Research Article

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S-shaped connected component of positive solutions for second-order discrete Neumann boundary value problems

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Abstract: By using the bifurcation method, we study the existence of an S-shaped connected component in the set of positive solutions for discrete second-order Neumann boundary value problem. By figuring the shape of unbounded connected component of positive solutions, we show that the Neumann boundary value problem has three positive solutions suggesting suitable conditions on the weight function and non-linearity.

Keywords: three positive solutions, discrete, Neumann boundary value problem, bifurcation

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1 Introduction

In this paper, we are concerned with the existence and multiplicity of positive solutions for the following second-order Neumann boundary value problem

$$\begin{cases} -\Delta^2 u(n-1) + q(n)u(n) = \lambda m(n)f(u(n)), & n \in [1, N]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(N) = 0, \end{cases} \quad (1.1)$$

where $N > 2$ is an integer, Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$, λ is a positive real parameter, and $f \in C([0, +\infty), [0, +\infty))$, $f(0) = 0$, $f(s) > 0$ for all $s > 0$, $m : [1, N]_{\mathbb{Z}} \rightarrow (0, +\infty)$ with $0 < m_* \leq m(n) < m^*$ for some $m_*, m^* \in (0, +\infty)$.

Recently, second-order Neumann boundary value problems have attracted the attention of many specialists both in differential equations and difference equations because of their interesting applications, see [1–10] and references therein. In [1], Feltrin and Sovrano studied the second-order Neumann boundary value problem

$$\begin{cases} u'' + a(t)g(u) = 0, & t \in [0, T], \\ u'(0) = u'(T) = 0, \end{cases} \quad (1.2)$$

where the weight function $a(\cdot)$ changes its sign. Based upon a shooting method, they showed that (1.2) has three positive solutions suggesting suitable conditions on the weight function and nonlinearity. In another paper, the same authors [2] successfully established some further multiplicity results for the positive

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solutions of problem (1.2). More precisely, they obtained at least eight positive solutions via the shooting method. Moreover, Boscaggin [3], Boscaggin and Zanolin [4] and Boscaggin and Garrione [6] also studied the existence and multiplicity of positive solutions for problem (1.2). For more details, we refer the reader to [11] and references therein.

On the other hand, in high-dimensional case, several results of existence and multiplicity can be found for Neumann boundary value problems associated with

$$-\Delta u + k(x)u = m(x)g(u), \quad x \in \Omega \subset \mathbb{R}^N, \quad (1.3)$$

where $k(x) > 0$ (see [12,13] for some classical results in this direction). Clearly, (1.1) is the one-dimensional discrete case of (1.3).

However, for problem (1.1), although it is quite relevant from the point of view of applications, not sufficiently developed yet. Recently, Long and Chen [8] established the existence of multiple solutions to problem (1.1). In fact, they considered the problem

$$\begin{cases} -\Delta(p(n-1)\Delta u(n-1)) + q(n)u(n) = \lambda f(n, u(n)), & n \in [1, N]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(N) = 0, \end{cases} \quad (1.4)$$

where $f : [1, N]_{\mathbb{Z}} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous in the second variable, $p : [0, N]_{\mathbb{Z}} \rightarrow (0, +\infty)$ satisfies $p(0) = p(1)$ and $q : [1, N]_{\mathbb{Z}} \rightarrow [0, +\infty)$. By using the invariant set of descending flow and variational method, they proved that problem (1.4) has at least three nontrivial solutions: one is positive, one is negative and one is sign-changing.

Motivated by the aforementioned studies, in this paper, we employ a bifurcation technique of Sim and Tanaka [14] and prove that an unbounded subcontinuum of positive solutions of (1.1) bifurcates from the trivial solution and grows to the right from the initial point, to the left at some point and to the right near $\lambda = +\infty$. Roughly speaking, we obtain that there exists an *S-shaped connected component* in the positive solution set of problem (1.1). As a by-product, we assert further that (1.1) has one, two or three positive solutions for λ lying in various intervals in $\mathbb{R}$.

