Research Article

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A new characterization of a proper type B semigroup

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Abstract: In this paper, we develop the elementary theory of inverse semigroups to the cases of type B semigroups. The main aim of this paper is to study proper type B semigroups. We introduce first the concept of a left admissible triple. After obtaining some basic properties of left admissible triple, we give the definition of a Q -semigroup and get a structure theorem of Q -semigroup. In particular, we introduce the notion of an admissible triple and give some characterization of proper type B semigroups. It is proved that an arbitrary Q -semigroup with an admissible triple is an E -unitary type B semigroup.

Keywords: type B semigroup, Q -semigroup, proper, E -unitary

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1 Introduction

Let S be a semigroup and denote the set of idempotents of S by $E(S)$. As in [1], the relations $\mathcal{R}^*$ and $\mathcal{L}^*$ are generalizations of Green's relations $\mathcal{R}$ and $\mathcal{L}$, respectively. We define $a\mathcal{R}^*b$ for the elements a, b of a semigroup S if and only if a, b are related by Green's relation $\mathcal{R}$ in T containing S as a subsemigroup. $\mathcal{L}^*$ is defined dually. In the study by Fountain [2], a semigroup S is *rpp* (resp., *lpp*) if each $\mathcal{L}^*$ -class (resp., $\mathcal{R}^*$ -class) of S contains an idempotent. A semigroup S is said to be *abundant* if it is both rpp and lpp. Fountain and others investigated some classes of abundant semigroups and got many interesting results (see [3–11]). From [2], a rpp semigroup S is said to be *right adequate* if $E(S)$ is a semilattice (i.e., any two elements of $E(S)$ commute). A *left adequate semigroup* is defined dually. A semigroup is *adequate* if it is both left and right adequate. Usually, we denote by a^* (resp., a^+) an idempotent $\mathcal{L}^*$ - (resp., $\mathcal{R}^*$ -) related to a . Following [2], a right adequate semigroup S is *right type B*, if it satisfies the following conditions (RB1) and (RB2):

(RB1) for all $e, f \in E(S^1)$, $a \in S$, $(efa)^* = (ea)^*(fa)^*$;

(RB2) if for all $a \in S$, $e \in E(S)$, $e \leq a^*$, then there is an element $f \in E(S^1)$ such that $e = (fa)^*$, where “ $\leq$ ” is a natural partial order on $E(S)$ (i.e., $(\forall g, h \in E(S)) g \leq h \Leftrightarrow g = gh = hg$).

Dually, a left adequate semigroup S is *left type B*, if it satisfies the following conditions (LB1) and (LB2):

(LB1) for all $e, f \in E(S^1)$, $a \in S$, $(aef)^+ = (ae)^+(af)^+$;

(LB2) if for all $a \in S$, $e \in E(S)$, $e \leq a^+$, then there is an element $f \in E(S^1)$ such that $e = (af)^+$.

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A semigroup is said to be *type B* if it is both right and left type B. Type B semigroups originally introduced by Fountain in [2] are generalizations of inverse semigroups in the range of abundant semigroups. Recently, Li and others investigated some classes of type B semigroups (see [12–18]).

Recall from Petrich [19] and Lawson [20] that a homomorphism ϕ of semigroups S and T is *idempotent-separated* if for all $e, f \in E(S)$, $e\phi f$ implies $e = f$, and that a semigroup S is said to be *E-unitary* if $a, ab \in E(S)$ implies $b \in E(S)$ and $a, ba \in E(S)$ implies $b \in E(S)$ for all $a, b \in S$. An inverse semigroup S is *proper* if $\mathcal{L} \cap \gamma = \mathcal{R} \cap \gamma = \iota_S$, where γ and ι_S are the least group congruence on S and the identity relation on S , respectively. It is well known that the class of *E-unitary* semigroups is one of the most important in inverse semigroup theory. There are two main reasons for this: first, McAlister theorem states that every inverse semigroup admits an *E-unitary* cover (i.e., every inverse semigroup is an idempotent-separated homomorphic image of an *E-unitary* semigroup); second, many naturally occurring inverse semigroups are *E-unitary*. Generally, a proper inverse semigroup is *E-unitary*, but the converse is not true. Therefore, it is an interesting thing to characterize a proper generalization inverse semigroup. Motivated by studying proper inverse semigroups and *E-unitary* inverse semigroups in Petrich (Inverse Semigroups, Wiley, New York, 1984), and as a continuation of Petrich and Lawson's work in inverse semigroups, in this paper, we shall characterize some proper type B semigroups.

This article is organized as follows. Section 2 provides some basic notions and some known results used in the sequel. In Section 3, we introduce the definitions of a left admissible triple and a Q -semigroup and give some basic properties. In Section 4, we give the notion of an admissible triple and obtain a new characterization of a proper type B semigroup. In particular, we prove that an arbitrary Q -semigroup with an admissible triple is an *E-unitary* type B semigroup.

2 Preliminaries

The following basic terminologies and notations are from [1,2,8,16,17,20].

Lemma 2.1. [1] *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

- (1) $a\mathcal{L}^*b$ ($a\mathcal{R}^*b$);
- (2) for all $x, y \in S^1$, $ax = ay$ ($xa = ya$) if and only if $bx = by$ ($xb = yb$).

Corollary 2.2. [1] *Let S be a semigroup and $a \in S$, $e \in E(S)$. Then the following statements hold:*

- (1) $a\mathcal{L}^*e$ ($a\mathcal{R}^*e$) if and only if $ae = a$ ($ea = a$) and for all $x, y \in S^1$, $ax = ay$ ($xa = ya$) implies $ex = ey$ ($xe = ye$);
- (2) $\mathcal{L}^*$ (resp., $\mathcal{R}^*$) is a right (resp., left) congruence on S .

For convenience, $\mathcal{L}_a^*$ and $\mathcal{R}_a^*$ denote the $\mathcal{L}^*$ -class and $\mathcal{R}^*$ -class containing a , respectively. Obviously, every left (resp., right) adequate semigroup S is $\mathcal{R}^*$ -unipotent (i.e., $|\mathcal{R}_a^* \cap E(S)| = 1$) (resp., $\mathcal{L}^*$ -unipotent) (i.e., $|\mathcal{L}_a^* \cap E(S)| = 1$).

