(\delta+1) \left(\frac{1}{m_0} A(t) \right)^{\frac{\delta+r_1}{\delta+1}} |\Omega|^{\frac{1-r_1}{\delta+1}} + \frac{3\sqrt{3}}{2} \eta c_2(\delta+1) \left(\frac{1}{\rho_0} \right)^{\frac{3}{2}} \\ &\quad \cdot \left(\frac{1}{m_0} A(t) \right)^{\frac{\delta+r_2}{\delta+1}+l} |\Omega|^{\frac{\delta+3-2r_2}{2(\delta+1)}} + \frac{\sqrt{6}}{9} \eta c_2(\delta+1)(\delta+r_2)^{\frac{3}{2}} \left(1 + \frac{d}{\rho_0} \right)^{\frac{3}{2}} \left(\int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} dx \int_{\Omega} \right)^{\frac{3}{4pq}} \\ &\quad \cdot \left(\frac{1}{m_0} A(t) \right)^{l+\frac{2pq(2\delta+2r_2-3)-3\delta+3}{4pq(\delta+1)}} |\Omega|^{\frac{pq(\delta-2r_2+6)-3}{2pq(\delta+1)}}. \end{aligned} \quad (34)$$

The Young inequality implies

$$\begin{aligned}
& \left(\int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} dx \right)^{\frac{3}{4pq}} A(t)^{l + \frac{2pq(2\delta+2r_2-3)-3\delta+3}{4pq(\delta+1)}} \\
&= \left(\varepsilon \int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} dx \right)^{\frac{3}{4pq}} \cdot \left(\frac{A(t)^{l + \frac{2pq(2\delta+2r_2-3)-3\delta+3}{4pq(\delta+1)}}}{\varepsilon^{\frac{3}{4pq-3}}} \right)^{\frac{4pq-3}{4pq}} \\
&\leq \frac{3\varepsilon}{4pq} \int_{\Omega} u^{\delta-1} |\nabla u|^{2pq} dx + \frac{4pq-3}{4pq} \varepsilon^{-\frac{3}{4pq-3}} \cdot A(t)^{\frac{4pql}{4pq-3} + \frac{2pq(2\delta+2r_2-3)-3\delta+3}{(4pq-3)(\delta+1)}}.
\end{aligned} \tag{35}$$

Substituting (35) into (34) and the definition of ε , we get

$$\begin{aligned}
A'(t) &\leq \eta c_1(\delta+1) \left(\frac{1}{m_0} A(t) \right)^{\frac{\delta+r_1}{\delta+1}} |\Omega|^{\frac{1-r_1}{\delta+1}} + \frac{3\sqrt{3}\eta c_2(\delta+1)}{2} \left(\frac{1}{\rho_0} \right)^{\frac{3}{2}} \\
&\cdot \left(\frac{1}{m_0} A(t) \right)^{\frac{\delta+r_2}{\delta+1}+l} |\Omega|^{\frac{\delta+3-2r_2}{2(\delta+1)}} + \frac{\sqrt{6}\eta c_2}{9} (\delta+1)(\delta+r_2)^{\frac{3}{2}} \left(1 + \frac{d}{\rho_0} \right)^{\frac{3}{2}} \\
&\cdot \left(\frac{1}{m_0} \right)^{l + \frac{2pq(2\delta+2r_2-3)-3\delta+3}{4pq(\delta+1)}} |\Omega|^{\frac{pq(\delta-2r_2+6)-3}{2pq(\delta+1)}} \frac{4pq-3}{4pq} \varepsilon^{-\frac{3}{4pq-3}} \cdot A(t)^{\frac{4pql}{4pq-3} + \frac{2pq(2\delta+2r_2-3)-3\delta+3}{(4pq-3)(\delta+1)}}.
\end{aligned} \tag{36}$$

Integrating (36) from 0 to t , we have

$$t \geq \int_{A(0)}^{A(t)} \frac{d\tau}{C_1 \tau^{\frac{\delta+r_1}{\delta+1}} + C_2 \tau^{l + \frac{\delta+r_2}{\delta+1}} + C_3 \tau^{\frac{4pql}{4pq-3} + \frac{2pq(2\delta+2r_2-3)-3\delta+3}{(4pq-3)(\delta+1)}}},$$

where C_1, C_2, C_3 are defined by (22)–(24). Thus, if u blows up in the measure $A(t)$ at some finite t^* , we pass to the limit as $t \rightarrow t^*$ to obtain

$$t^* \geq \int_{A(0)}^{+\infty} \frac{d\tau}{C_1 \tau^{\frac{\delta+r_1}{\delta+1}} + C_2 \tau^{l + \frac{\delta+r_2}{\delta+1}} + C_3 \tau^{\frac{4pql}{4pq-3} + \frac{2pq(2\delta+2r_2-3)-3\delta+3}{(4pq-3)(\delta+1)}}}.$$

The proof is complete. $\square$

4 Applications

As applications, two examples are presented to illustrate our main results.