Let S be the set of sequences $u = (u(0), u(1), \dots, u(N), u(N+1))$. Let $W = \{u \in S \mid \Delta u(0) = \Delta u(N) = 0\}$ with the inner product

$$\langle u, v \rangle = \sum_{n=1}^N (\Delta u(n-1)\Delta v(n-1) + q(n)u(n)v(n)).$$

Then W is an N -dimensional Hilbert space and the induced norm is

$$\|u\|_1 = \left(\sum_{n=1}^N |\Delta u(n-1)|^2 + q(n)|u(n)|^2 \right)^{\frac{1}{2}}, \quad u \in S.$$

Let E be an n -dimensional Hilbert space equipped with the usual inner product $(\cdot, \cdot)$ and the usual norm $\|\cdot\|$, then W is isomorphic to E , $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent.

We first study the following eigenvalue problem:

$$\begin{cases} -\Delta^2 u(t-1) + q(t)u(t) = \lambda m(t)u(t), & t \in [1, N]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(N) = 0. \end{cases} \quad (1.5)$$

Lemma 1.1. [15] Problem (1.5) possesses N real and simple eigenvalues $\mu_k(q)$, which can be ordered as

$$0 < \mu_1(q) < \dots < \mu_N(q),$$

and the corresponding eigenfunction ϕ_k changes its sign exactly $k-1$ times on $[1, N]_{\mathbb{Z}}$. For simplicity, denote $\mu_1(q)$ to be μ_1 , and the associated eigenfunction ϕ is positive.

Remark 1.1. The eigenvalue μ_1 is the minimum of the “Rayleigh quotient,” that is

$$K(v) = \frac{\sum_{j=1}^N (|\Delta v(j)|^2 + q(j)v^2(j))}{\sum_{j=1}^N m(j)v^2(j)}$$

and

$$\mu_1 = \inf \left\{ K(v) \mid v \in W \text{ and } \sum_{j=1}^N m(j)v^2(j) > 0 \right\}.$$

Furthermore, we assume that

(H1) there exist $\alpha > 0$, $f_0 > 0$ and $f_1 > 0$ such that $\lim_{s \rightarrow 0^+} \frac{f(s) - f_0 s}{s^{1+\alpha}} = -f_1$;

(H2) $f_\infty := \lim_{s \rightarrow \infty} \frac{f(s)}{s} = 0$.

Let η_1 be the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta^2 u(t-1) = \eta m(t)u(t), & t \in [1, \hat{t}-1]_{\mathbb{Z}}, \\ u(0) = u(\hat{t}) = 0, \end{cases} \quad (1.6)$$

and the corresponding eigenfunction ω_1 is positive, $\frac{N}{2} \leq \hat{t} \leq \frac{N+1}{2}$.

Furthermore, we assume that

(H3) there exists $s_0 > 0$ such that

$$\min_{s \in [\sigma s_0, s_0]} \frac{f(s)}{s} \geq \frac{f_0}{\mu_1 m_*} (\eta_1 + \hat{q}),$$

where $\hat{q} = \max_{s \in [0, N+1]_{\mathbb{Z}}} q(s)$, $0 < \sigma < 1$ be as in (2.2).

It is easy to find that if (H1) holds, then

$$\lim_{s \rightarrow 0^+} \frac{f(s)}{s} = f_0.$$

Our main results are as follows.

Theorem 1.1. (see Figure 1) Assume that (H1)–(H3) hold. Then there exist $\lambda_* \in \left(0, \frac{\mu_1}{f_0}\right)$ and $\lambda^* > \frac{\mu_1}{f_0}$ such that

- (i) (1.1) has at least one positive solution if $\lambda = \lambda_*$;
- (ii) (1.1) has at least two positive solutions if $\lambda_* < \lambda \leq \frac{\mu_1}{f_0}$;
- (iii) (1.1) has at least three positive solutions if $\frac{\mu_1}{f_0} < \lambda < \lambda^*$;
- (iv) (1.1) has at least two positive solutions if $\lambda = \lambda^*$;
- (v) (1.1) has at least one positive solution if $\lambda > \lambda^*$.

Figure 1: S-shaped connected component of positive solutions of problem (1.1).

For other results concerning the existence of an S-shaped connected component in the set of solutions for diverse boundary value problems, see [16–18].

