Research Article

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Finite groups with some weakly pronormal subgroups

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Abstract: A subgroup H of a finite group G is called weakly pronormal in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is pronormal in G . In this paper, we investigate the structure of the finite groups in which some subgroups are weakly pronormal. Our results improve and generalize many known results.

Keywords: weakly pronormal subgroups, hypercenter, minimal subgroups, saturated formations

MSC 2010: 20D10, 20D20

1 Introduction

All groups considered in this paper are finite. Our notation and terminology are standard, and the reader is referred to [1,2]. Recall that a formation $\mathfrak{F}$ is a class of groups which is closed by taking epimorphic images and such that every group G has a smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$ -residual of G , and it is denoted by $G^{\mathfrak{F}}$. A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. Throughout this paper, $\mathfrak{N}_p$, $\mathfrak{N}$ and $\mathfrak{U}$ denote the classes of p -nilpotent groups, nilpotent groups and supersolvable groups, respectively.

The ways in which minimal subgroups can be embedded in a nilpotent, p -nilpotent or supersolvable group have been investigated by many scholars. For example, a famous result of Itô [3, III, 5.3 Satz] proved that a group G of odd order is nilpotent if every subgroup of G of prime order lies in the center of G . An extension of Itô's result is the following statement [3, IV, 5.5 Satz]: Let p be a prime dividing the order of a group G . If every element of G of order p lies in $Z(G)$ and, when $p = 2$, every element of G of order 4 also lies in $Z(G)$, then G is p -nilpotent. In 1970, Buckley [4, Theorem 3] proved that a group G of odd order is supersolvable if every minimal subgroup of G is normal in G . Later on, many authors studied the structure of a group G , under the assumption that some minimal subgroups of G have various embedding properties, for example, see [5–9].

Recall that a subgroup H of a group G is called pronormal in G if for each $g \in G$, the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$. Since P. Hall introduced the concept of pronormal subgroup, there has been much interest in investigating the topic on the pronormality [10–14]. Recently, Asaad in [15, Definition 1.1] introduced the definition of weakly pronormal subgroups, which is a generalization of pronormal subgroups. A subgroup H of a group G is called weakly pronormal in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is pronormal in G . It has been proved that the concept of weakly pronormal

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subgroup is suitable for describing the structure of groups [15,16]. In this paper, we go further to study the influence of weak pronormality of some subgroups on the structure of a group G . More precisely, we prove the following theorems:

Theorem 1.1. *Let $\mathfrak{F}$ be a saturated formation containing the class of all nilpotent groups $\mathfrak{N}$. Then $G \in \mathfrak{F}$ if and only if for every prime p dividing the order of $F^*(G^{\mathfrak{F}})$ and for every Sylow p -subgroup P of $F^*(G^{\mathfrak{F}})$, every minimal subgroup of $P \cap G^{\mathfrak{N}_p}$ is contained in the $\mathfrak{F}$ -hypercenter $Z_{\mathfrak{F}}(G)$ of G and every cyclic subgroup of $P \cap G^{\mathfrak{N}_p}$ of order 4 (if $p = 2$) is weakly pronormal in G .*

Theorem 1.2. *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{U}$, and let G be a group such that G is S_4 -free and quaternion-free. Then $G \in \mathfrak{F}$ if and only if for every prime p dividing the order of $F^*(G^{\mathfrak{F}})$ and for every Sylow p -subgroup P of $F^*(G^{\mathfrak{F}})$, every minimal subgroup of $P \cap G^{\mathfrak{N}_p}$ is weakly pronormal in $N_G(P)$.*

Remark 1.3. Theorem 1.2 is not true for saturated formations which do not contain $\mathfrak{U}$. For example, if $\mathfrak{F}$ is the saturated formation of all nilpotent groups, then the symmetric group of degree 3 is a counterexample.

2 Preliminaries

In this section, we show some lemmas, which are required in the proofs of our main results.

