Research Article

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A modified Tikhonov regularization method based on Hermite expansion for solving the Cauchy problem of the Laplace equation

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Abstract: In this paper, a Cauchy problem for the Laplace equation is considered. We develop a modified Tikhonov regularization method based on Hermite expansion to deal with the ill posed-ness of the problem. The regularization parameter is determined by a discrepancy principle. For various smoothness conditions, the solution process of the method is uniform and the convergence rate can be obtained self-adaptively. Numerical tests are also carried out to verify the effectiveness of the method.

Keywords: Cauchy problem for Laplace equation, ill-posed problem, Tikhonov regularization, Hermite approximation, discrepancy principle

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1 Introduction

The Cauchy problem for the Laplace equation appears in many applications such as non-destructive testing [1,2], engineering problems in geophysics and seismology [3], bioelectric field problems [4,5], and cardiology [6]. In general, the Cauchy problem for the Laplace equation is ill posed: the solution (if it exists) does not depend continuously on the boundary data, i.e., a small perturbation in the Cauchy data may lead to enormous error in its numerical approximation. Thus, some regularization techniques have to be introduced to obtain stable numerical solution.

Let $L^2(\mathbb{R})$, $H^p(\mathbb{R})$ be the usual Lebesgue and Sobolev spaces. $\|\cdot\|$ and $\|\cdot\|_p$ denote their corresponding norms in $L^2(\mathbb{R})$ and $H^p(\mathbb{R})$, respectively. In this paper, the following Cauchy problem for the Laplace equation in a strip domain is considered [3,7–10]:

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & -\infty < x < \infty, 0 < y < 1, \\ u(x, 0) &= g(x), & -\infty < x < \infty, \\ u_y(x, 0) &= 0, & -\infty < x < \infty. \end{aligned} \quad (1)$$

We need to determine $u(\cdot, y)$ for $0 < y \leq 1$ from the noisy measurement data $g^\delta(x)$ which satisfies

$$\|g^\delta - g\| \leq \delta, \quad (2)$$

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where $\delta > 0$ represents a bound on the measurement error. We take the definition of the Fourier transforms of a function g to be

$$\hat{g}(\xi) = \mathcal{F}[g(x)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} g(x) dx.$$

By applying the Fourier transform technique, it is easy to deduce that the solution of (1) can be given by

$$u(x, y) = \mathcal{F}^{-1}[\hat{g}(\xi) \cosh(y|\xi|)] =: T_y g(x). \quad (3)$$

It is obvious that $\hat{g}(\xi)$ must decay rapidly as $|\xi| \rightarrow \infty$. But for the Fourier transform of noisy data, such a decay cannot be expected. Some techniques have been developed for solving linear ill-posed inverse problems in partial differential equations: wavelet regularization method [3,9], mollification method [7,8], Fourier regularization method [10,11], dynamical regularization method [12], etc.

In [13], authors of this paper have proposed a truncated Hermite expansion method for problem (1). The method is effective but the a priori smoothness assumption on the exact data which is used to obtain convergence result is not natural. It is not easy to verify in practical application. In this paper, we focus on finding a new approach to overcome this limitation. Similar to [9,10], we assume for some $p \geq 0$, the following a priori bound exists

$$\|u(\cdot, 1)\|_p \leq E. \quad (4)$$

In fact, under conditions (2) and (4), we can obtain the stable solution of the problem by using the classical Tikhonov method: let f_α^δ be the minimizer of the Tikhonov functional

$$\|T_1^{-1}f - g^\delta\|^2 + \alpha \|f\|_p^2, \quad (5)$$

where $\alpha > 0$ is the regularization parameter. Then

$$u^\delta(x, y) = \mathcal{F}^{-1} \left[\hat{f}_\alpha^\delta(\xi) \frac{\cosh(y|\xi|)}{\cosh(|\xi|)} \right] \quad (6)$$

can be used as the approximation of $u(x, y)$. And if α is determined by

$$\|T_1^{-1}f_\alpha^\delta - g^\delta\| = C\delta$$

with $C > 1$, then the convergence result can be obtained. The main problem with this procedure is that the value of p is usually unknown in practical applications. In this paper, we present a modified Tikhonov regularization method based on Hermite expansion. An improved functional without the value of p will be given and the convergence result can be obtained adaptively for various p .