The rest of this paper is arranged as follows. In Section 2, we state some notations and preliminary results and obtain a global bifurcation phenomenon from the trivial branch with the rightward direction. In Section 3, we show the change of direction of bifurcation and complete the proof of Theorem 1.1.

2 Preliminaries and rightward bifurcation

Define $\mathcal{L} : W \rightarrow W$ by setting

$$\mathcal{L}u := -\Delta^2 u(t-1) + q(t)u(t).$$

Here $\mathcal{L} := P + Q$, where

$$P = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{N \times N}, \quad Q = \begin{pmatrix} q(1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & q(2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & q(3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q(N-1) & 0 \\ 0 & 0 & 0 & \cdots & 0 & q(N) \end{pmatrix}_{N \times N}.$$

Next, we consider the following boundary value problem:

$$\begin{cases} -\Delta^2 u(n-1) + q(n)u(n) = h(n), & n \in [1, N]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(N) = 0, \end{cases} \quad (2.1)$$

where $h : [1, N]_{\mathbb{Z}} \rightarrow \mathbb{R}$. Then (2.1) and the system of linear algebra equations $(P + Q)u = h$ are equivalent. Therefore, the unique solution of (2.1) is

$$u = (P + Q)^{-1}h.$$

In addition, we have the following lemma.

Lemma 2.1. [7, Lemma 2.1] *Let $A = \frac{1}{2}(q(n) + 2 + \sqrt{q^2(n) + 4q(n)})$, $\rho = (A^N - A^{-N})(A^2 - 1)$, then the unique solution of (2.1) can be expressed by*

$$u(n) = \sum_{s=1}^N G(n, s)h(s), \quad n \in [0, N+1]_{\mathbb{Z}},$$

where

$$G(n, s) = \frac{1}{\rho} \begin{cases} (A^s + A^{-s+1})(A^{n-N} + A^{N-n+1}) & 0 \leq s \leq n \leq N+1, \\ (A^n + A^{-n+1})(A^{s-N} + A^{N-s+1}) & 0 \leq n \leq s \leq N. \end{cases}$$

In addition, $G(n, s) > 0$ for all $(n, s) \in [0, N+1]_{\mathbb{Z}} \times [1, N]_{\mathbb{Z}}$.

Denote

$$m = \min_{0 \leq n, s \leq N+1} G(n, s), \quad M = \max_{0 \leq n, s \leq N+1} G(n, s), \quad \sigma = m/M. \quad (2.2)$$

Obviously, $0 < m < M$ and $0 < \sigma < 1$ (see [7]).

We extend f to a continuous function $\tilde{f}$ defined by

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \geq 0, \\ -f(-s) & \text{if } s < 0. \end{cases}$$

Obviously, within the context of positive solutions, problem (1.1) is equivalent to the same problem with f replaced by $\tilde{f}$. Furthermore, $\tilde{f}$ is an odd function for $s \in \mathbb{R}$. In the sequel of the proof we shall replace f with $\tilde{f}$. However, for the sake of simplicity, the modified function $\tilde{f}$ will still be denoted by f .

Lemma 2.2. $\left(\frac{\mu_1}{f_0}, 0\right)$ is a bifurcation point of problem (1.1) and the corresponding bifurcation branch C in $\mathbb{R} \times W$ whose closure contains $\left(\frac{\mu_1}{f_0}, 0\right)$ is either unbounded or contains a pair $(\bar{\lambda}, 0)$, where $\bar{\lambda}$ is an eigenvalue of problem (1.5) and $\bar{\lambda} \neq \mu_1$.

Proof. Let $\xi, \zeta \in C(\mathbb{R}, \mathbb{R})$ be such that $f(u) = f_0 u + \xi(u)$, $f(u) = f_\infty u + \zeta(u)$, clearly

$$\lim_{u \rightarrow 0} \frac{\xi(u)}{u} = 0, \quad \lim_{u \rightarrow \infty} \frac{\zeta(u)}{u} = 0.$$

Let us consider

$$\begin{cases} -\Delta^2 u(n-1) + q(n)u(n) = \lambda m(n)(f_0 u + \xi(u)), & n \in [1, N]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(N) = 0, \end{cases} \quad (2.3)$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

By Lemma 2.1, problem (2.3) can be equivalently written as

$$u(n) = \lambda Lu + H(\lambda, u),$$

where

$$Lu = f_0 \sum_{j=1}^N G(n, j)m(j)u(j), \quad H(\lambda, u) = \sum_{j=1}^N G(n, j)\lambda m(j)\xi(u(j)).$$

Then it is well known that L and $H : \mathbb{R} \times E \rightarrow E$ are completely continuous (see [8]).