Lemma 2.3. [8] *Let S be an abundant semigroup. Define the natural partial order relation “ $\leq$ ” on S as follows:*

$$(\forall a, b \in S) a \leq b \Leftrightarrow (\exists e, f \in E(S)) a = eb = bf.$$

Lemma 2.4. [16] *Let S be a left type B semigroup. Define a relation “ σ ” on S as follows:*

$$(a, b) \in \sigma \Leftrightarrow (\exists e \in E(S)) ae = be.$$

Then σ is the least right cancellative monoid congruence on S .

Lemma 2.5. [17] *Let S be a type B semigroup. Define a relation “ ξ ” on S as follows:*

$$(a, b) \in \xi \Leftrightarrow (\exists e \in E(S)) eae = ebe.$$

Then ξ is the least cancellative monoid congruence on S .

In this paper, we call a left type B semigroup S *proper* if $\sigma \cap \mathcal{R}^* = \iota_S$, and call a type B semigroup S *proper* if $\xi \cap \mathcal{L}^* = \iota_S$ and $\xi \cap \mathcal{R}^* = \iota_S$, where ι_S is the identity relation on S .

Lemma 2.6. [18] *Let S be a proper type B semigroup. Then S is E -unitary.*

Definition 2.1. [20] Let T be a monoid with identity 1 and A be a nonempty set. Define a mapping μ as follows:

$$\mu : T \times A \rightarrow A, \quad (t, a) \mapsto ta$$

such that $1 \cdot a = a$ and $(st)a = s(ta)$. Then A is a left T -act or T acts on A on the left.

3 Definitions and basic results

First, we recall some terminologies and notations of partial order set. Suppose that X is a partial order set. If $a, b \in X$ have a greatest lower bound in S , it will be denoted by $a \wedge b$. Let Y be a subset of X with $a, b \in Y$. Then $a \wedge b \in Y$ means that a and b have a greatest lower bound in X that it belongs to Y . In particular, if for all $a, b \in Y$, $a \wedge b \in Y$, then the nonempty subset Y of X is said to be a *subsemilattice* of X .

Let X be a partial order set and let Y be a subsemilattice of X . Suppose that $i \in X$ and for all $a \in Y$, $a \leq i$. Let T be a right cancellative monoid with identity 1. Let T act on X on the left such that for all $a, b \in X$, $t \in T$ $a \leq b$ implies $ta \leq tb$. We call (T, X, Y) with the above properties a *triple*.

Definition 3.1. Let (T, X, Y) be a triple satisfying the following conditions (Q1) to (Q6):

(Q1) $TY^i = X$, where $Y^i = Y \cup \{i\}$;

(Q2) For all $t \in T$, there exists $b \in Y$ such that $b \leq ti$;

(Q3) If $a, b \in Y$, $t \in T$ and $a \leq ti$, then $a \wedge tb \in Y$;

(Q4) If $a, b, c \in Y$, $t, u \in T$ and $a \leq ti$, $b \leq ui$, then $(a \wedge tb) \wedge tuc = a \wedge t(b \wedge uc)$;

(Q5) If $a, b \in Y$, $t \in T$ with $a \leq ti$ and $b \leq ti$, then $a \wedge t(a \wedge b) = b \wedge t(a \wedge b)$;

(Q6) If $a, b \in Y$ and $b = a \wedge b$, then for all $t \in T$ there exists $c \in Y^i$ such that $b = a \wedge tc$.

Then we call (T, X, Y) with the aforementioned conditions (Q1) to (Q6) a *left admissible triple*.

Now, we give an example which satisfies conditions (Q1) to (Q6) listed in Definition 3.1.

Example 3.1. Let $X = \{I, A, A', O\}$ be the four element Boolean algebra with top element I , bottom element O and A' the complement of A . Put $Y = \{A, O\}$. Let $T = \{1, g\}$ with $g^2 = 1$, where 1 is the identity automorphism of X and $gI = I$, $gO = O$, $gA = A'$, $gA' = A$. Obviously, X is a partial order set. It is easy to see that Y is a subsemilattice of X satisfying $TY^I = X$, where I is the element i listed in Definition 3.1. That is, (T, X, Y) satisfies condition (Q1) of Definition 3.1. Note that $1I = I = gI$. We have that $b \leq tI = I$ for all $t \in T$ and $b \in Y$, which implies that condition (Q2) holds. Similarly, it is easily seen that conditions (Q3) and (Q4) hold. On the other hand, for $A, O \in Y$, we have that $A \leq tI = I$ and $O \leq tI = I$ for all $t \in T$. Again, for all $t \in T$,

$$A \wedge t(A \wedge O) = A \wedge tO = A \wedge O = O,$$

$$O \wedge t(A \wedge O) = O \wedge tO = O \wedge O = O.$$

Thus, (T, X, Y) satisfies condition (Q5). Finally, it is easily seen that $O = O \wedge O = O \wedge tO$, $A = A \wedge A = A \wedge I = A \wedge tI$ and $O = A \wedge O = A \wedge tO$. Therefore, condition (Q6) is true.

Summarizing the aforementioned arguments, we conclude that (T, X, Y) is a left admissible triple.

Definition 3.2. Let (T, X, Y) be a left admissible triple and we define:

$$Q(T, X, Y) = \{(a, t) \in Y \times T \mid a \leq ti\}$$

with a multiplication

$$(a, t)(b, u) = (a \wedge tb, tu).$$

Obviously, $Q(T, X, Y)$ is a semigroup with respect to the above multiplication.

Example 3.2. In our Example 3.1, put $Q = \{(A, 1), (O, 1), (A, g), (O, g)\}$. Obviously, $A \leq 1I = I$, $O \leq 1I = I$, $A \leq gI = I$ and $O \leq gI = I$. This means that Q satisfies the property $Q(T, X, Y) = \{(a, t) \in Y \times T \mid a \leq ti\}$ listed in Definition 3.2. Furthermore, it is easy to check that Q is a semigroup with respect to the multiplication of Definition 3.2.