The structure of the paper is as follows. We give the basic description of the method in Section 2. Error estimate can be found in Section 3 and we show some numerical tests to verify the effectiveness of the method in Section 4.

2 A modified Tikhonov regularization method based on Hermite expansion

Let $H_k(x)$ be the normalized Hermite function of degree k defined by the recursion

$$\begin{aligned} H_0(x) &\equiv \pi^{-1/4} \exp(-(1/2)x^2), \\ H_1(x) &\equiv \pi^{-1/4} \sqrt{2}x \exp(-(1/2)x^2), \\ H_{k+1}(x) &= \sqrt{\frac{2}{k+1}} x H_k(x) - \sqrt{\frac{k}{k+1}} H_{k-1}(x), \quad k \geq 1. \end{aligned}$$

They satisfy the orthogonality relations

$$\int_{\mathbb{R}} H_k(x) H_l(x) dx = \begin{cases} 1, & \text{if } k = l, \\ 0, & \text{otherwise.} \end{cases}$$

For any $h \in \mathbb{L}^2(\mathbb{R})$, we may write

$$h(x) = \sum_{k=0}^{\infty} \mathbf{h}_k H_k(x),$$

where

$$\mathbf{h}_k = \int_{\mathbb{R}} h(x) H_k(x) dx.$$

For any Fourier-Hermite coefficients vector $\vec{\mathbf{h}} = (\mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_n, \dots)^T \in l^2$, we define the following operators:

$$(\mathcal{H}\vec{\mathbf{h}})(x) = \sum_{k=0}^{\infty} \mathbf{h}_k H_k(x), \quad \mathcal{R}\vec{\mathbf{h}} = \mathcal{H}^{-1} \mathcal{F}^{-1} \left[\widehat{\mathcal{H}\vec{\mathbf{h}}}(\xi) \cosh(\xi) \right], \quad \mathcal{P}_N \vec{\mathbf{h}} = \mathcal{H}^{-1} \mathcal{F}^{-1} \left[\widehat{\mathcal{H}\vec{\mathbf{h}}}(\xi) \chi_N(\xi) \right], \quad (7)$$

where χ_N is the characteristic function of the interval $[-N, N]$.

Let $K = T_1^{-1}$, now we consider to solve the ill-posed operator equation

$$Kh = g^\delta. \quad (8)$$

To this end, for $\alpha > 0$, we denote by $\vec{\mathbf{h}}_\alpha^\delta$ the minimizer of the modified functional

$$\|\mathcal{K}\vec{\mathbf{h}} - g^\delta\|^2 + \alpha \|\mathcal{R}\vec{\mathbf{h}}\|_{l^2}^2, \quad (9)$$

where $\mathcal{K} = K\mathcal{H}$. Then

$$h_\alpha^\delta = \mathcal{H}\vec{\mathbf{h}}_\alpha^\delta$$

will be chosen as the approximation solution of equation (8) and

$$u_\alpha^\delta(x, y) = T_y K h_\alpha^\delta(x) \quad (10)$$

will be used as the approximation solution of (1).