Lemma 2.1. [15, Lemma 2.2] *Let H be a weakly pronormal subgroup of a group G . Then the following statements are true:*

- (1) H is a weakly pronormal subgroup in K for every subgroup K of G with $H \leq K$.
- (2) Let N be a normal subgroup of G . Then HN/N is weakly pronormal in G/N if $N \leq H$ or $(|H|, |N|) = 1$.

Lemma 2.2. [1, Lemma 6.3] *If a subgroup H of a group G is both pronormal and subnormal in G , then H is normal in G .*

Lemma 2.3. *Let P be a normal p -subgroup of a group G . Then P is contained in the hypercenter $Z_{\infty}(G)$ of G if and only if $O^p(G) \leq C_G(P)$, where $O^p(G)$ is the unique smallest normal subgroup of G for which the corresponding factor group is a p -group.*

Proof. If $O^p(G) \leq C_G(P)$, then $G/C_G(P)$ is a p -group and so $P \leq Z_{\infty}(G)$ by [17, Appendix C, Theorem 6.3]. The converse is [18, Lemma 2.8]. $\square$

Recall that the generalized Fitting subgroup $F^*(G)$ of a group G is the set of all elements x of G which induce an inner automorphism on every chief factor of G (see [19, X. 13.9]). We list the basic properties which will be used as follows.

Lemma 2.4. *Let M be a subgroup of a group G .*

- (1) *If M is normal in G , then $F^*(M) \leq F^*(G)$.*
- (2) *If $G \neq 1$, then $F^*(G) \neq 1$.*
- (3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*
- (4) *If $M \leq Z(G)$, then $F^*(G/M) = F^*(G)/M$.*

Proof. (1)–(3) can be found in [19, X. 13]. (4) can be found in [20, Lemma 2.3]. $\square$

Let $\mathfrak{F}$ be a saturated formation. A chief factor H/K of a group G is said to be $\mathfrak{F}$ -central if $G/C_G(H/K) \in \mathfrak{F}$. A normal subgroup N of G is called $\mathfrak{F}$ -hypercentral in G provided either $N = 1$ or $N \neq 1$ and every chief factor

of G below N is $\mathfrak{F}$ -central in G . The product of all $\mathfrak{F}$ -hypercentral subgroups of G is again an $\mathfrak{F}$ -hypercentral subgroup of G . It is denoted by $Z_{\mathfrak{F}}(G)$ and called the $\mathfrak{F}$ -hypercenter of G . For the formation $\mathfrak{N}$, $Z_{\infty}(G)$ becomes $Z_{\mathfrak{N}}(G)$ in this notation (see [1, IV, 6.8]).

Lemma 2.5. [1, IV, Theorem 6.10] *If $\mathfrak{F}$ is a saturated formation, then $[G^{\mathfrak{F}}, Z_{\mathfrak{F}}(G)] = 1$.*

Lemma 2.6. [21, Lemma 2] *Let $\mathfrak{F}$ be a saturated formation. Assume that G is a non- $\mathfrak{F}$ -group and there exists a maximal subgroup M of G such that $M \in \mathfrak{F}$ and $G = F(G)M$, where $F(G)$ is the Fitting subgroup of G . Then*

- (i) $G^{\mathfrak{F}}/(G^{\mathfrak{F}})'$ is a chief factor of G .
- (ii) $G^{\mathfrak{F}}$ is a p -group for some prime p .
- (iii) $G^{\mathfrak{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$.
- (iv) $G^{\mathfrak{F}}$ is either an elementary abelian group or $(G^{\mathfrak{F}})' = Z(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$ is an elementary abelian group.

Lemma 2.7. *Let G be an S_4 -free group. Also let p be the smallest prime dividing the order of G and let P be a Sylow p -subgroup of G . If every minimal subgroup of $P \cap G^{\mathfrak{N}_p}$ is weakly pronormal in $N_G(P)$ and when $p = 2$ P is quaternion-free, then G is p -nilpotent.*

Proof. This is a special case of [16, Main Theorem] since we always assume that $P \cap P^x \cap G^{\mathfrak{N}_p} = P \cap G^{\mathfrak{N}_p}$ when $G \setminus N_G(P)$ is empty. $\square$

3 Main results

For the purpose of giving proof of Theorem 1.1, we first prove the following result for a group to be p -nilpotent.