Example 4.1. Let u be a nonnegative classical solution of the following equation:

$$\begin{cases} (3u + \ln(2+u))_t = \nabla \cdot (|\nabla u|^7 \nabla u) + (3 - e^{-t})(15 - |x|^2)u^2 \left(\int_{\Omega} u^3 dx \right)^{\frac{1}{4}}, & (x, t) \in \Omega \times (0, T), \\ \frac{\partial u}{\partial n} = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = 1 + (1 - |x|^2)^2, & x \in \bar{\Omega}, \end{cases}$$

where $\Omega = \{x = (x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 < 1\}$. Then we can obtain u will blow up at T . And it satisfies

$$1.6521 \times 10^{-3} \leq T < 0.161115.$$

Proof. Compared with (1), we have

$$g(u) = 3u + \ln(2 + u), \quad \rho(|\nabla u|^3) = |\nabla u|^6, \quad f(x, u) = (15 - |x|^2)u^2 \left(\int_{\Omega} u^3 dx \right)^{\frac{1}{4}},$$

$$k(t) = 3 - e^{-t}, \quad u_0(x) = 1 + (1 - |x|^2)^2, \quad p = 3.$$

First, we show the solution $u(x, t)$ blows up at a finite time under appropriate measure sense $\Phi(t)$. At the same time, an upper bound on blow-up time is obtained. According to (3)–(4), we have

$$F(x, u) = (15 - |x|^2) \int_0^u y^2 \left(\int_{\Omega} y^3 dx \right)^{\frac{1}{4}} dy,$$

$$G(u) = 6 \int_0^u yg'(y) dy = 6u + 9u^2 + 6 \ln 4 - 12 \ln(2 + u),$$

$$P(|\nabla u|^3) = \frac{1}{3} |\nabla u|^9,$$

$$\Phi(t) = \int_{\Omega} G(u) dx = \int_{\Omega} (6u + 9u^2 + 6 \ln 4 - 12 \ln(2 + u)) dx,$$

$$\Psi(t) = - \int_{\Omega} |\nabla u|^9 dx + 3(3 - e^{-t}) \int_{\Omega} F(x, u) dx.$$

Set $\beta = \alpha = 2$. Clearly, conditions (5)–(7) in Theorem 1 are satisfied. Now we verify assumption (8). We calculate that

$$\begin{aligned} \Phi(0) &= \int_{\Omega} G(u_0(x)) dx = \int_{\Omega} (6(1 + (1 - |x|^2)^2) + 9(1 + (1 - |x|^2)^2)^2 + 6 \ln 4 - 12 \ln(3 + (1 - |x|^2)^2)) dx \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 (6(1 + (1 - r^2)^2) + 9(1 + (1 - r^2)^2)^2 + 6 \ln 4 - 12 \ln(3 + (1 - r^2)^2)) r^2 \sin \varphi dr \\ &\approx 60.0509. \end{aligned}$$

Since $1 \leq u_0 \leq 2$, we get

$$\begin{aligned} F(x, u_0) &= (15 - |x|^2) \int_0^{u_0} y^2 \left(\int_{\Omega} y^3 dx \right)^{\frac{1}{4}} dy \geq (15 - |x|^2) \int_1^{u_0} y^2 \left(\int_{\Omega} dx \right)^{\frac{1}{4}} dy = \frac{1}{3} (15 - |x|^2) |\Omega|^{\frac{1}{4}} (u_0^3 - 1), \\ \Psi(0) &= - \int_{\Omega} |\nabla u_0|^9 dx + 6 \int_{\Omega} F(x, u_0) dx \\ &= - \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 4^9 (1 - r^2)^9 r^{11} \sin \varphi dr + 2 \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 (15 - r^2) |\Omega|^{\frac{1}{4}} \approx 136.654 > 0. \end{aligned}$$

Applying Theorem 2, we can obtain u will blow up at $t^* < T$ in the measure $\Phi(t)$. And

$$T < T^* = \frac{\Phi(0)}{(1 + \alpha)(\alpha - 1)\Psi(0)} \approx 0.161115,$$

which is an upper bound for the blow-up time.