Theorem 3.1. Let (T, X, Y) be a left admissible triple. Then the following statements are true:

- (1) $E(Q(T, X, Y)) = Y \times \{1\}$;
- (2) $Q(T, X, Y)$ is a left type B semigroup;
- (3) Let $(a, t), (b, u) \in Q(T, X, Y)$. Then $(a, t)\mathcal{R}^*(b, u)$ if and only if $a = b$.

Proof.

- (1) Let $(a, t) \in E(Q(T, X, Y))$. Then $(a, t) = (a, t)^2 = (a \wedge ta, t^2)$ from Definition 3.2. Hence, $t = t^2$. Note that T is a right cancellative monoid with identity 1 and so $t = 1$. That is, $(a, t) = (a, 1) \in Y \times \{1\}$. Conversely, for all $a \in Y$ we have $a \leq i$. So $(a, 1) \in Q(T, X, Y)$ and $(a, 1)$ is an idempotent of $Q(T, X, Y)$. Hence, the elements in $Y \times \{1\}$ are the idempotents of $Q(T, X, Y)$. Therefore, (1) holds.
- (2) Obviously, $Q(T, X, Y)$ is a semigroup from Definition 3.2. Now, we prove that $Q(T, X, Y)$ is left abundant. To see it, let $(a, t) \in Q(T, X, Y)$, $(b, s), (c, k) \in Q(T, X, Y)^1$ such that $(b, s)(a, t) = (c, k)(a, t)$. Then $(b \wedge sa, st) = (c \wedge ka, kt)$, and so $b \wedge sa = c \wedge ka$, $st = kt$. Note that T is a right cancellative monoid with identity 1. We have $s = k$. Hence,

$$(b, s)(a, 1) = (b \wedge sa, s) = (c \wedge ka, k) = (c, k)(a, 1).$$

But $(a, 1)(a, t) = (a \wedge a, t) = (a, t)$. We have $(a, t)\mathcal{R}^*(a, 1) \in E(Q(T, X, Y))$ from Corollary 2.2. This gives that $Q(T, X, Y)$ is left abundant. Clearly, $E(Q(T, X, Y))$ is a semilattice from (1). Therefore, $Q(T, X, Y)$ is a left adequate semigroup. That is, $(a, t)^+ = (a, 1)$ for all $(a, t) \in Q(T, X, Y)$.

Now, we prove that $Q(T, X, Y)$ is a left type B semigroup. First, we verify that $Q(T, X, Y)$ satisfies condition (LB1). To see it, let $(a, t) \in Q(T, X, Y)$, $(b, 1), (c, 1) \in E(Q(T, X, Y))^1$. Then

$$(a, t)(b, 1)(c, 1) = (a \wedge tb, t)(c, 1) = (a \wedge tb \wedge tc, t).$$

Note that

$$[(a, t)(b, 1)(c, 1)]^+ = (a \wedge tb \wedge tc, t)^+ = (a \wedge tb \wedge tc, 1),$$

$$[(a, t)(b, 1)]^+ = [(a \wedge tb, t)]^+ = (a \wedge tb, 1),$$

$$[(a, t)(c, 1)]^+ = [(a \wedge tc, t)]^+ = (a \wedge tc, 1).$$

But $(a \wedge tb, 1)(a \wedge tc, 1) = (a \wedge tb \wedge tc, 1)$. Hence, $[(a, t)(b, 1)(c, 1)]^+ = [(a, t)(b, 1)]^+[(a, t)(c, 1)]^+$. That is, $Q(T, X, Y)$ satisfies condition (LB1).

Next, we prove that $Q(T, X, Y)$ satisfies condition (LB2). To see it, let $(a, t) \in Q(T, X, Y)$, $(b, 1) \in E(Q(T, X, Y))$ and $(b, 1) \leq (a, t)^+$. That is, $(b, 1) \leq (a, 1)$. Then $(b, 1) = (b, 1)(a, 1) = (a, 1)(b, 1)$ and so $b = a \wedge b = b \wedge a$. By condition (Q6), we know there exists $c \in Y$ such that $b = a \wedge tc$. Hence, $(b, 1) = (a \wedge tc, 1) = (a \wedge tc, t)^+ = [(a, t)(c, 1)]^+$ and $(c, 1) \in E(Q(T, X, Y))$. This shows that $Q(T, X, Y)$ satisfies condition (LB2). Therefore, $Q(T, X, Y)$ is a left type B semigroup.

- (3) Let $(a, t), (b, u) \in Q(T, X, Y)$ and $(a, t)\mathcal{R}^*(b, u)$. Then, by the proof of (2), we obtain that $(a, 1)\mathcal{R}^*(b, 1)$. By (2), each $\mathcal{R}^*$ class of $Q(T, X, Y)$ is unipotent. Hence, $(a, 1) = (b, 1)$. That is, $a = b$. The converse is true from the proof of (2). $\square$

Remark 3.1. From Lemma 2.4, it can be seen that every left type B semigroup has a least right cancellative monoid congruence. Therefore, there exists a least right cancellative monoid congruence σ (refer to Lemma 2.4) on $Q(T, X, Y)$ from Theorem 3.1(2).

Theorem 3.2. Let (T, X, Y) be a left admissible triple and let σ be the least right cancellative monoid congruence on $Q(T, X, Y)$. Then the following statements are true:

- (1) If $(a, t), (b, u) \in Q(T, X, Y)$, then $(a, t)\sigma(b, u)$ if and only if $t = u$;
- (2) $\sigma \cap \mathcal{R}^* = \iota_Q$, where ι_Q is the identity relation on $Q(T, X, Y)$;
- (3) $Q(T, X, Y)/\sigma \cong T$.

Proof.