It can be deduced that the minimizer $\vec{\mathbf{h}}_\alpha^\delta$ can be obtained by solving the following equation:

$$(\mathcal{K}^* \mathcal{K} + \alpha \mathcal{R}^2) \vec{\mathbf{h}} = \mathcal{K}^* g^\delta. \quad (11)$$

Lemma 1. [14] If we let $C = \mathcal{K}\mathcal{R}^{-1}$, the regularized solution $\mathbf{f}_\beta^\delta$ defined by (11) possesses the representation

$$\vec{\mathbf{h}}_\alpha^\delta = \mathcal{R}^{-1} r_\alpha(C^* C) C^* g^\delta \quad \text{with} \quad r_\alpha(\beta) = \frac{1}{\beta + \alpha}. \quad (12)$$

In addition, the function $r_\alpha : (0, \|C\|^2] \rightarrow (0, \infty)$ obey the properties

$$\sup_{\beta > 0} \beta^{1/2} |r_\alpha(\beta)| \leq \frac{1}{2\sqrt{\alpha}}, \quad \sup_{\beta > 0} \beta |r_\alpha(\beta)| \leq 1 \quad (13)$$

and

$$\sup_{\beta > 0} \beta^{1/2} |1 - \beta r_\alpha(\beta)| \leq \frac{\sqrt{\alpha}}{2}, \quad \sup_{\beta > 0} |1 - \beta r_\alpha(\beta)| \leq 1. \quad (14)$$

3 Error estimate of regularization solution

Now we begin to derive the convergence result of the regularization solution. Let

$$f(x) = u(x, 1)$$

and $\vec{\mathbf{f}} = (\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_n, \dots)^T$ be its Fourier-Hermite coefficients vector. We take

$$\vec{\mathbf{f}}_N = \mathcal{P}_N \vec{\mathbf{f}}, \quad f_N = \mathcal{H} \vec{\mathbf{f}}_N. \quad (15)$$

First, we give some auxiliary results.

Lemma 2. [15] *Given the function $f(\rho) : (0, a] \rightarrow \mathbb{R}$ described by*

$$f(\rho) = \rho^b \left(d \ln \frac{1}{\rho} \right)^{-c} \quad (16)$$

with a constant $c \in \mathbb{R}$ and positive constants $a < 1$, b and d , then for the inverse function $f^{-1}(\eta)$ we have

$$f^{-1}(\rho) = \rho^{\frac{1}{b}} \left(\frac{d}{b} \ln \frac{1}{\rho} \right)^{\frac{c}{b}} (1 + o(1)), \quad \text{for } \rho \rightarrow 0. \quad (17)$$

Lemma 3. *If $\|f\|_p \leq E$ and $\vec{\mathbf{f}}$ and $\vec{\mathbf{f}}_N$ are defined as (15), then we have*

$$\begin{aligned} \|f - f_N\| &\leq N^{-p}E, \\ \|\mathcal{K} \vec{\mathbf{f}} - \mathcal{K} \vec{\mathbf{f}}_N\| &\leq 2e^{-N}N^{-p}E, \end{aligned} \quad (18)$$

and

$$\|\mathcal{R} \vec{\mathbf{f}}_N\|_{l^2} \leq C_N E,$$

where

$$C_N = \max \left(1, \frac{e^N}{2N^p} \right).$$

Proof. By using Parseval's formula, (7) and (3), we have

$$\begin{aligned} \|f - f_N\|^2 &= \|(I - \mathcal{P}_N) \vec{\mathbf{f}}\|_2^2 = \int_{|\xi| > N} |\hat{f}|^2 d\xi \\ &\leq N^{-2p} \int_{|\xi| > N} (1 + \xi^2)^p |\hat{f}|^2 d\xi \\ &\leq N^{-2p} \int_{\mathbb{R}} (1 + \xi^2)^p |\hat{f}|^2 d\xi = N^{-2p} \|f\|_p^2, \\ \|\mathcal{K}(\vec{\mathbf{f}} - \vec{\mathbf{f}}_N)\|^2 &= \int_{|\xi| > N} \cosh^{-2}(\xi) |\hat{f}(\xi)|^2 d\xi \\ &= \int_{|\xi| > N} \cosh^{-2}(\xi) (1 + |\xi|^2)^{-p} (1 + |\xi|^2)^p |\hat{f}(\xi)|^2 d\xi \\ &\leq 4e^{-2N} N^{-2p} \int_{|\xi| > N} (1 + |\xi|^2)^p |\hat{f}(\xi)|^2 d\xi \leq 4e^{-2N} N^{-2p} \|f\|_p^2, \end{aligned}$$