Theorem 3.1. *Suppose that H is a normal subgroup of a group G such that G/H is p -nilpotent, where p is a fixed prime. Let P be a Sylow p -subgroup of H . Then G is p -nilpotent if and only if every minimal subgroup of $P \cap G^{\mathfrak{N}_p}$ is contained in $Z_{\infty}(G)$ and every cyclic subgroup of $P \cap G^{\mathfrak{N}_p}$ of order 4 (if $p = 2$) is weakly pronormal in G .*

Proof. If G is p -nilpotent, then $G^{\mathfrak{N}_p} = 1$, nothing needs to be proved. So we only need to prove that the converse is true.

Assume that the result is false and let G be a counter example of minimal order. Let K be a proper subgroup of G . It is clear that $K/(K \cap H) \cong HK/H \leq G/H$ is p -nilpotent. Since $K/(K \cap G^{\mathfrak{N}_p}) \cong KG^{\mathfrak{N}_p}/G^{\mathfrak{N}_p} \leq G/G^{\mathfrak{N}_p}$, we have $K^{\mathfrak{N}_p} \leq G^{\mathfrak{N}_p}$. Now let P_1 be a Sylow p -subgroup of $K \cap H$. Without loss of generality, we may assume that $P_1 \leq P$. If A is a minimal subgroup of $P_1 \cap K^{\mathfrak{N}_p}$, then $A \leq K \cap Z_{\infty}(G) \leq Z_{\infty}(K)$. By Lemma 2.1, we can conclude that every cyclic subgroup of $P_1 \cap K^{\mathfrak{N}_p}$ with order 4 (if $p = 2$) is weakly pronormal in K . Hence, K satisfies the hypotheses of the theorem for its normal subgroup $K \cap H$. The minimality of G implies that K is p -nilpotent. By a result of Schmidt [2, Theorem 9.1.9 and Exercise 9.1.11], we know that G has a normal Sylow p -subgroup G_p and a cyclic Sylow q -subgroup G_q such that $G = G_p \rtimes G_q$, G_p is of exponent p when p is odd and of exponent at most 4 when $p = 2$. Moreover, $G_p/\Phi(G_p)$ is a minimal normal subgroup of $G/\Phi(G_p)$ and $\Phi(G_p) \leq Z(G)$.

Since G/G_p is nilpotent, we obtain that $G^{\mathfrak{N}_p} \leq G_p$. If $G^{\mathfrak{N}_p} < G_p$, then we can see that $G^{\mathfrak{N}_p}\Phi(G_p)/\Phi(G_p) \trianglelefteq G/\Phi(G_p)$, which means that $G_p = G^{\mathfrak{N}_p}\Phi(G_p) \leq \Phi(G)$. Therefore, $G/\Phi(G)$ is nilpotent and so is G , which is a contradiction. It follows that $G_p = G^{\mathfrak{N}_p}$. Furthermore, we have that $G/(G_p \cap H)$ is p -nilpotent as $G/(G_p \cap H) \leq G/G_p \times G/H$. If $G_p \not\leq H$, then $G_p \cap H < G_p$ and $G_q(G_p \cap H) < G$. The minimal choice of G yields that $G_q(G_p \cap H)$ is nilpotent. On the other hand, we obtain that $G_q(G_p \cap H)/(G_p \cap H) \trianglelefteq G/(G_p \cap H)$ by the p -nilpotency of $G/(G_p \cap H)$. Thus $G_q \trianglelefteq G$, which is a contradiction. Hence, we may assume that $G_p \leq H$ and so $G_p = P$.