- (1) Let $(a, t), (b, u) \in Q(T, X, Y)$ such that $(a, t)\sigma(b, u)$. Then, by Lemma 2.4, there exists $(c, 1) \in E(Q(T, X, Y))$ such that $(a, t)(c, 1) = (b, u)(c, 1)$. That is, $(a \wedge tc, t) = (b \wedge uc, u)$. Thus, $t = u$.
Conversely, let $(a, t), (b, t) \in Q(T, X, Y)$. Then $a \leq ti, b \leq ti$. By condition (Q5) of Definition 3.1, we have that $a \wedge t(a \wedge b) = b \wedge t(a \wedge b)$ and so $(a, t)(a \wedge b, 1) = (b, t)(a \wedge b, 1)$. In other words, $(a, t)\sigma(b, t)$.
- (2) It follows directly from Theorems 3.1(3) and 3.2(1).
- (3) Define the mapping ϕ as follows:

$$\phi : Q(T, X, Y)/\sigma \rightarrow T, \quad (a, t)\sigma \mapsto t.$$

Then ϕ is well-defined. In fact, let $(a, t)\sigma, (b, u)\sigma \in Q(T, X, Y)/\sigma$ and $(a, t)\sigma = (b, u)\sigma$. It follows that $t = u$ from (1). This shows that ϕ is well-defined. Clearly, ϕ is a homomorphism from $Q(T, X, Y)/\sigma$ onto T . Next, we only show that ϕ is injective. To see it, let $(a, t)\sigma, (b, u)\sigma \in Q(T, X, Y)/\sigma$ such that $\phi[(a, t)\sigma] = \phi[(b, u)\sigma]$. Then $t = u$ from the definition of ϕ . It follows that $(a, t)\sigma = (b, u)\sigma$ from Theorem 3.2(1). Therefore, ϕ is injective.

Summarizing the aforementioned arguments, we conclude that $Q(T, X, Y)/\sigma \cong T$. This completes the proof. $\square$

For convenience, we denote $Q(T, X, Y)$ by Q and call $Q(T, X, Y)$ a Q -semigroup.

Theorem 3.3. Every proper left type B semigroup is isomorphic to a Q -semigroup.

Proof. Let S be a proper left type B semigroup. Then $E(S)$ is a semilattice. Put $S/\sigma = T$, where σ is the least right cancellative monoid congruence on S . Then T is a right cancellative monoid and $E(S)$ is the identify of T . Define a relation “ $\triangleleft$ ” on $E(S) \times T$ as follows:

$$(\forall e, f \in E(S), a, b \in S) (e, a\sigma) \triangleleft (f, b\sigma) \Leftrightarrow (\exists c \in S) c^+ = e, c = cf, \text{ and } b\sigma = a\sigma c\sigma.$$

Obviously, “ $\triangleleft$ ” is reflexive. Next, we prove that “ $\triangleleft$ ” is transitive. To see it, let $(e, a\sigma), (f, b\sigma), (g, c\sigma) \in E(S) \times T$ with $(e, a\sigma) \triangleleft (f, b\sigma), (f, b\sigma) \triangleleft (g, c\sigma)$. Then there exist $x, y \in S$ such that $b\sigma = a\sigma x\sigma$ and $c\sigma = b\sigma y\sigma$ and that $x^+ = e, x = xf, y^+ = f$ and $y = yg$. Hence, $c\sigma = a\sigma x\sigma y\sigma = a\sigma(xy)\sigma$. Since $y\mathcal{R}^*f$ and $\mathcal{R}^*$ is a left congruence. We have $xy\mathcal{R}^*xf = x$. Note that each $\mathcal{R}^*$ class of S contains only one idempotent. We get $(xy)^+ = x^+ = e$. But $xyg = x(yg) = xy$. That is, $(e, a\sigma) \triangleleft (g, c\sigma)$. We deduce that “ $\triangleleft$ ” is transitive. However, the relation “ $\triangleleft$ ” is not a partial order. Now, we define a relation “ π ” on $E(S) \times T$ as follows:

$$(e, a\sigma)\pi(f, b\sigma) \Leftrightarrow (e, a\sigma) \triangleleft (f, b\sigma) \text{ and } (f, b\sigma) \triangleleft (e, a\sigma).$$

Obviously, “ π ” is an equivalence relation. For $A, B \in (E(S) \times T)/\pi$, we give a definition as follows:

$$A \leq B \Leftrightarrow (\exists (e, a\sigma) \in A, (f, b\sigma) \in B) (e, a\sigma) \triangleleft (f, b\sigma).$$

It is clear that $\leq$ is a partial order of $(E(S) \times T)/\pi$. We write $X = (E(S) \times T)/\pi$, which is a partial order set by $\leq$. Note that for all $A, B \in X$. We have that:

$$A \leq B \Leftrightarrow (\forall (e, a\sigma) \in A, (f, b\sigma) \in B) (e, a\sigma) \triangleleft (f, b\sigma).$$

For $(e, a\sigma)\pi \in X$ and $b\sigma \in T$, we define $b\sigma(e, a\sigma)\pi = (e, ba\sigma)\pi$. Obviously, it is well-defined. In fact, let $(e, a\sigma)\pi = (f, c\sigma)\pi$ and $b\sigma = d\sigma$. Then there exist $x, y \in S$ such that $c\sigma = a\sigma x\sigma$ and $a\sigma = c\sigma y\sigma$, where $x^+ = e, x = xf, y^+ = f$ and $y = ye$. Hence, we have

$$(dc)\sigma = d\sigma a\sigma x\sigma = (ba)\sigma x\sigma, \quad (ba)\sigma = b\sigma a\sigma = b\sigma c\sigma y\sigma = (dc)\sigma y\sigma.$$

Therefore, $(e, ba\sigma)\pi = (f, dc\sigma)\pi$. It means that for all $A, B \in X$, $t \in T$, $A \leq B$ implies $tA \leq tB$.

Now, we put $Y = \{(e, E(S))\pi \mid e \in E(S)\}$. Then, by the definition of “ π ,” we obtain that $(e, E(S))\pi \triangleleft (f, E(S))\pi \Leftrightarrow e \leq f$, where $\leq$ is the partial order of $E(S)$. Hence,

$$(e, E(S))\pi \leq (f, E(S))\pi \Leftrightarrow e \leq f \quad (3.1)$$

Therefore, the largest element of Y is $(1, E(S))\pi$.