By a simple calculation, we have

$$\lim_{u \rightarrow 0} \frac{\|H(\lambda, u)\|}{\|u\|} = 0$$

with respect to λ varying in bounded intervals.

Denote by S the closure in $\mathbb{R} \times W$ of the set of all non-trivial solutions (λ, u) of (2.3) with $\lambda > 0$. Theorem 1.3 in [19] yields the existence of a maximal closed connected set C in S such that $\left(\frac{\mu_1}{f_0}, 0\right) \in C$ and at least one of the following conditions holds:

- (i) C is unbounded in $\mathbb{R} \times W$;
- (ii) there exists a characteristic value of $\mathcal{L}$, with $\hat{\mu} \neq \mu_1$, such that $(\hat{\mu}, 0) \in C$. The proof is complete. $\square$

Next, we shall prove that the first choice of the alternative of Lemma 2.2 is the only possibility.

Let P denote the set of functions in W which are positive in $[0, N+1]_{\mathbb{Z}}$. Let $K = \mathbb{R} \times P$ under the product topology.

Lemma 2.3. Let $C \subset \left(K \cup \left\{\left(\frac{\mu_1}{f_0}, 0\right)\right\}\right)$, then the last alternative of Lemma 2.2 is impossible.

Proof. Suppose on the contrary, if there exists $(\lambda_n, u_n) \rightarrow (\bar{\lambda}, 0)$, when $n \rightarrow +\infty$ with $(\lambda_n, u_n) \in C$, $u_n \neq 0$, $\bar{\lambda} \neq \mu_1/f_0$. Let $v_n := u_n/\|u_n\|$, then $\|v_n\| = 1$ and v_n is the solution of the problem:

$$v(n) = \mathcal{L}^{-1}\lambda_n m(n) \left(f_0 v(n) + \frac{\xi(u(n))}{\|u_n\|} \right).$$

Because v_n is bounded, so there is a subsequence of v_n , again denote by $\{v_n\}$, such that $v_n \rightarrow v_0$ as $n \rightarrow +\infty$. It is easy to see that $(\bar{\lambda}, v_0)$ verifies problem (1.5). On the other hand, we can easily show that the bifurcation points must be eigenvalues. Thus, C does not join to $(0, 0)$ because 0 is not the eigenvalue of problem (1.5).

Clearly, $(0, 0)$ is the only solution of problem (1.1) for $\lambda = 0$. Hence, we have $C \cap (\{0\} \times W) = \emptyset$. It follows that $\bar{\lambda} \neq \mu_1/f_0$. Lemma 1.1 follows v_0 must change its sign, and as a consequence for some n large enough, u_n must change sign. This is a contradiction. $\square$

Lemma 2.4. Assume (H1). Let $\{(\lambda_n, u_n)\}$ be a sequence of positive solutions to (1.1) which satisfies $\lambda_n \rightarrow \frac{\mu_1}{f_0}$ and $\|u_n\| \rightarrow 0$. Let ϕ be the positive eigenfunction corresponding to μ_1 , which satisfies $\|\phi\| = 1$. Then there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $\frac{u_n}{\|u_n\|}$ converges uniformly to ϕ on $[0, N+1]_{\mathbb{Z}}$.

Proof. Set $v_n := \frac{u_n}{\|u_n\|}$, since v_n is bounded, a subsequence of $\{v_n\}$ uniformly converges to a limit $v \in W$ with $\|v\| = 1$, and we again denote by $\{v_n\}$ the subsequence.