and

$$\|\mathcal{R} \vec{\mathbf{f}}_N\|_l^2 = \int_{|\xi| \leq N} \cosh^2(\xi) |\hat{f}(\xi)|^2 d\xi = \int_{|\xi| \leq N} \frac{\cosh^2(\xi)}{(1 + \xi^2)^p} (1 + \xi^2)^p |\hat{f}(\xi)|^2 d\xi \leq \max\left(1, \frac{e^{2N}}{4N^{2p}}\right) \|f\|_p^2. \quad \square$$

Lemma 4. Suppose that the vector sequence $\vec{\mathbf{f}}^\delta = (\mathbf{f}_0^\delta, \mathbf{f}_1^\delta, \dots, \mathbf{f}_n^\delta, \dots)^T$ satisfies

$$\|\mathcal{K} \vec{\mathbf{f}}^\delta\| \leq k_1 \delta, \quad \|\mathcal{R} \vec{\mathbf{f}}^\delta\|_{l^2} \leq \frac{k_2}{\left\{ \left(\ln \frac{k_3}{\delta} \right) \ln \left[\frac{k_3}{\delta} \left(\ln \frac{k_3}{\delta} \right)^{-p} \right] \right\}^p \delta}, \quad \delta \rightarrow 0, \quad (19)$$

where k_1, k_2, k_3 are some fixed constants, then there exists a constant $M > 0$ such that

$$\|\mathcal{H} \vec{\mathbf{f}}^\delta\|_p \leq M. \quad (20)$$

Proof. Let

$$N_0 = \ln \left[\frac{k_3}{\delta} \left(\ln \frac{k_3}{\delta} \right)^{-p} \right], \quad (21)$$

then by using the triangle inequality

$$\|\mathcal{H} \vec{\mathbf{f}}^\delta\| \leq \|\mathcal{H}(I - \mathcal{P}_{N_0}) \vec{\mathbf{f}}^\delta\| + \|\mathcal{H} \mathcal{P}_{N_0} \vec{\mathbf{f}}^\delta\| = I_1 + I_2.$$

For the first term I_1 , we have

$$\begin{aligned} I_1^2 &= \int_{|\xi| > N_0} (1 + |\xi|^2)^p \left| \widehat{\mathcal{H} \vec{\mathbf{f}}^\delta}(\xi) \right|^2 d\xi \\ &\leq \int_{|\xi| > N_0} \frac{(1 + |\xi|^2)^p}{\cosh^2(\xi)} \left| \cosh(\xi) \widehat{\mathcal{H} \vec{\mathbf{f}}^\delta}(\xi) \right|^2 d\xi \\ &\leq \frac{(N_0 + 1)^{2p}}{\cosh^2(N_0)} \int_{|\xi| > N_0} \left| \cosh(\xi) \widehat{\mathcal{H} \vec{\mathbf{f}}^\delta}(\xi) \right|^2 d\xi \\ &\leq \frac{4N_0^2}{e^{2(N_0-1)}} \|\mathcal{R} \vec{\mathbf{f}}^\delta\|_{l^2}^2 \\ &\leq 4e^{-2} k_2^2 k_3^{-2}. \end{aligned}$$

And the second term I_2 can be estimated as

$$\begin{aligned} I_2^2 &= \int_{|\xi| \leq N_0} (1 + |\xi|^2)^p \left| \widehat{\mathcal{H} \vec{\mathbf{f}}^\delta}(\xi) \right|^2 d\xi \\ &= \int_{|\xi| \leq N_0} (1 + |\xi|^2)^p \cosh^2(|\xi|) \left| \cosh^{-2}(\xi) \widehat{\mathcal{H} \vec{\mathbf{f}}^\delta}(\xi) \right|^2 d\xi \\ &\leq N_0^{2p} e^{2N_0} \int_{|\xi| \leq N_0} \left| \cosh^{-2}(\xi) \widehat{\mathcal{H} \vec{\mathbf{f}}^\delta}(\xi) \right|^2 d\xi \\ &\leq N_0^{2p} e^{2N_0} \|\mathcal{K} \vec{\mathbf{f}}^\delta\|^2 \\ &= k_3 \left(\ln \frac{k_3}{\delta} \right)^{-p} \left\{ \ln \left[\frac{k_3}{\delta} \left(\ln \frac{k_3}{\delta} \right)^{-p} \right] \right\}^p \rightarrow 0. \end{aligned}$$