Next, we verify that for all $f \in E(S)$, $a \in S$ we have

$$(f, a\sigma)\pi \wedge (a^+, E(S))\pi = ((af)^+, E(S))\pi. \quad (3.2)$$

In fact, from $(af)^+a^+ = (af)^+$, we have that $(af)^+ \leq a^+$. By (3.1), $((af)^+, E(S))\pi \leq (a^+, E(S))\pi$. Again since $a\sigma = (af)\sigma = E(S) \cdot (af)\sigma$ and $af = aff$. We have that $((af)^+, E(S))\pi \triangleleft (f, a\sigma)\pi$ from the definition of “ π .” Hence, $((af)^+, E(S))\pi \leq (f, a\sigma)\pi$. This shows that $((af)^+, E(S))\pi$ is a lower bound of $(f, a\sigma)\pi$ and $(a^+, E(S))\pi$. Next, we show that $((af)^+, E(S))\pi$ is the greatest lower bound of $(f, a\sigma)\pi$ and $(a^+, E(S))\pi$. To see it, suppose that $(k, b\sigma)\pi$ is another lower bound of $(f, a\sigma)\pi$ and $(a^+, E(S))\pi$. Then $(k, b\sigma)\pi \leq (a^+, E(S))\pi$. Hence, there exists $c \in S$ such that $E(S) = b\sigma \cdot c\sigma$, $c^+ = k$ and $ca^+ = c$. Note that T is a right cancellative monoid. We have that $b\sigma \cdot c\sigma$ is the identify of T and that the inverse of $b\sigma$ is $c\sigma$. Hence, there exists $d \in S$ such that $(ca)\sigma = E(S) \cdot d\sigma$, $d^+ = k$ and $df = d$. Again since $ca^+ = c$ and $\mathcal{R}^*$ is a left congruence, $ca\mathcal{R}^*ca^+ = c$. However, $c\mathcal{R}^*k\mathcal{R}^*d$. Hence, $(ca, d) \in \mathcal{R}^* \cap \sigma$. Since S is a proper type B semigroup, we have that $ca = d$. Furthermore, $caf = df = d = ca$. That is, $ca = c(af)\mathcal{R}^*c(af)^+$. And since $ca\mathcal{R}^*k$, we have that $c(af)^+\mathcal{R}^*c$. But $c(af)^+\mathcal{R}^*\sigma c$ and S is proper. We have that $c = c(af)^+$. These together with $c\mathcal{R}^*k$, we obtain that $(k, E(S))\pi \leq ((af)^+, E(S))\pi$. Therefore, $(k, b\sigma)\pi \leq ((af)^+, E(S))\pi$. That is, $((af)^+, E(S))\pi$ is the greatest lower bound of $(f, a\sigma)\pi$ and $(a^+, E(S))\pi$. Thus, (3.2) holds.

In particular, if we choose $a = e$ in (3.2), then

$$(\forall e, f \in E(S)) (e, E(S))\pi \wedge (f, E(S))\pi = (ef, E(S))\pi.$$

Hence, Y is a subsemilattice of X .

Define a mapping as follows:

$$\theta : E(S) \rightarrow Y, \quad e \mapsto (e, E(S))\pi.$$

It is easy to see that θ is an order automorphism mapping. Now, we define another mapping as follows:

$$\psi : S \rightarrow Y \times T, \quad x \mapsto ((x^+, E(S))\pi, x\sigma).$$

Then we can obtain that ψ is an injective mapping. Define a multiplication on $\text{Im}\psi$ as follows:

$$((x^+, E(S))\pi, x\sigma)((y^+, E(S))\pi, y\sigma) = ((x^+, E(S))\pi \wedge x\sigma(y^+, E(S))\pi, xy\sigma).$$

Since $\mathcal{R}^*$ is a left congruence, we have that $xy\mathcal{R}^*xy^+$ and each $\mathcal{R}^*$ class of S contains only one idempotent. We have that $(xy)^+ = (xy^+)^+$. Hence,

$$\begin{aligned} ((x^+, E(S))\pi, x\sigma)((y^+, E(S))\pi, y\sigma) &= ((x^+, E(S))\pi \wedge x\sigma(y^+, E(S))\pi, xy\sigma) \\ &= ((x^+, E(S))\pi \wedge (y^+, x\sigma)\pi, xy\sigma) \\ &= (((xy^+)^+, E(S))\pi, xy\sigma) \\ &= (((xy)^+, E(S))\pi, xy\sigma). \end{aligned}$$

Therefore, $\text{Im}\psi$ is closed with respect to the above multiplication. This means that $\text{Im}\psi$ is a monoid with identify $((1, E(S))\pi, E(S))$.

Finally, we consider the Q -semigroup $Q = Q(T, X, Y)$, where

$$T = S/\sigma, \quad X = (E(S) \times S/\sigma)/\pi, \quad Y = \{(e, E(S))\pi \mid e \in E(S)\}.$$

It is clear that Q is a subsemigroup of $Y \times T$ and that $\text{Im}\psi$ is a subsemigroup of Q . We get $\text{Im}\psi = Q$ from (3.2). On the other hand, it is easily seen that $\psi : S \rightarrow \text{Im}\psi$ is an isomorphism. Therefore, $S \cong Q$. This completes the proof. $\square$

Remark 3.2.

- (1) From Theorem 3.3, S is an arbitrary proper left type B semigroup with the least right cancellative monoid congruence σ on it. Put

$$T = S/\sigma, X = (E(S) \times S/\sigma)/\pi, Y = \{(e, E(S))\pi \mid e \in E(S)\},$$

where π is an equivalence relation on $E(S) \times T$. In our Theorem 3.3, we have proved that Y is a subsemilattice of a partial set X . It is easy to see that $Y = Y^i$, where $i = (1, E(S))\pi$.