For every (λ_n, u_n) , we have

$$u_n(t) = \lambda_n \sum_{j=1}^N G(t, j) m(j) (f_0 u_n(j) + \xi(u_n(j))). \quad (2.4)$$

Dividing the both sides of (2.4) by $\|u_n\|$, we get

$$v_n(t) = \lambda_n \sum_{j=1}^N G(t, j) m(j) \left(f_0 v_n(j) + \frac{\xi(u_n(j))}{\|u_n\|} \right).$$

Since $u_n(t) \rightarrow 0$ for all $t \in [0, N+1]_{\mathbb{Z}}$, we conclude that $\frac{\xi(u_n(t))}{\|u_n\|} \rightarrow 0$ for each fixed $t \in [0, N+1]_{\mathbb{Z}}$.

Lebesgue's dominated convergence theorem shows that

$$v(t) = \mu_1 \sum_{j=1}^N G(t, j) m(j) v(j),$$

for each fixed $t \in [0, N+1]_{\mathbb{Z}}$, which means that v is a nontrivial solution of (1.5) with $\lambda = \mu_1$, and hence $v \equiv \phi$. $\square$

Lemma 2.5. Let $u, \varphi \in W$. Then

$$\sum_{n=1}^N \varphi(n) \Delta^2 u(n-1) = - \sum_{n=1}^N \Delta u(n) \Delta \varphi(n).$$

Proof. Since $\Delta u(0) = \Delta u(N) = 0$, we have

$$\begin{aligned} \sum_{l=1}^N \varphi(l) \Delta^2 u(l-1) &= \sum_{j=0}^{N-1} \varphi(j+1) (\Delta u(j+1) - \Delta u(j)) = \sum_{l=1}^N \Delta u(l) \varphi(l) - \sum_{j=0}^{N-1} \Delta u(j) \varphi(j+1) \\ &= \left(\Delta u(N) \varphi(N) + \sum_{l=1}^{N-1} \Delta u(l) \varphi(l) \right) - \left(\Delta u(0) \varphi(1) + \sum_{j=1}^{N-1} \Delta u(j) \varphi(j+1) \right) \\ &= (\Delta u(N) \varphi(N) - \Delta u(0) \varphi(1)) + \sum_{l=1}^{N-1} \Delta u(l) (\varphi(l) - \varphi(l+1)) \\ &= - \sum_{l=1}^N \Delta u(l) \Delta \varphi(l). \quad \square \end{aligned}$$

Next, we give an important Lemma which will be used later.

Lemma 2.6. Assume (H1). Then there exists $\delta > 0$ such that for all $(\lambda, u) \in C$ and $|\lambda - \frac{\mu_1}{f_0}| + \|u\| \leq \delta$ imply $\lambda > \frac{\mu_1}{f_0}$.

Proof. Assume to the contrary that there exists a sequence $\{(\lambda_n, u_n)\} \subset C$ such that $\lambda_n \rightarrow \frac{\mu_1}{f_0}$, $\|u_n\| \rightarrow 0$ and $\lambda_n \leq \frac{\mu_1}{f_0}$. By Lemma 2.4, there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $\frac{u_n}{\|u_n\|}$ converges

uniformly to ϕ on $[0, N + 1]_{\mathbb{Z}}$, where ϕ is the positive eigenfunction corresponding to μ_1 , which satisfies $\|\phi\| = 1$. Multiplying equation (1.1) with $(\lambda, u) = (\lambda_n, u_n)$ by u_n and summation it over $[1, N]_{\mathbb{Z}}$, we obtain

$$\lambda_n \sum_{j=1}^N m(j)f(u_n(j))u_n(j) = \sum_{j=1}^N (-\Delta^2 u_n(j-1) + q(j)u_n(j))u_n(j).$$

By simple computation and using the definition of μ_1 in Remark 1.1, we get

$$\lambda_n \sum_{j=1}^N m(j)f(u_n(j))u_n(j) = \sum_{j=1}^N (\Delta u_n(j))^2 + \sum_{j=1}^N q(j)u_n^2(j) \geq \mu_1 \sum_{j=1}^N m(j)u_n^2(j),$$

that is,

$$\sum_{j=1}^N m(j) \frac{f(u_n(j)) - f_0 u_n(j)}{(u_n(j))^{1+\alpha}} \left| \frac{u_n(j)}{\|u_n\|} \right|^{2+\alpha} \geq \frac{\mu_1 - f_0 \lambda_n}{\lambda_n \|u_n\|^\alpha} \sum_{j=1}^N h(j) \frac{u_n^2(j)}{\|u_n\|^2}.$$