In the following, we give a lower bound for the blow-up time. Set $b_1 = 1$, $b_2 = r_1 = c_1 = 0$, $r_2 = 2$, $\eta = m_0 = 3$, $c_2 = 15$, $q = \frac{3}{2}$, $l = \frac{1}{4}$. We can compute $\rho_0 = 1$, $d = 1$. From (22)–(24), it is easy to get $\varepsilon \approx 0.0604$, $C_1 = 0$, $C_2 \approx 78.2033$, $C_3 \approx 870.249$. Moreover, we obtain

$$B(s) = 3 \int_0^s \left(3 + \frac{1}{2+y} \right) y^2 dy = -6s + \frac{3s^2}{2} + 3s^3 - 3 \ln(16) + 12 \ln(2+s),$$

$$A(t) = \int_{\Omega} B(u) dx = \int_{\Omega} -6u + \frac{3u^2}{2} + 3u^3 - 3 \ln(16) + 12 \ln(2+u) dx.$$

Then

$$\begin{aligned} A(0) &= \int_{\Omega} -6u_0 + \frac{3u_0^2}{2} + 3u_0^3 - 3 \ln(16) + 12 \ln(2-u_0) dx \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 \left(-6(1 + (1-r^2)^2) + \frac{3(1 + (1-r^2)^2)^2}{2} + 3(1 + (1-r^2)^2)^3 \right. \\ &\quad \left. - 3 \ln(16) + 12 \ln(3 + (1-r^2)^2) \right) r^2 \sin \varphi dr \approx 9.0238. \end{aligned}$$

From Theorem 2, we can get that u becomes unbounded in the measure $A(t)$ at some finite time T . Then the blow-up time T is bounded from below as follows:

$$T \geq \int_{A(0)}^{\infty} \frac{d\tau}{78.2033\tau^{\frac{19}{12}} + 870.249\tau^{\frac{79}{30}}} \approx 1.6521 \times 10^{-3}.$$

Hence, we have

$$1.6521 \times 10^{-3} \leq T \leq 0.161115.$$

□

Example 4.2. Let u be a nonnegative classical solution of the following equation:

$$\begin{cases} (u + 2 \ln(1+u))_t = \nabla \cdot ((1 + |\nabla u|^8) \nabla u^2 \nabla u) + \left(2 - \frac{1}{1+t} \right) \left((4 - |x|^2) u^{\frac{1}{2}} + u^2 \int_{\Omega} u^3 dx \right), & (x, t) \in \Omega \times (0, T), \\ \frac{\partial u}{\partial n} = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = 2 + (4 - |x|^2)^2, & x \in \bar{\Omega}, \end{cases}$$

where $\Omega = \{x = (x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 < 4\}$. Then we can obtain u will blow up at T . And it satisfies

$$5.07653 \times 10^{-12} \leq T \leq 2.54846 \times 10^{-10}.$$

Proof. Compared with (1), we have

$$g(u) = u + 2 \ln(1+u), \quad \rho(|\nabla u|^4) = 1 + |\nabla u|^8, \quad f(x, u) = (4 - |x|^2) u^{\frac{1}{2}} + u^2 \int_{\Omega} u^3 dx,$$

$$k(t) = 2 - \frac{1}{1+t}, \quad u_0(x) = 2 + (4 - |x|^2)^2, \quad p = 4.$$

First, we present the solution $u(x, t)$ blows up at a finite time under appropriate measure sense $\Phi(t)$. At the same time, an upper bound on blow-up time is obtained. According to (3)–(4), we have

$$F(x, u) = \frac{2}{3} (4 - |x|^2) u^{\frac{3}{2}} + \int_0^u y^2 \left(\int_{\Omega} y^3 dx \right) dy,$$

$$G(u) = 8 \int_0^u yg'(y) dy = 4(4u + u^2 - 4 \ln(1 + u)),$$

$$P(|\nabla u|^4) = |\nabla u|^4 + \frac{1}{3}|\nabla u|^{12},$$

$$\Phi(t) = \int_{\Omega} G(u) dx = \int_{\Omega} 16u + 4u^2 - 16 \ln(1 + u) dx,$$

$$\Psi(t) = -4 \int_{\Omega} |\nabla u|^4 + \frac{1}{3}|\nabla u|^{12} dx + 4 \left(2 - \frac{1}{1+t} \right) \int_{\Omega} F(x, u) dx.$$