If $p > 2$, then $P \leq Z_\infty(G)$ from the hypotheses. Thus, $G = PG_q = P \times G_q$ is nilpotent by Lemma 2.3, against the choice of G . Consequently, $p = 2$. Let $x \in P \setminus \Phi(P)$. If x has order 2, then $\langle x^G \rangle \Phi(P) / \Phi(P)$ is normal in $G / \Phi(P)$, which implies that $P = \langle x^G \rangle \Phi(P) = \langle x^G \rangle \leq Z_\infty(G)$, a contradiction. Hence, x is of order 4. Since $\langle x \rangle$ is weakly pronormal in G , there exists a subgroup K of G such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K$ is pronormal in G . If $[G : K] = 4$, then, using the fact that $x^2 \in \Phi(P) \leq Z(G)$, we obtain that $K\langle x^2 \rangle$ is a subgroup of G with index 2 and therefore $K\langle x^2 \rangle$ is normal in G . This implies that the Sylow q -subgroup of $K\langle x^2 \rangle$ is normal in G and therefore G is nilpotent, which is a contradiction. If $[G : K] = 2$, then K itself is a normal subgroup of G with index 2. We still obtain a contradiction. It follows that $G = K$. The pronormality of $\langle x \rangle$ in G combined with Lemma 2.2 yields $\langle x \rangle \trianglelefteq G$. Thus, $P = \langle x \rangle$ is cyclic. Again we infer that G is 2-nilpotent, and this leads to the final contradiction. $\square$

Since a group G is nilpotent if and only if G is p -nilpotent for every $p \in \pi(G)$, by Theorem 3.1, we have:

Theorem 3.2. *Suppose that H is a normal subgroup of a group G such that G/H is nilpotent. Then G is nilpotent if and only if for every prime p dividing the order of H and for every Sylow p -subgroup P of H , every minimal subgroup of $P \cap G^{\mathfrak{N}_p}$ is contained in $Z_\infty(G)$ and every cyclic subgroup of $P \cap G^{\mathfrak{N}_p}$ of order 4 (if $p = 2$) is weakly pronormal in G .*

Moreover, we have the following result.

Theorem 3.3. *Let H be a normal subgroup of G such that G/H is nilpotent. Then G is nilpotent if and only if for every prime p dividing the order of $F^*(H)$ and for every Sylow p -subgroup P of $F^*(H)$, every minimal subgroup of $P \cap G^{\mathfrak{N}_p}$ is contained in $Z_\infty(G)$ and every cyclic subgroup of $P \cap G^{\mathfrak{N}_p}$ of order 4 (if $p = 2$) is weakly pronormal in G .*

Proof. We only need to prove the sufficiency. Assume that the result is false and let G be a counterexample of minimal order. We split the proof into the following steps:

- (1) Every proper normal subgroup of G is nilpotent and $F(G)$ is the unique maximal normal subgroup of G .

Let M be a maximal normal subgroup of G . Then it is easy to see that $M/(M \cap H) \cong MH/H \leq G/H$ is nilpotent. Employing Lemma 2.4, $F^*(M \cap H) \leq F^*(H)$. Now let P_1 be a Sylow p -subgroup of $F^*(M \cap H)$. Without loss of generality, we may assume that $P_1 \leq P$. For any minimal subgroup A of $P_1 \cap M^{\mathfrak{N}_p}$, we have $A \leq M \cap Z_\infty(G) \leq Z_\infty(M)$. By Lemma 2.1, we can see that every cyclic subgroup of $P_1 \cap M^{\mathfrak{N}_p}$ with order 4 (if $p = 2$) is weakly pronormal in M . So M satisfies the hypotheses of the theorem for its normal subgroup $M \cap H$. The minimal choice of G implies that M is nilpotent. Since the class of all nilpotent groups is a Fitting class, G has a unique maximal normal subgroup M . The nilpotency of M means that $M = F(G)$.

- (2) $H = G = G'$ and $F^*(G) = F(G) < G$.