These finish the proof. $\square$

Lemma 5. Suppose that the function sequences $r^\delta(x)$ satisfy

$$\|r^\delta\| \leq k_4 \delta \text{ and } \|T_1 r^\delta\|_p \leq k_5, \quad \delta \rightarrow 0, \quad (22)$$

where k_4, k_5 are two fixed constants, then we have

$$\|T_y r^\delta\| \sim O \left(\left[1 + \left(\frac{\ln \frac{1}{\delta}}{\ln \frac{1}{\delta} + \ln \left(\ln \frac{1}{\delta} \right)^{-p}} \right)^p \right] \delta^{1-y} \left(\ln \frac{1}{\delta} \right)^{-py} \right). \quad (23)$$

Proof. Let

$$\tau = \ln \left[\frac{1}{\delta} \left(\ln \frac{1}{\delta} \right)^{-p} \right], \quad (24)$$

then by using Parseval's formula and the triangle inequality

$$\begin{aligned} \|T_y r^\delta\| &= \|\cosh(y\xi) \widehat{r^\delta}(\xi)\| \\ &\leq \|\cosh(y\xi) \widehat{r^\delta}(\xi) \chi_\tau(\xi)\| + \|\cosh(y\xi) \widehat{r^\delta}(\xi) - \cosh(y\xi) \widehat{r^\delta}(\xi) \chi_\tau(\xi)\| \\ &= \left(\int_{|\xi| \leq \tau} \cosh^2(y\xi) |\widehat{r^\delta}(\xi)|^2 d\xi \right)^{1/2} + \left(\int_{|\xi| > \tau} \cosh^2(y\xi) |\widehat{r^\delta}(\xi)|^2 d\xi \right)^{1/2} \\ &\leq \left(\int_{|\xi| \leq \tau} \cosh^2(y\xi) |\widehat{r^\delta}(\xi)|^2 d\xi \right)^{1/2} + \left(\int_{|\xi| > \tau} \frac{\cosh^2(y\xi)}{(1 + |\xi|^2)^p \cosh^2(\xi)} (1 + |\xi|^2)^p |\cosh(\xi) \widehat{r^\delta}(\xi)|^2 d\xi \right)^{1/2} \\ &\leq \cosh(y\tau) \|r^\delta\| + \frac{\cosh(y\tau)}{\tau^p \cosh(\tau)} \|T_1 r^\delta\|_p \\ &\leq e^{y\tau} \|r^\delta\| + 2\tau^{-p} e^{(y-1)\tau} \|T_1 r^\delta\|_p. \end{aligned} \quad (25)$$

□

Now the statement of the theorem can be obtained by (24) and (25).

The main result of this paper is given as follows:

Theorem 6. Suppose that conditions (2) and (4) hold, u_α^δ is defined by (10). If we choose the regularization parameter α as the solution of scalar equation

$$\|\mathcal{K} \vec{\mathbf{h}}_\alpha^\delta - \mathbf{g}^\delta\| = C\delta \quad (26)$$

with $C > 1$, then

$$\|u_\alpha^\delta(\cdot, y) - u(\cdot, y)\| \sim O \left(\delta^{1-y} \left(\ln \frac{1}{\delta} \right)^{-py} \right), \quad \forall 0 \leq y \leq 1.$$

