Research Article

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Results on analytic functions defined by Laplace-Stieltjes transforms with perfect ϕ -type

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Abstract: In this paper, we introduce the concept of the perfect ϕ -type to describe the growth of the maximal molecule of Laplace-Stieltjes transform by using the more general function than the usual. Based on this concept, we investigate the approximation and growth of analytic functions $F(s)$ defined by Laplace-Stieltjes transforms convergent in the half plane and obtain some results about the necessary and sufficient conditions on analytic functions $F(s)$ defined by Laplace-Stieltjes transforms with perfect ϕ -type, which are some generalizations and improvements of the previous results given by Kong [*On generalized orders and types of Laplace-Stieltjes transforms analytic in the right half-plane*, Acta Math. Sin. **59A** (2016), 91–98], Singhal and Srivastava [*On the approximation of an analytic function represented by Laplace-Stieltjes transformations*, Anal. Theory and Appl. **31** (2015), 407–420].

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1 Introduction

Let $L(s, F, \alpha)$ be a class of Laplace-Stieltjes transforms

$$F(s) = \int_0^{+\infty} e^{sx} d\alpha(x), \quad s = \sigma + it, \quad (1.1)$$

where $\alpha(x)$ is a bounded variation on any finite interval $[0, Y]$ ($0 < Y < +\infty$), and σ and t are real variables. If we choose a suitable $\alpha(t)$, then $F(s)$ can be represented as a Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}$, where a_n ($n = 1, 2, \dots$) are nonzero complex numbers, and the sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfies

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n \uparrow +\infty. \quad (1.2)$$

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Denote

$$A_n^* = \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^x e^{ity} d\alpha(y) \right|,$$

if

$$\limsup_{n \rightarrow +\infty} \frac{\log A_n^*}{\lambda_n} = 0 \quad (1.3)$$

and

$$\limsup_{n \rightarrow +\infty} (\lambda_{n+1} - \lambda_n) = h < +\infty, \quad \limsup_{n \rightarrow +\infty} \frac{n}{\lambda_n} = D < \infty, \quad (1.4)$$

then in view of the Valiron-Knopp-Bohr formula [1], it leads to $\sigma_u^F = 0$, that is, $F(s)$ is analytic in the left half plane, where σ_u^F is the abscissa of uniform convergence of $F(s)$. If (1.3) is replaced by

$$\limsup_{n \rightarrow +\infty} \frac{\log A_n^*}{\lambda_n} = -\infty, \quad (1.5)$$

then it yields $\sigma_u^F = +\infty$, that is, $F(s)$ is called as an entire function. For convenience, for $-\infty < \beta < \infty$, we denote

$$\begin{aligned} \bar{L}_\beta &:= \{F(s) \in L(s, F, \alpha) \mid \sigma_u^F = \beta, \text{ and } \{\lambda_n\} \text{ satisfy (1.2), (1.4)}\}, \\ L_0 &:= \{F(s) \in L(s, F, \alpha) \mid A_n^*, \{\lambda_n\} \text{ satisfy (1.2)–(1.4)}\} \end{aligned}$$

and

$$L_\infty := \{F(s) \in L(s, F, \alpha) \mid A_n^*, \{\lambda_n\} \text{ satisfy (1.2), (1.4), (1.5)}\}.$$

Thus, if $F(s) \in L_0$ and $-\infty < \beta < 0$, then $F(s) \in \bar{L}_\beta$.

In order to estimate the growth of $F(s)$, Yu [1] introduced the concepts of the maximal term $\mu(\sigma, F)$, the maximal molecule $M_u(\sigma, F)$ and the order of $F(s)$ and also studied that the value distribution of entire functions defined by Laplace-Stieltjes transforms converge in the complex plane. After his wonderful works, many scholars studied the value distribution and the growth of analytic functions represented by Laplace-Stieltjes transforms converge in the whole plane or the half plane, and obtained a large number of important and interesting results (see [2–14]).

Define

$$\mu(\sigma, F) = \max_{n \in \mathbb{N}} \{A_n^* e^{\lambda_n \sigma}\}, \quad M_u(\sigma, F) = \sup_{0 < x < +\infty, -\infty < t < +\infty} \left| \int_0^x e^{(\sigma+it)y} d\alpha(y) \right|, \quad (\sigma < 0).$$

For $F(s) \in L_0$, in view of $M_u(\sigma, F) \rightarrow +\infty$ as $\sigma \rightarrow 0^-$, the concepts of order and type can be usually used in estimating the growth of $F(s)$ precisely.

Definition 1.1. [15, Definition 1.1] If Laplace-Stieltjes transform $F(s) \in L_0$ satisfies, we define the order and the lower order of $F(s)$ as follows:

$$\rho = \limsup_{\sigma \rightarrow 0^-} \frac{\log^+ \log^+ M_u(\sigma, F)}{-\log(-\sigma)}, \quad \mu = \limsup_{\sigma \rightarrow 0^-} \frac{\log^+ \log^+ M_u(\sigma, F)}{-\log(-\sigma)}, \quad 0 \leq \mu \leq \rho \leq +\infty,$$

where $\log^+ x = \max\{\log x, 0\}$. Furthermore, if $\rho \in (0, +\infty)$, the type and the lower type of $F(s)$ are defined by

$$T = \limsup_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\left(-\frac{1}{\sigma}\right)^\rho}, \quad t = \liminf_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\left(-\frac{1}{\sigma}\right)^\rho}, \quad 0 \leq t \leq T \leq +\infty.$$

Remark 1.1. However, if $\rho = 0$ and $\rho = +\infty$, we cannot estimate the growth of such functions precisely by using the concept of type.