If $H < G$, then H is nilpotent by (1), and so $F^*(H) = F(H) = H$. By Theorem 3.2, we have that G is nilpotent, against the choice of G . Hence, we may assume that $H = G$. From (1), we know that $G/F(G)$ is a simple group. If $G/F(G)$ is cyclic of prime order, then from Theorem 3.2 G is nilpotent, which is a contradiction. Thus, $G/F(G)$ is a non-abelian simple group. We apply $G' \not\leq F(G)$ to obtain $G' = G$. If $F(G) < F^*(G)$, then $F^*(H) = F^*(G) = G$. Again, by Theorem 3.2, we see that G is nilpotent, this contradiction shows $F^*(G) = F(G)$.

- (3) Let p be the smallest prime dividing the order of $F(G)$ and let P^* be the Sylow p -subgroup of $F(G)$. Then $G/C_G(P^*)$ is a p -group.

Obviously, P^* is normal in G . Let Q be a Sylow q -subgroup of G , where $q > p$. Then P^*Q is a subgroup of G . We claim that P^*Q is nilpotent. Since $Z_\infty(G) \cap P^*Q \leq Z_\infty(P^*Q)$, our hypotheses and Lemma 2.1 combine to give that every cyclic subgroup of $P^* \cap (P^*Q)^{\mathfrak{N}_p}$ of order 4 is weakly pronormal in P^*Q and every minimal subgroup of $P^* \cap (P^*Q)^{\mathfrak{N}_p}$ is contained in $Z_\infty(P^*Q)$. Theorem 3.1 yields that P^*Q is p -nilpotent. Thus, Q is normal in P^*Q and so $P^*Q = P^* \times Q$. Consequently, $O^p(G) \leq C_G(P^*)$ and therefore $G/C_G(P^*)$ is a p -group.

(4) The final contradiction.

By (2) and (3), we have that $G = C_G(P^*)$ and so $P^* \leq Z(G)$. Set $\bar{G} = G/P^*$. It is clear from Lemma 2.4 that $F^*(\bar{G}) = F^*(G)/P^*$. Let R be a Sylow r -subgroup of $F^*(G)$ with $r \neq p$. We can see that $\bar{R} \cap \bar{G}^{\mathfrak{N}_p} = (R \cap G^{\mathfrak{N}_p})P^*/P^*$ as $\bar{G}^{\mathfrak{N}_p} = G^{\mathfrak{N}_p}P^*/P^*$. Furthermore, every minimal subgroup of $\bar{R} \cap \bar{G}^{\mathfrak{N}_p}$ is contained in $Z_{\infty}(\bar{G})$ as $Z_{\infty}(\bar{G}) = Z_{\infty}(G)/P^*$, and every cyclic subgroup of $\bar{R} \cap \bar{G}^{\mathfrak{N}_p}$ of order 4 (if $p = 2$) is weakly pronormal in $\bar{G}$ by Lemma 2.1. Therefore, $\bar{G}$ satisfies the hypotheses of our theorem. The minimality of G means that $\bar{G}$ is nilpotent and so is G , which is a contradiction. $\square$

Proof of Theorem 1.1. Certainly, it is only the sufficiency of the condition that is in doubt.

Assume that the result is false and let G be a counterexample of minimal order. Then $G^{\mathfrak{F}} \neq 1$ and $Z_{\mathfrak{F}}(G) < G$. If $G^{\mathfrak{F}} \leq \Phi(G)$, then $G/\Phi(G) \cong (G/G^{\mathfrak{F}})/(\Phi(G)/G^{\mathfrak{F}}) \in \mathfrak{F}$ and so is G as $\mathfrak{F}$ is saturated, and this contradiction forces that there exists a maximal subgroup M of G such that $G = MG^{\mathfrak{F}}$. Let P be a Sylow p -subgroup of $F^*(G^{\mathfrak{F}})$. If $\langle x \rangle$ is a subgroup of $P \cap (G^{\mathfrak{F}})^{\mathfrak{N}_p}$ of order p , then from our hypotheses and Lemma 2.5 $\langle x \rangle$ is contained in $Z(G^{\mathfrak{F}}) \leq Z_{\infty}(G^{\mathfrak{F}})$. By Lemma 2.1, every cyclic subgroup of $P \cap (G^{\mathfrak{F}})^{\mathfrak{N}_p}$ of order 4 (if $p = 2$) is weakly pronormal in $G^{\mathfrak{F}}$. Theorem 3.3 implies that $G^{\mathfrak{F}}$ is nilpotent. Hence, $F^*(G^{\mathfrak{F}}) = G^{\mathfrak{F}} \leq F(G)$ and $G = MF(G)$. Furthermore, we claim that $P \leq G^{\mathfrak{N}_p}$. Let P^* be a Sylow p -subgroup of G . As $G/G^{\mathfrak{N}_p}$ is p -nilpotent, we obtain that $P^*G^{\mathfrak{N}_p} \cap O^p(G)G^{\mathfrak{N}_p} = G^{\mathfrak{N}_p}$ and so $P^* \cap O^p(G) \leq G^{\mathfrak{N}_p}$, which means that $P \cap O^p(G) = P \cap G^{\mathfrak{N}_p}$. A similar argument shows that $P \cap O^p(G) = P \cap G^{\mathfrak{N}_1}$. On the other hand, $P \leq G^{\mathfrak{F}} \leq G^{\mathfrak{N}_1}$ since $G/G^{\mathfrak{N}_1}$ is nilpotent. Thus, our claim holds.