Lebesgue's dominated convergence theorem, condition (H1) and Lemma 2.4 imply that

$$\sum_{j=1}^N m(j) \frac{f(u_n(j)) - f_0 u_n(j)}{(u_n(j))^{1+\alpha}} \left| \frac{u_n(j)}{\|u_n\|} \right|^{2+\alpha} \rightarrow -f_1 \cdot \sum_{j=1}^N m(j) \phi^{2+\alpha}(j) < 0$$

and

$$\sum_{j=1}^N m(j) \frac{u_n^2(j)}{\|u_n\|^2} \rightarrow \sum_{j=1}^N m(j) \phi^2(j) > 0.$$

This contradicts $\lambda_n < \frac{\mu_1}{f_0}$. □

Definition 2.1. [20] We say that a solution $y(t)$ of problem (1.1) has a generalized zero at t_0 provided that $y(t_0) = 0$ if $t_0 = 0$ and if $t_0 > 0$ either $y(t_0) = 0$ or $y(t_0 - 1)y(t_0) < 0$.

By an argument similar to proving [20, Theorem 6.5] or [21, Lemma 3.3] with obvious changes, we may obtain the following result.

Lemma 2.7. Let $P_k \geq p_k$ for $k \in [m, n + 1]_{\mathbb{Z}}$. Also let $y(k), z(k)$ be solutions of the following difference equations:

$$\Delta^2 y(k) + p_k y(k + 1) = 0,$$

$$\Delta^2 z(k) + P_k z(k + 1) = 0,$$

respectively. If $y(m) = y(n + 1) = 0$ but without any generalized zeros in $[m + 1, n]_{\mathbb{Z}}$, then either there exists $\tau \in [m + 1, n]_{\mathbb{Z}}$ such that τ is a generalized zero of z or $P_k = p_k$ and $\frac{\Delta y(k)}{y(k)} = \frac{\Delta z(k)}{z(k)}$.

3 Direction turns of component and proof of Theorem 1.1

In this section, we show that there is a direction turn of the bifurcation under condition (H3) and accordingly we finish the proof of Theorem 1.1.

Lemma 3.1. Assume (H3). Let u be a positive solution of (1.1) with $\|u\| = s_0$, then $\lambda < \frac{\mu_1}{f_0}$.

Proof. Let u be a positive solution of (1.1) with $\|u\| = s_0$. Without loss of generality, assume that there exists $x_0 \in [0, N + 1]_{\mathbb{Z}}$ such that $\|u\| = u(x_0)$. This together with the fact

$$\min_{t \in [0, N+1]_{\mathbb{Z}}} u(t) \geq \sigma \|u\|, \quad 0 < \sigma < 1$$

implies that

$$\sigma s_0 = \sigma \|u\| \leq u(t) \leq \|u\| = s_0, \quad t \in [0, \hat{t}]_{\mathbb{Z}}.$$

We note that u is a solution of

$$\Delta^2 u(t-1) - q(t)u(t) + \lambda m(t) \frac{f(u(t))}{u(t)} u(t) = 0, \quad t \in [0, \hat{t}]_{\mathbb{Z}}.$$

Now we assume $\lambda \geq \frac{\mu_1}{f_0}$. Then for $t \in [0, \hat{t}]_{\mathbb{Z}}$, by (H3), we get

$$\lambda m(t) \frac{f(u(t))}{u(t)} - q(t) \geq \frac{\mu_1}{f_0} m_* \frac{f_0}{\mu_1 m_*} (\eta_1 + \hat{q}) - \hat{q} = \eta_1.$$

Since η_1 is the principal eigenvalue of the eigenvalue problem (1.6). By Lemma 2.7, we know that u has at least one generalized zero on $[0, \hat{t}]_{\mathbb{Z}}$. This contradicts the fact that $u(t) > 0$ on $[0, \hat{t}]_{\mathbb{Z}}$. $\square$

Lemma 3.2. Assume that (H1)–(H3) hold. Then C joins $\left(\frac{\mu_1}{f_0}, 0\right)$ to $(-\infty, +\infty)$.