Proof. If $f_N, \vec{\mathbf{f}}_N$ are defined by (15), due to (2), (26) and by using the triangle inequality

$$\|\mathcal{K}(\vec{\mathbf{h}}_\alpha^\delta - \vec{\mathbf{f}}_N)\| \leq \|\mathcal{K} \vec{\mathbf{h}}_\alpha^\delta - \mathbf{g}^\delta\| + \|\mathbf{g}^\delta - \mathbf{g}\| + \|\mathcal{K}(\vec{\mathbf{f}} - \vec{\mathbf{f}}_N)\| \leq (C + 1)\delta + 2e^{-N} N^{-p} E. \quad (27)$$

If we define $\vec{\mathbf{f}}_{\alpha, N} = \mathcal{R}^{-1} r_\alpha(C^* C) C^* f_N$, then we have

$$\begin{aligned} \mathcal{R}(\vec{\mathbf{h}}_\alpha^\delta - \vec{\mathbf{f}}_{\alpha, N}) &= r_\alpha(C^* C) C^* (\mathbf{g}^\delta - \mathcal{K} \vec{\mathbf{f}}_N), \\ \mathcal{R}(\vec{\mathbf{f}}_N - \vec{\mathbf{f}}_{\alpha, N}) &= [I - r_\alpha(C^* C) C^* C] \mathcal{R} \vec{\mathbf{f}}_N. \end{aligned} \quad (28)$$

Hence, in terms of the triangle inequality, (13), (18), and (28)

$$\begin{aligned}
\|\mathcal{R}(\vec{\mathbf{h}}_\alpha^\delta - \vec{\mathbf{f}}_N)\|_{l^2} &\leq \|\mathcal{R}(\vec{\mathbf{h}}_\alpha^\delta - \vec{\mathbf{f}}_{\alpha,N})\|_{l^2} + \|\mathcal{R}(\vec{\mathbf{f}}_N - \vec{\mathbf{f}}_{\alpha,N})\|_{l^2} \\
&\leq \frac{1}{2\sqrt{\alpha}} \|g^\delta - \mathcal{K}\vec{\mathbf{f}}_N\| + \|\mathcal{R}\vec{\mathbf{f}}_N\|_{l^2} \\
&\leq \frac{1}{2\sqrt{\alpha}} (\delta + 2e^{-N}N^{-p}E) + C_N E.
\end{aligned} \tag{29}$$

Let $S_\alpha = I - r_\alpha(CC^*)CC^*$, note that $g^\delta - \mathcal{K}\vec{\mathbf{h}}_\alpha^\delta = S_\alpha g^\delta$, then from the triangle inequality, (14), and (18), we have

$$\begin{aligned}
\|\mathcal{K}\vec{\mathbf{h}}_\alpha^\delta - g^\delta\| &\leq \|S_\alpha(g^\delta - g)\| + \|S_\alpha(g - \mathcal{K}\vec{\mathbf{f}}_N)\| + \|S_\alpha\mathcal{K}\vec{\mathbf{f}}_N\| \\
&\leq \delta + \|g - \mathcal{K}\vec{\mathbf{f}}_N\| + \|S_\alpha C\| \cdot \|\mathcal{R}\vec{\mathbf{f}}_N\| \\
&\leq \delta + 2e^{-N}N^{-p}E + \frac{\sqrt{\alpha}}{2} C_N E.
\end{aligned}$$

Denote

$$\rho =: e^{-N},$$

then

$$N = \ln \frac{1}{\rho}$$

and (31) becomes

$$\rho \left(\ln \frac{1}{\rho} \right)^{-p} = \frac{C-1}{4} \delta,$$

i.e., $b = 1$, $d = 1$, and $c = -p$ in (16). Then by using (17), we obtain

$$\rho = \frac{C-1}{4} \delta \left(\ln \frac{4}{(C-1)\delta} \right)^p (1 + o(1)). \tag{30}$$