By [22, Theorem 3.5], we may choose M to be an $\mathfrak{F}$ -critical maximal subgroup and $G/M_G \notin \mathfrak{F}$, it follows that $Z_{\mathfrak{F}}(G) \leq M$. Moreover, the G -chief factor A/B below $Z_{\mathfrak{F}}(G)$ is actually an M -chief factor and $\text{Aut}_M(A/B)$ is isomorphic to $\text{Aut}_G(A/B)$ because $F(G)$ centralizes A/B . Consequently, $Z_{\mathfrak{F}}(G)$ is contained in $Z_{\mathfrak{F}}(M)$. Since $M/(M \cap G^{\mathfrak{F}}) \cong MG^{\mathfrak{F}}/G^{\mathfrak{F}} \leq G/G^{\mathfrak{F}} \in \mathfrak{F}$, we have that $M^{\mathfrak{F}} \leq M \cap G^{\mathfrak{F}}$. Now, every subgroup of $P_1 \cap M^{\mathfrak{N}_p}$ of order p is contained in $Z_{\mathfrak{F}}(M)$, and every cyclic subgroup of $P_1 \cap G^{\mathfrak{N}_p}$ of order 4 is weakly pronormal in M by Lemma 2.1 again, where $P_1 \in \text{Syl}_p(F^*(M^{\mathfrak{F}}))$. It follows that M satisfies the hypotheses of the theorem. The minimality of G implies that $M \in \mathfrak{F}$.

By Lemma 2.6, $G^{\mathfrak{F}}$ is a p -group for some prime p . If $G^{\mathfrak{F}}$ is elementary abelian, then $G^{\mathfrak{F}} = \Omega_1(G^{\mathfrak{F}}) \leq Z_{\mathfrak{F}}(G)$ and so $G/Z_{\mathfrak{F}}(G) \in \mathfrak{F}$, this would imply that $G \in \mathfrak{F}$, which is a contradiction. Hence, $p = 2$ and $G^{\mathfrak{F}}$ has exponent at most 4. Let $x \in G^{\mathfrak{F}} \setminus \Phi(G^{\mathfrak{F}})$ such that x is of order 2. Then $\langle x^G \rangle \Phi(G^{\mathfrak{F}})/\Phi(G^{\mathfrak{F}})$ is normal in $G/\Phi(G^{\mathfrak{F}})$, which means that $G^{\mathfrak{F}} = \langle x^G \rangle \Phi(G^{\mathfrak{F}}) = \langle x^G \rangle \leq Z_{\mathfrak{F}}(G)$ as $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G , which is a contradiction. So for any $x \in G^{\mathfrak{F}} \setminus \Phi(G^{\mathfrak{F}})$, x has order 4. By hypotheses, there exists a subgroup K of G such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K$ is pronormal in G . Suppose that $\langle x \rangle$ is not normal in G . If $K = G$, then, from Lemma 2.2 and the subnormality of $\langle x \rangle$ in G , $\langle x \rangle \trianglelefteq G$, and this contradiction forces that $K < G$. Let G_1 be a maximal subgroup of G containing K . Since $x^2 \in \Phi(G^{\mathfrak{F}}) \leq \Phi(G) \leq G_1$, we can see that $|G : G_1| = 2$. It follows that $G_1 \trianglelefteq G$ and G/G_1 is a 2-group, which implies $x \in G^{\mathfrak{F}} \leq G_1$, a contradiction. Hence, $\langle x \rangle$ is normal in G . Now we see that $G = \langle x \rangle M$ and $x^2 \in M$, so M is normal in G . We can conclude from the fact that G/M is a 2-group that $G^{\mathfrak{F}} \leq M$, which is a final contradiction. This completes the proof. $\square$