Remark 1.2. We say that $F(s)$ is of regular growth, if $\rho = \mu$; furthermore, $F(s)$ is of perfect regular growth, if $T = t$, that is,

$$T = \lim_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\left(-\frac{1}{\sigma}\right)^\rho}.$$

If Laplace-Stieltjes transform (1.1) satisfies $A_n^* = 0$ for $n \geq k + 1$, and $A_k^* \neq 0$, then $F(s)$ may be called as an exponential polynomial of degree k usually denoted by p_k , i.e., $p_k(s) = \int_0^{\lambda_k} \exp(sy) d\alpha(y)$. If we choose a suitable function $\alpha(y)$, the function $p_k(s)$ may be reduced to a polynomial in terms of $\exp(s\lambda_i)$, that is, $\sum_{i=1}^k b_i \exp(s\lambda_i)$. We denote Π_k to be the class of all exponential polynomials of degree almost k , that is,

$$\Pi_k = \left\{ \sum_{i=1}^k b_i \exp(s\lambda_i) : (b_1, b_2, \dots, b_k) \in \mathbb{C}^k \right\}.$$

For $F(s) \in \bar{L}_\beta$, $-\infty < \beta < 0$, we denote by $E_n(F, \beta)$ the error in approximating the function $F(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(F, \beta) = \inf_{p \in \Pi_n} \|F - p\|_\beta, \quad n = 1, 2, \dots,$$

where

$$\|F - p\|_\beta = \max_{-\infty < t < +\infty} |F(\beta + it) - p(\beta + it)|.$$

For convenience, the error $E_{n-1}(F, \beta)$ is always supposed to be not null.

For $F(s) \in L_0$, Singhal and Srivastava [16] in 2015 studied the approximation of $F(s)$ with finite order, and obtained as follows.

Theorem 1.1. [16] *If Laplace-Stieltjes transform $F(s) \in L_0$, and is of order ρ ($0 < \rho < \infty$) and of type T , then for any real number $-\infty < \beta < 0$, we have*

$$\rho = \limsup_{n \rightarrow +\infty} \frac{\log^+ \log^+ [E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}]}{\log \lambda_n - \log^+ \log^+ [E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}]}$$

and

$$T = \limsup_{n \rightarrow +\infty} \frac{(\log^+ [E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}])^{\rho+1}}{\left(\frac{\rho+1}{\rho}\right)^{\rho+1} \rho(\lambda_n)^\rho}. \quad (1.6)$$

From Remark 1.2 and Theorem 1.1, the following question will be suggested naturally:

Question 1.1. What will happen when $F(s)$ is of perfect regular growth, that is, $T = t$ in Theorem 1.1?

To answer Question 1.1, we first introduce the following definition of ϕ -type of Laplace-Stieltjes transform $F(s)$, which can estimate the growth of $M_u(\sigma, F)$ or $\mu(\sigma, F)$ more widely by utilizing more general function than $\left(-\frac{1}{\sigma}\right)^\rho$ in Definition 1.1.

Definition 1.2. Let $\mathcal{E}_0$ be the set of positive unbounded function ϕ on $(-\infty, 0)$ such that the derivative ϕ' is positive, continuous and increasing to $+\infty$ on $(-\infty, 0)$. If $\phi \in \mathcal{E}_0$ and Laplace-Stieltjes transform $F(s) \in L_0$ satisfies

$$T_\phi = \limsup_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\phi(\sigma)}, \quad 0 \leq T_\phi \leq +\infty,$$

then T is called the ϕ -type of $F(s)$. Similarly, the lower ϕ -type of $F(s)$ is defined by

$$t_\phi = \liminf_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\phi(\sigma)}, \quad 0 \leq t_\phi \leq +\infty.$$

Remark 1.3. Obviously, $0 \leq t_\phi \leq T_\phi \leq +\infty$. Besides, if $t_\phi = T_\phi$, then we say that $F(s)$ is of perfect ϕ -type, that is,

$$\lim_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\phi(\sigma)} = T_\phi.$$

Let φ be the inverse function of ϕ' , then φ is continuous on $(0, +\infty)$ and increases to 0, and let $\phi \in \Xi_0$ and $\psi(x) = x - \frac{\phi(x)}{\phi'(x)}$, then in view of [17], ψ is an increasing function on $(-\infty, 0)$, and $\psi(x) \rightarrow 0$ as $x \rightarrow 0$. Besides, let ψ^{-1} be the inverse function of ψ . Then ψ^{-1} is an increasing function on $(-\infty, 0)$ and $\phi'(\psi^{-1}(\sigma))$ increases to $+\infty$ on $(-\infty, 0)$. For $0 < a < b < +\infty$ and $q > 0$, let

$$G_1(a, b, q, \phi) = \frac{ab}{b-a} \int_a^b \frac{\phi(\varphi(qt))}{t^2} dt$$

and

$$G_2(a, b, q, \phi) = \phi \left(\frac{1}{b-a} \int_a^b \varphi(qt) dt \right).$$

Now, we list our main results below to show the relations among the perfect ϕ -type, the error $E_n(F, \beta)$, λ_n and A_n^* for Laplace-Stieltjes transforms $F(s)$ with the perfect ϕ -type.