Taking the principal part of ρ given by (30) and if we choose N with

$$e^{-N}N^{-p}E = \frac{C-1}{4} \delta, \tag{31}$$

then we have

$$e^{-N} = \frac{C-1}{4} \delta \left(\ln \frac{4}{(C-1)\delta} \right)^p$$

and

$$N = \ln \left[\frac{4}{(C-1)\delta} \left(\ln \frac{4}{(C-1)\delta} \right)^{-p} \right],$$

then there exist constants k_1, k_2

$$\begin{aligned}
\|\mathcal{K}(\vec{\mathbf{f}}_\alpha^\delta - \vec{\mathbf{f}}_N)\| &\leq k_1 \delta, \\
\|\mathcal{R}(\vec{\mathbf{f}}_\alpha^\delta - \vec{\mathbf{f}}_N)\|_{l^2} &\leq \frac{k_2}{\left\{ \left(\ln \frac{k_3}{\delta} \right) \ln \left[\frac{k_3}{\delta} \left(\ln \frac{k_3}{\delta} \right)^{-p} \right] \right\}^p \delta}.
\end{aligned}$$

Hence, by using Lemma 4, there exists a constant M

$$\|\mathcal{H}(\vec{\mathbf{f}}_\alpha^\delta - \vec{\mathbf{f}}_N)\|_p \leq M.$$

So we can deduce that

$$\|\mathcal{H}\vec{\mathbf{f}}_{\alpha}^{\delta} - g\| = \|\mathcal{H}(\vec{\mathbf{f}}_{\alpha}^{\delta} - \vec{\mathbf{g}}_N)\| + \|g - g_N\| \leq \|\mathcal{H}(\vec{\mathbf{f}}_{\alpha}^{\delta} - \vec{\mathbf{g}}_N)\| + \|g\| \leq M + E. \quad (32)$$

From (2), (26), and by using the triangle inequality

$$\|\mathcal{H}\vec{\mathbf{f}}_{\alpha}^{\delta} - g\| \leq \|\mathcal{H}\vec{\mathbf{f}}_{\alpha}^{\delta} - g^{\delta}\| + \|g^{\delta} - g\| \leq (C + 1)\delta. \quad (33)$$

So the assertion is proved by (32), (33), and Lemma 5. $\square$

4 Numerical tests

In this section, to examine the effectiveness of the proposed method, we present numerical results of some examples. The discretization knots are $x_i = -B + ih$, $i = 0, 1, \dots, m$; $h = 2B/m$ with $m = 256$, where B is a positive constant and satisfies that $g(x)$ approach zero as $|x| > B$. The perturbed discrete data are given by

$$g^{\delta} = g + \text{randn}(\text{size}(g)) \times \delta_1, \quad (34)$$

where “randn(·)” is a Matlab function which generates normally distributed random numbers. The following relative errors are used to estimate the computational error of approximate solution.

$$\varepsilon_y = \frac{\sqrt{\frac{1}{m} \sum_{i=1}^m |u_{\alpha}^{\delta}(x_i, y) - u(x_i, y)|^2}}{\sqrt{\frac{1}{m} \sum_{i=1}^m |u(x_i, y)|^2}}. \quad (35)$$

Example 1. This example is given by Fu and his coworkers in [10]. It is easy to see that the function

$$u(x, y) = e^{y^2 - x^2} \cos(2xy)$$

is the exact solution of problem (1) with

$$g(x) = e^{-x^2}.$$

From Table 1, it can be seen that when the noise level δ_1 is decreased from 0.1 to 0.0001, the relative errors will decrease too. All of these numerical results show that the proposed method is effective. The comparison of the exact solution of problem (1) and its approximation for different noise levels and different locations y are shown in Figure 1. Here, the solid curves represent the exact solution and the dotted curves indicate approximation solutions. It is easy to see that the numerical results become worse with the increase of y . This accords with our theoretical results.