Proof. Assume on the contrary that there exists λ_M a blow up point and $\lambda_M < +\infty$. Then there exists a sequence $\{(\lambda_n, u_n)\}$ satisfying $\lambda_n + \|u_n\| \rightarrow +\infty$, such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda_M$ and $\lim_{n \rightarrow \infty} \|u_n\| = +\infty$. We note that $\lambda_n > 0$ for all $n \in \mathbb{N}$, since $(0, 0)$ is the only solution of problem (2.3) for $\lambda = 0$ and $C \cap (\{0\} \times W) = \emptyset$.

Since $(\lambda_n, u_n) \in C$, we divide the equation

$$\mathcal{L}u_n = \lambda_n m(t) f_{\infty} u_n(t) + \lambda_n m(t) \zeta(u_n(t))$$

by $\|u_n\|$ and set $\omega_n := \frac{u_n}{\|u_n\|}$. Since ω_n is bounded, after taking a subsequence if necessary, we have that $\omega_n \rightarrow \omega$ for some $\omega \in W$ with $\|\omega\| = 1$. Let $\tilde{\zeta}(u) = \max_{u \leq s \leq 2u} |\zeta(s)|$, then $\tilde{\zeta}$ is nondecreasing and

$$\lim_{u \rightarrow \infty} \frac{\tilde{\zeta}(u)}{u} = 0.$$

Since

$$\frac{|\zeta(u_n(t))|}{\|u_n\|} \leq \frac{|\tilde{\zeta}(u_n(t))|}{\|u_n\|} \leq \frac{|\tilde{\zeta}(\|u_n(t)\|)|}{\|u_n\|}.$$

Then

$$\lim_{n \rightarrow \infty} \frac{|\zeta(u_n(t))|}{\|u_n\|} = 0.$$

Combining the fact that $f_{\infty} = 0$, we have

$$\omega(t) = \lambda_M \sum_{j=1}^N G(t, j) m(j) f_{\infty} \omega(j) = \lambda_M \sum_{j=1}^N G(t, j) m(j) 0 \omega(j) = 0,$$

again choosing a subsequence and relabeling if necessary. This contradicts with $\|\omega\| = 1$. $\square$

Proof of Theorem 1.1. By Lemmas 2.3 and 2.4, C is bifurcating from $\left(\frac{\mu_1}{f_0}, 0\right)$ and goes rightward. By Lemma 3.2, there exists $(\lambda_0, u_0) \in C$ such that $\|u_0\| = s_0$ and $\lambda_0 < \frac{\mu_1}{f_0}$. By Lemmas 2.4, 3.1 and 3.2, C passes through some points $\left(\frac{\mu_1}{f_0}, v_1\right)$ and $\left(\frac{\mu_1}{f_0}, v_2\right)$ with $\|v_1\| < s_0 < \|v_2\|$, and there exist $\underline{\lambda}$ and $\bar{\lambda}$ which satisfy $0 < \underline{\lambda} < \frac{\mu_1}{f_0} < \bar{\lambda}$ and both (i) and (ii):

- (i) if $\lambda \in \left(\frac{\mu_1}{f_0}, \bar{\lambda}\right]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in C$ and $\|u\| < \|v\| < s_0$;
- (ii) if $\lambda \in \left[\underline{\lambda}, \frac{\mu_1}{f_0}\right]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in C$ and $\|u\| < s_0 < \|v\|$.

Define $\lambda^* = \sup\{\bar{\lambda} : \bar{\lambda} \text{ satisfies (i)}\}$ and $\lambda_* = \inf\{\underline{\lambda} : \underline{\lambda} \text{ satisfies (ii)}\}$. Then (1.1) has a positive solution u_{λ} at $\lambda = \lambda_*$ and u_{λ^*} at $\lambda = \lambda^*$, respectively. Clearly, the component C turns to the left at $(\lambda^*, \|u_{\lambda^*}\|_{\infty})$ and to the right at $(\lambda_*, \|u_{\lambda_*}\|_{\infty})$. This completes the proof. $\square$

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