- (2) In fact, in the first half proof of Theorem 3.3, we can see that (T, X, Y) satisfies conditions (Q1–Q6) listed in Definition 3.1. For example, let $a\sigma \in T$, $(e, E(S))\pi \in Y^i$. Then $a\sigma \cdot (e, E(S))\pi = (e, a\sigma E(S))\pi = (e, a\sigma)\pi \in X$ since $E(S)$ is the identity of T , which implies that $TY^i \subseteq X$. Conversely, for all $(e, a\sigma)\pi \in X$, we have $(e, a\sigma)\pi = (e, a\sigma E(S))\pi = a\sigma \cdot (e, E(S))\pi \in TY^i$. That is, condition (Q1) holds. Similarly, let $a\sigma \in T$ then $a\sigma \cdot i = a\sigma \cdot (1, E(S))\pi = (1, a\sigma)\pi$. From the proof of Theorem 3.3, it is easy to see that there is $(a^+, E(S))\pi \in Y$ such that $(a^+, E(S))\pi \leq (1, a\sigma)\pi$ since $(a^+, E(S))\pi$ is the greatest lower bound of $(1, a\sigma)\pi$ and $(1, E(S))\pi$. This means that condition (Q2) holds. Again, let $(e, E(S))\pi, (f, E(S))\pi \in Y$ and $a\sigma \in T$ such that $(e, E(S))\pi \leq a\sigma \cdot i = a\sigma(1, E(S))\pi = (1, a\sigma)\pi$. By the proof of Theorem 3.3, $((af)^+, E(S))\pi$ is the greatest lower bound of $(f, a\sigma)\pi$ and $(a^+, E(S))\pi$. Again, since $(a^+, E(S))\pi$ is the greatest lower bound of $(1, a\sigma)\pi$ and $(a^+, E(S))\pi$. We have that $(e, E(S))\pi \leq (a^+, E(S))\pi$, and so that

$$\begin{aligned} ((e, E(S))\pi \wedge a\sigma(f, E(S))\pi) &= ((e, E(S))\pi \wedge (f, a\sigma E(S))\pi) \\ &= ((e, E(S))\pi \wedge (f, a\sigma)\pi) \\ &= ((e, E(S))\pi \wedge (a^+, E(S))\pi \wedge (f, a\sigma)\pi) \\ &= ((e, E(S))\pi \wedge ((af)^+, E(S))\pi) \\ &= (e(af)^+, E(S))\pi \in Y. \end{aligned}$$

This means that (T, X, Y) satisfies condition (Q3) of Definition 3.1. By using a simple calculation similar to the proof of (Q3), it is easily seen that conditions (Q5) and (Q6) hold.

Finally, we explain (Q4) is also true. To see it, let $(e, E(S))\pi, (f, E(S))\pi, (g, E(S))\pi \in Y$ and $a\sigma, b\sigma \in T$ such that $(e, E(S))\pi \leq a\sigma(1, E(S))\pi$ and $(f, E(S))\pi \leq b\sigma(1, E(S))\pi$. That is, $(e, E(S))\pi \leq (1, a\sigma)\pi$ and $(f, E(S))\pi \leq (1, b\sigma)\pi$. Then $(e, E(S))\pi \leq (a^+, E(S))\pi, (f, E(S))\pi \leq (b^+, E(S))\pi$ and $(f, a\sigma)\pi \leq (1, (ab)\sigma)\pi$. Hence, $(f, a\sigma)\pi \leq ((ab)^+, E(S))\pi$, and so

$$\begin{aligned} &[(e, E(S))\pi \wedge a\sigma(f, E(S))\pi] \wedge [(a\sigma \cdot b\sigma)(g, E(S))\pi] \\ &= (e, E(S))\pi \wedge (f, a\sigma)\pi \wedge (g, (ab)\sigma)\pi \\ &= (e, E(S))\pi \wedge (a^+, E(S))\pi \wedge (f, a\sigma)\pi \wedge ((ab)^+, E(S))\pi \wedge (g, (ab)\sigma)\pi \\ &= (e, E(S))\pi \wedge ((af)^+, E(S))\pi \wedge ((abg)^+, E(S))\pi \\ &= (e(af)^+(abg)^+, E(S))\pi \\ &= (e(af)^+(abg)^+, E(S))\pi \quad (\text{since } (abg)^+ = (a(bg)^+)^+) \\ &= (e(af(bg)^+)^+, E(S))\pi \quad (\text{by (LB1)}) \end{aligned}$$

and

$$\begin{aligned} &(e, E(S))\pi \wedge a\sigma[(f, E(S))\pi \wedge b\sigma(g, E(S))\pi] \\ &= (e, E(S))\pi \wedge a\sigma[(f, E(S))\pi \wedge (g, b\sigma)\pi] \\ &= (e, E(S))\pi \wedge a\sigma[(f, E(S))\pi \wedge (b^+, E(S))\pi \wedge (g, b\sigma)\pi] \\ &= (e, E(S))\pi \wedge a\sigma[(f, E(S))\pi \wedge ((bg)^+, E(S))\pi] \\ &= (e, E(S))\pi \wedge a\sigma(f(bg)^+, E(S))\pi \\ &= (e, E(S))\pi \wedge (f(bg)^+, a\sigma)\pi \\ &= (e, E(S))\pi \wedge (a^+, E(S))\pi \wedge (f(bg)^+, a\sigma)\pi \\ &= (e, E(S))\pi \wedge ((af(bg)^+)^+, E(S))\pi \\ &= (e(af(bg)^+)^+, E(S))\pi. \end{aligned}$$

Therefore, (T, X, Y) listed in Theorem 3.3 satisfies condition (Q4) of Definition 3.1. In other words, (T, X, Y) listed in Theorem 3.3 is a left admissible triple.

4 A characterization of a proper type B semigroup

In this section, we give a new characterization of a proper type B semigroup. For this purpose, we first give a characterization of Green $*$ relation $\mathcal{L}^*$ on the Q -semigroup $Q = Q(T, X, Y)$ and replace the condition “ T is a right cancellative monoid” by “ T is a cancellative monoid.”

Proposition 4.1. *Let $Q = Q(T, X, Y)$ be a Q -semigroup and let T be a cancellative monoid with identify 1. If $(b, 1), (a, t) \in Q$, then $(b, 1)\mathcal{L}^*(a, t)$ if and only if it satisfies the following conditions:*

- (1) $a \leq tb$;
- (2) For all $c, d \in Y$, $a \wedge tc = a \wedge td$ implies $b \wedge c = b \wedge d$.