In general, an explicit analytical solution to (1) is difficult to obtain, we set forth the example as follows: take a $\psi(x) \in L^2(\mathbb{R})$ and solve the well-posed problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & -\infty < x < +\infty, & \quad 0 < y < 1, \\ u(x, 1) &= \psi(x), & -\infty < x < +\infty, \\ u_y(x, 0) &= 0, & -\infty < x < +\infty, \end{aligned} \quad (36)$$

to get an approximation for $g(x)$. Then put the noise to $g(x)$ to get g^{δ} .

Example 2. In this example, we take ψ as the following function:

$$\psi(x) = \begin{cases} x + 4, & -4 \leq x < 0; \\ -x + 4, & 0 \leq x \leq 4. \end{cases}$$

Table 1: Relative errors ε_y of Example 1

δ_1	α	ε_0	$\varepsilon_{0.5}$	ε_1
1×10^{-1}	8.23×10^{-03}	8.11×10^{-2}	1.34×10^{-1}	3.13×10^{-1}
1×10^{-2}	6.16×10^{-5}	1.02×10^{-2}	2.33×10^{-2}	9.3×10^{-2}
1×10^{-3}	2.18×10^{-3}	1.20×10^{-3}	3.58×10^{-3}	2.28×10^{-2}
1×10^{-4}	2.42×10^{-4}	1.36×10^{-4}	5.66×10^{-4}	7.10×10^{-3}

Table 2: Relative errors ε_y of Example 2

δ_1	α	ε_0	$\varepsilon_{0.5}$	ε_1
1×10^{-1}	6.13×10^{-1}	6.87×10^{-2}	7.17×10^{-2}	9.09×10^{-2}
1×10^{-2}	1.14×10^{-1}	6.83×10^{-3}	1.11×10^{-2}	3.60×10^{-2}
1×10^{-3}	1.42×10^{-4}	6.75×10^{-4}	2.24×10^{-3}	1.77×10^{-2}
1×10^{-4}	1.86×10^{-7}	6.74×10^{-5}	4.83×10^{-4}	9.59×10^{-3}

Table 3: Relative errors ε_y of Example 3

δ_1	α	ε_0	$\varepsilon_{0.5}$	ε_1
1×10^{-1}	4.44×10^{-2}	2.26×10^{-2}	3.66×10^{-2}	1.74×10^{-1}
1×10^{-2}	3.19×10^{-4}	2.30×10^{-3}	9.23×10^{-3}	1.28×10^{-1}
1×10^{-3}	1.42×10^{-7}	2.28×10^{-4}	2.40×10^{-3}	1.04×10^{-1}
1×10^{-4}	4.16×10^{-11}	2.30×10^{-5}	6.65×10^{-4}	8.97×10^{-2}

Example 3. In this example, we take ψ as the following function:

$$\psi(x) = \begin{cases} -5, & x < 0; \\ 5, & x \geq 0. \end{cases}$$

Tables 2, 3 and Figures 2, 3 have given the results of Examples 2 and 3. All of the results show that the method is also effective.

5 Conclusion

A Hermite extension method with a modified Tikhonov regularization for the Cauchy problem of the Laplace equation has been presented in this paper. The numerical results show that the method works well and coincides with the theoretical results. The main advantage of this method is that the convergence rates of the method are self-adaptive. Moreover, we point out that the framework of Hermite extension method can be applied to other ill-posed problems.

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Figure 1: The exact solution, the regularization solution and error for $\delta_1 = 0.01$ (Example 1). (a) $u(x, y)$, (b) $u_\alpha^\delta(x, y)$, and (c) $u_\alpha^\delta(x, y) - u(x, y)$.

Figure 2: The exact solution, the regularization solution and error for $\delta_1 = 0.01$ (Example 2). (a) $u(x, y)$, (b) $u_\alpha^\delta(x, y)$, and (c) $u_\alpha^\delta(x, y) - u(x, y)$.

Figure 3: The exact solution, the regularization solution and error for $\delta_1 = 0.01$ (Example 3). (a) $u(x, y)$, (b) $u_\alpha^\delta(x, y)$, and (c) $u_\alpha^\delta(x, y) - u(x, y)$.

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