Proof of Theorem 1.2. Only the sufficiency needs to be shown.

Let P be a Sylow p -subgroup of $F^*(G^{\mathfrak{F}})$, where $p \in \pi(F^*(G^{\mathfrak{F}}))$. Our result is given in the following three steps:

(1) $F^*(G^{\mathfrak{F}}) = F(G^{\mathfrak{F}})$.

Suppose that p is a smallest prime dividing the order of $F^*(G^{\mathfrak{F}})$. By Lemma 2.1, we know that every minimal subgroup of $P \cap F^*(G^{\mathfrak{F}})^{\mathfrak{N}_p}$ is weakly pronormal in $N_{F^*(G^{\mathfrak{F}})}(P)$. Applying Lemma 2.7, we have that $F^*(G^{\mathfrak{F}})$ is p -nilpotent. In particular, $F^*(G^{\mathfrak{F}})$ is solvable, by Lemma 2.4, $F^*(G^{\mathfrak{F}}) = F(G^{\mathfrak{F}})$.

(2) $F(G^{\mathfrak{F}}) \leq Z_{\mathfrak{F}}(G)$.

For completing our statement, we only need to show that $P \leq Z_{\mathfrak{F}}(G)$. Employing (1), we obtain that $P \leq G^{\mathfrak{N}_p}$ and so every minimal subgroup of $F(G^{\mathfrak{F}})$ is weakly pronormal in $N_G(P) = G$. If $p = 2$, then PQ is

a subgroup of G , where Q is a Sylow q -subgroup of G and $q \neq 2$. In view of Lemma 2.7, we can see that PQ is 2-nilpotent. It follows that $PQ = P \times Q$. Now by using Lemma 2.3, we have $P \leq Z_{\infty}(G)$. It follows from [1, IV, Proposition 3.11] that $P \leq Z_{\mathfrak{F}}(G)$. Hence, we may assume that $p > 2$. Let A be a minimal subgroup of $F(G^{\mathfrak{F}})$, then from our hypotheses G has a subgroup K such that $G = AK$ and $A \cap K$ is pronormal in G . If $A \cap K \neq 1$, then the subnormality of A in G and Lemma 2.2 combine to give $A = A_G$. We apply [23, Theorem 1.1] to obtain that $P \leq Z_{\mathfrak{U}}(G)$ and so $P \leq Z_{\mathfrak{F}}(G)$ by [1, IV, Proposition 3.11] again.

(3) Conclusion

By Lemma 2.5 and (2), we obtain that $G^{\mathfrak{F}} \leq C_G(Z_{\mathfrak{F}}(G)) \leq C_G(F(G^{\mathfrak{F}}))$, which means $G/C_G(F(G^{\mathfrak{F}})) \in \mathfrak{F}$. It follows that $G/C_{G^{\mathfrak{F}}}(F(G^{\mathfrak{F}})) \in \mathfrak{F}$. Since $C_{G^{\mathfrak{F}}}(F(G^{\mathfrak{F}})) \leq F(G^{\mathfrak{F}})$, we see that $G/F(G^{\mathfrak{F}}) \in \mathfrak{F}$. (2) leads to $G \in \mathfrak{F}$. $\square$

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