Proof. (Necessity) Let $(b, 1), (a, t) \in Q$ and $(b, 1)\mathcal{L}^*(a, t)$. Then $(a, t)(b, 1) = (a, t)$ from Corollary 2.2. That is, $(a \wedge tb, t) = (a, t)$. Hence, $a \wedge tb = a$ and so $a \leq tb$. This shows that condition (1) is satisfied. Let $(c, u), (d, v) \in Q^1$ with $(a, t)(c, u) = (a, t)(d, v)$. Then $a \wedge tc = a \wedge td$ and $tu = tv$. Note that T is a cancellative monoid. We have that $u = v$. Again since $(b, 1)\mathcal{L}^*(a, t)$, it follows that $(b, 1)(c, u) = (b, 1)(d, v)$ and so $b \wedge c = b \wedge d$. Therefore, condition (2) is satisfied.

(Sufficiency) Suppose that conditions (1) and (2) hold. We have that $(a, t)(b, 1) = (a \wedge tb, t) = (a, t)$ from condition (1). On the other hand, if there are $(c, u), (d, v) \in Q^1$ such that $(a, t)(c, u) = (a, t)(d, v)$, then $(a \wedge tc, tu) = (a \wedge td, tv)$, and so $a \wedge tc = a \wedge td$ and $tu = tv$. Hence, $b \wedge c = b \wedge d$ from condition (2) and $u = v$ since T is cancellative. Therefore, $(b \wedge c, u) = (b \wedge d, v)$. That is, $(b, 1)(c, u) = (b, 1)(d, v)$. By Corollary 2.2, we have that $(b, 1)\mathcal{L}^*(a, t)$. $\square$

Remark 4.1. In our Proposition 4.1, if both $(b, 1)$ and $(b', 1)$ satisfy the conditions of Proposition 4.1, that is, $(b, 1)\mathcal{L}^*(a, t)\mathcal{L}^*(b', 1)$, then $a \leq tb$, $a \leq tb'$ and $a \wedge tb = a \wedge tb' = a$. By condition (2), we obtain that $b \wedge b = b \wedge b'$ and $b' \wedge b = b' \wedge b'$. Hence, $b = b'$. Thus, the element $(b, 1)$ is uniquely determined by the properties in Proposition 4.1. For given $(a, t) \in Q$, we denote b by $[a, t]$. That is, $b = [a, t]$. In other words, $(a, t)^* = ([a, t], 1)$.

For convenience, we restate the conditions in Proposition 4.1 as follows:

(Q7) Let T be a cancellative monoid with identify 1. For all $t \in T$, $a \in Y$ with $a \leq ti$ and there exists a unique $[a, t] \in Y$ such that:

- (1) $a \leq t[a, t]$;
- (2) For all $c, d \in Y$, $a \wedge tc = a \wedge td$ implies that $[a, t] \wedge c = [a, t] \wedge d$.

It is easily seen that every Q -semigroup $Q(T, X, Y)$ satisfying conditions (Q1) to (Q7) is adequate.

Theorem 4.2. *Let $Q = Q(T, X, Y)$ be a Q -semigroup satisfying conditions (Q1) to (Q7). Then Q is a type B semigroup if and only if the following two conditions hold:*

- (Q8)** For all $a, b \in Y, t \in T$, $[a \wedge b \wedge c, t] = [a \wedge b, t] \wedge [c \wedge b, t]$;
- (Q9)** If for all $c \in Y, t \in T$, $c = c \wedge [a, t]$, then there exists $b \in Y$ such that $c = [b \wedge a, t]$.

Proof. (Necessity) Let $Q(T, X, Y)$ be a type B semigroup. Then for all $(b, t) \in Q, (a, 1), (c, 1) \in E(Q^1)$ we have

$$\begin{aligned} [(a, 1)(c, 1)(b, t)]^* &= (a \wedge b \wedge c, t)^* \\ &= ([a \wedge b \wedge c, t], 1) \quad (\text{by Proposition 4.1}) \\ &= [(a, 1)(b, t)]^* [(c, 1)(b, t)]^* \quad (\text{by condition (RB1)}) \\ &= (a \wedge b, t)^* (c \wedge b, t)^* \\ &= ([a \wedge b, t], 1)([c \wedge b, t], 1) \quad (\text{by Proposition 4.1}) \\ &= ([a \wedge b, t] \wedge [c \wedge b, t], 1). \end{aligned}$$

Hence, $[a \wedge b \wedge c, t] = [a \wedge b, t] \wedge [c \wedge b, t]$. Therefore, condition (Q8) holds.

On the other hand, if for all $(a, t) \in Q$, $(c, 1) \in E(Q^1)$ such that $(c, 1) \leq (a, t)^*$, then $(c, 1) \leq ([a, t], 1)$ from Proposition 4.1. Hence, $(c, 1) = (c, 1)([a, t], 1) = (c \wedge [a, t], 1)$ and so $c = c \wedge [a, t]$. Since $Q(T, X, Y)$ is type B, we have that $(c, 1) = [(b, 1)(a, t)]^*$ for some $(b, 1) \in E(Q^1)$. That is, $(c, 1) = (b \wedge a, t)^* = ([b \wedge a, t], 1)$ from Proposition 4.1. Hence, $c = [b \wedge a, t]$. Therefore, condition (Q9) holds.

(Sufficiency) By conditions (Q1) to (Q6), Q is a left type B semigroup from Theorem 3.1(2). Furthermore, it is easy to see that Q is an adequate semigroup from condition (Q7). Finally, by conditions (Q8) and (Q9), Q is a right type B semigroup. Therefore, Q is a type B semigroup. $\square$

Proposition 4.3. *Let $Q = Q(T, X, Y)$ be a Q -semigroup satisfying conditions (Q1) to (Q9). Then $\xi = \sigma$.*

Proof. Note that Q is a type B semigroup from Theorem 4.2. By Lemmas 2.4 and 2.5, it is easy to see that $\sigma \subseteq \xi$. Now, we prove that $\xi \subseteq \sigma$. To see it, let $(a, t), (b, u) \in Q$ such that $(a, t)\xi(b, u)$. Then, by Lemma 2.5, there exists $(c, 1) \in E(Q^1)$ satisfying $(c, 1)(a, t)(c, 1) = (c, 1)(b, u)(c, 1)$. That is, $(c \wedge a \wedge tc, t) = (c \wedge b \wedge uc, u)$. Hence, $t = u$. By Theorem 3.2(1), $(a, t)\sigma(b, u)$. This gives that $\xi \subseteq \sigma$. Therefore, $\xi = \sigma$. This completes the proof. $\square$

Remark 4.2. From Proposition 4.3, it is easily seen that the least right cancellative monoid congruence on a Q -semigroup $Q(T, X, Y)$ is the least cancellative monoid congruence on $Q(T, X, Y)$ (and vice versa) if $Q(T, X, Y)$ satisfies conditions (Q1) to (Q9). Therefore, in the remaining, we can write $\xi = \sigma$ on $Q(T, X, Y)$ satisfying conditions (Q1) to (Q9).