Set $\beta = \alpha = 2$. Clearly, conditions (5)–(7) in Theorem 1 are satisfied. Now we verify assumption (8). We calculate that

$$\begin{aligned} \Phi(0) &= \int_{\Omega} G(u_0(x)) dx = \int_{\Omega} 16(2 + (4 - |x|^2)^2) + 4(2 + (4 - |x|^2)^2)^2 - 16 \ln(3 + (4 - |x|^2)^2) dx \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^2 16(2 + (4 - r^2)^2) + 4(2 + (4 - r^2)^2)^2 - 16 \ln(3 + (4 - r^2)^2) r^2 \sin \varphi dr \approx 2.44862 \times 10^4. \end{aligned}$$

Since $2 \leq u_0 \leq 3$, we get

$$\begin{aligned} F(x, u_0) &\geq \frac{2}{3}(4 - |x|^2)u_0^{\frac{3}{2}} + \int_2^{u_0} y^2 \left(\int_{\Omega} y dx \right) dy \geq \frac{2}{3}(4 - |x|^2)u_0^{\frac{3}{2}} + \frac{8}{3}|\Omega|(u_0^3 - 8), \\ \Psi(0) &= -4 \int_{\Omega} |\nabla u_0|^4 + \frac{1}{3}|\nabla u_0|^{12} dx + 4 \int_{\Omega} F(x, u_0) dx \\ &\geq -4 \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^2 16^2(4 - r^2)^4 r^6 \sin \varphi dr + \frac{16^6}{3} \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^2 (4 - r^2)^{12} r^{14} \sin \varphi dr \\ &\quad + \frac{8}{3} \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^2 (4 - r^2)(2 + (4 - r^2)^2)^{\frac{3}{2}} r^2 \sin \varphi dr \\ &\quad + \frac{32}{3}|\Omega| \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^2 ((2 + (4 - r^2)^2)^3 - 8) r^2 \sin \varphi dr \approx 3.20275 \times 10^{13} > 0. \end{aligned}$$

Applying Theorem 2, we can obtain u will blow up at $t^* < T$ in the measure $\Phi(t)$. And

$$T < T^* = \frac{\Phi(0)}{(1 + \alpha)(\alpha - 1)\Psi(0)} \approx 2.54846 \times 10^{-10},$$

which is an upper bound for the blow-up time.

Now, we estimate a lower bound for the blow-up time. Set $b_1 = c_2 = l = m_0 = 1$, $b_2 = 0$, $r_1 = \frac{1}{2}$, $\delta = r_2 = \eta = 2$, $c_1 = 4$, $q = \frac{3}{2}$. Obviously, the assumptions in Theorem 2 are satisfied. And we can compute $\rho_0 = 2$, $d = 2$. From (22)–(24) in Theorem 2, it is easy to get $\varepsilon \approx 0.16747$, $C_1 \approx 43.0931$, $C_2 \approx 9.8958$, $C_3 \approx 228.911$. Moreover, we obtain

$$B(s) = -6s + s^2(3 + s) + 6 \ln(1 + s),$$

$$A(t) = \int_{\Omega} B(u) dx = \int_{\Omega} -6u + u^2(3 + u) + 6 \ln(1 + u) dx.$$

Then

$$\begin{aligned} A(0) &= \int_{\Omega} -6u_0 + u_0^2(3 + u_0) + 6 \ln(1 + u_0) dx \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^2 (-6(2 + (4 - r^2)^2) + (2 + (4 - r^2)^2)^2(5 + (4 - r^2)^2) \\ &\quad + 6 \ln(3 + (4 - r^2)^2)) r^2 \sin \varphi dr \approx 20742.4. \end{aligned}$$

According to Theorem 2, we can get that u becomes unbounded in the measure $A(t)$ at some finite time T . Then the blow-up time T is bounded from below as follows:

$$T \geq \int_{A(0)}^{\infty} \frac{d\tau}{43.0931\tau^{\frac{5}{6}} + 9.8958\tau^{\frac{7}{3}} + 228.911\tau^3} \approx 5.07653 \times 10^{-12}.$$

Therefore, we have

$$5.07653 \times 10^{-12} \leq T \leq 2.54846 \times 10^{-10}.$$

□

5 Conclusions

Nonlinear reaction diffusion model plays an important role in the fields of physics, chemistry, biology, and engineering, as its global and blow-up solutions always reflect the stability and instability of heat and mass transport process. In this paper, we study a class of blow-up analyses in nonlocal reaction diffusion equations with time-dependent coefficients under Neumann boundary conditions. By means of some suitable auxiliary functions and differential inequality techniques, we establish some sufficient conditions to ensure that the solution $u(x, t)$ blows up at a finite time under appropriate measure sense. And we estimate an upper and a lower bound on blow-up time under some appropriate assumptions. In a case, our results are extension of those in [13,17,19].

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