Theorem 4.4. *Every Q -semigroup $Q(T, X, Y)$ satisfying conditions (Q1) to (Q9) is proper type B if and only if it satisfies the following condition:*

(Q10) *For all $a, b \in Y$, if $a \leq ti$ and $b \leq ti$, then $[a, t] = [b, t]$ implies $a = b$.*

Proof. (Sufficiency) Clearly, $Q(T, X, Y)$ is a type B semigroup from Theorem 4.2. Suppose that condition (Q10) is satisfied. Let $(a, t), (b, s) \in Q$ with $(a, t)(\sigma \cap \mathcal{L}^*)(b, s)$. Then, we get $t = s$ from Theorem 3.2(1). By Proposition 4.1, $[a, t] = [b, s]$. Hence, we have that $a = b$ from the hypothesis. This means that $(a, t) = (b, s)$. That is, $\sigma \cap \mathcal{L}^* = \iota_Q$. On the other hand, we obtain that $\sigma \cap \mathcal{R}^* = \iota_Q$ from Theorem 3.2(2). Therefore, $Q(T, X, Y)$ is a proper type B semigroup.

(Necessity) Let $a, b \in Y, t \in T$ such that $a \leq ti$ and $b \leq ti$. Then $(a, t), (b, t) \in Q$ from the definition of $Q(T, X, Y)$. Hence, by Theorem 2.2(2), $(a, t)\sigma(b, t)$. Suppose that $[a, t] = [b, t]$. By the proof of Proposition 4.1, we have $(a, t)^* = (b, t)^*$. Hence, $(a, t)\mathcal{L}^*(b, t)$ and so $(a, t)(\sigma \cap \mathcal{L}^*)(b, t)$. Note that Q is proper. We have that $(a, t) = (b, t)$, which implies that $a = b$. Therefore, condition (Q10) is satisfied. $\square$

Definition 4.1. Let (T, X, Y) be a triple. Then (T, X, Y) is said to be an admissible triple if it satisfies conditions (Q1) to (Q10).

Example 4.1. In our Example 3.1, denote $Q = \{(A, 1), (O, 1), (O, g)\}$. It is easy to see that Q satisfies the condition of Definition 3.2 with $E(Q) = \{(A, 1), (O, 1)\}$ and that Q satisfies conditions (Q1) to (Q6). Obviously, T is a cancellative monoid and that $(O, g)\mathcal{L}^*(O, 1)$. By a simple calculation, we have that $[A, 1] = A, [O, 1] = O = [O, g]$. Clearly, Q satisfies condition (Q7). Note that $E(Q)$ is a semilattice and that $(O, g)\mathcal{R}^*(O, 1), (A, 1)\mathcal{R}^*(A, 1)$. We have that Q is an adequate semigroup. By Theorem 3.1, Q is left type B. It is easily seen that Q is also right type B. In other words, Q is a type B semigroup.

Next, we prove that Q satisfies conditions (Q8) to (Q10). In fact, for $A, O \in Y$, we have

$$[A \wedge O \wedge O, g] = [O, g] = O = [A \wedge O, g] \wedge [O \wedge O, g]$$

and

$$[A \wedge O \wedge O, 1] = [O, 1] = O = [A \wedge O, 1] \wedge [O \wedge O, 1],$$

which implies that condition (Q8) holds. On the other hand, for all $A, O \in Y, 1, g \in T$, we have

$$\begin{aligned}
A &= A \wedge [A, 1] \Rightarrow A = [A \wedge A, 1], \\
O &= O \wedge [O, g] \Rightarrow O = [A \wedge O, g], \\
O &= O \wedge [O, 1] \Rightarrow O = [A \wedge O, 1], \\
O &= O \wedge [A, 1] \Rightarrow O = [O \wedge A, 1].
\end{aligned}$$

This means that condition (Q9) holds. Finally, by Theorems 3.1(3) and 3.2(1), it is easy to check that Q is proper. Furthermore, we have that

$$\begin{aligned}
[A, 1] &= [A, 1], \quad A \leq 1I \quad \text{and} \quad A = A, \\
[O, 1] &= [O, 1], \quad O \leq 1I \quad \text{and} \quad O = O, \\
[O, g] &= [O, g], \quad O \leq gI \quad \text{and} \quad O = O.
\end{aligned}$$

Therefore, condition (Q10) holds.

Summarizing the aforementioned arguments, Q is an admissible triple satisfying conditions (Q1) to (Q10).

Theorem 4.5. *Let S be a proper type B semigroup. Then for some admissible triple (T, X, Y) , $S \cong Q(T, X, Y)$. Conversely, every Q -semigroup constructed by an admissible triple of Definition 4.1 is a proper type B semigroup.*

Proof. Let S be a proper type B semigroup. Then, by Theorem 3.3, $S = Q(T, X, Y)$ for some left admissible triple (T, X, Y) . By Theorem 3.2(3), $Q(T, X, Y)/\sigma \cong T$. But S is a type B semigroup and so $Q(T, X, Y)/\sigma$ is a cancellative monoid. Thus, T is a cancellative monoid. Then, by Proposition 4.1, Theorems 4.2 and 4.4, $Q(T, X, Y)$ is a proper type B semigroup, where (T, X, Y) is an admissible triple. Conversely, it follows directly from Proposition 4.1, Theorems 4.2 and 4.4. $\square$

Corollary 4.6. *Let $Q = Q(T, X, Y)$ be a Q -semigroup satisfying conditions (Q1) to (Q10). Then Q is an E -unitary type B semigroup. Conversely, every E -unitary type B semigroup can be constructed so.*

Proof. It follows directly from Lemma 2.6 and Theorem 4.5. $\square$

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