Theorem 1.2. If Laplace-Stieltjes transform $F(s) \in L_0$ satisfies

$$\limsup_{\sigma \rightarrow 0^-} \frac{-\log(-\sigma)}{\phi(\sigma)} = 0, \quad (1.7)$$

then for any real number $-\infty < \beta < 0$ and $0 < T_\phi < +\infty$, we have

$$\lim_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\phi(\sigma)} = T_\phi \Leftrightarrow (i) \quad \limsup_{n \rightarrow +\infty} \frac{\lambda_n}{\Omega_n(F, \beta, \lambda_n, \phi', \psi^{-1})} = T_\phi;$$

(ii) There exists a non-decreasing positive integer sequence $\{n_v\}$ satisfying

$$\lim_{v \rightarrow +\infty} \frac{\lambda_{n_v}}{\Omega_{n_v}(F, \beta, \lambda_{n_v}, \phi', \psi^{-1})} = T_\phi \quad (1.8)$$

and

$$\lim_{v \rightarrow +\infty} \frac{G_1(\lambda_{n_v}, \lambda_{n_{v+1}}, T_\phi, \phi)}{G_2(\lambda_{n_v}, \lambda_{n_{v+1}}, T_\phi, \phi)} = 1, \quad (1.9)$$

where

$$\Omega_n(F, \beta, \lambda_n, \phi', \psi^{-1}) = \phi' \left(\psi^{-1} \left(\frac{1}{\lambda_n} \log \frac{1}{E_{n-1}(F, \beta) \exp\{-\beta \lambda_n\}} \right) \right)$$

and

$$\Omega_{n_v}(F, \beta, \lambda_{n_v}, \phi', \psi^{-1}) = \phi' \left(\psi^{-1} \left(\frac{1}{\lambda_{n_v}} \log \frac{1}{E_{n_v-1}(F, \beta) \exp\{-\beta \lambda_{n_v}\}} \right) \right).$$

Theorem 1.3. If Laplace-Stieltjes transform $F(s) \in L_0$ satisfies (1.7), then

$$\lim_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\phi(\sigma)} = T_\phi \Leftrightarrow (i) \quad \limsup_{n \rightarrow +\infty} \frac{\lambda_n}{\Omega_n(F, A_n^*, \lambda_n, \phi', \psi^{-1})} = T_\phi;$$

(ii) There exists a non-decreasing positive integer sequence $\{n_v\}$ satisfying (1.9) and

$$\lim_{v \rightarrow +\infty} \frac{\lambda_{n_v}}{\Omega_{n_v}(F, A_{n_v}^*, \lambda_{n_v}, \phi', \psi^{-1})} = T_\phi, \quad (1.10)$$

where

$$\Omega_n(F, A_n^*, \lambda_n, \phi', \psi^{-1}) = \phi' \left(\psi^{-1} \left(\frac{1}{\lambda_n} \log \frac{1}{A_n^*} \right) \right)$$

and

$$\Omega_{n_v}(F, A_{n_v}^*, \lambda_{n_v}, \phi', \psi^{-1}) = \phi' \left(\psi^{-1} \left(\frac{1}{\lambda_{n_v}} \log \frac{1}{A_{n_v}^*} \right) \right).$$

Particularly, if $F(s) \in L_0$ is of finite order ρ , let $\phi(\sigma) = \left(-\frac{1}{\sigma}\right)^\rho$, in view of Theorems 1.2 and 1.3, we can obtain the following corollaries.

Corollary 1.1. If $F(s) \in L_0$ is of finite order ρ ($0 < \rho < \infty$), then for any real number $-\infty < \beta < 0$, we have

$$\lim_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\left(-\frac{1}{\sigma}\right)^\rho} = T \Leftrightarrow (i) \quad \limsup_{n \rightarrow +\infty} \frac{\log^+[E_{n-1}(F, \beta) \exp\{-\beta \lambda_n\}]}{\left(\frac{\lambda_n}{\log^+[E_{n-1}(F, \beta) \exp\{-\beta \lambda_n\}]}\right)^\rho} = \frac{(\rho + 1)^{\rho+1}}{\rho^\rho} T;$$

(ii) There exists a non-decreasing positive integer sequence $\{n_v\}$ satisfying

$$\limsup_{v \rightarrow +\infty} \frac{\log^+[E_{n_v-1}(F, \beta) \exp\{-\beta \lambda_{n_v}\}]}{\left(\frac{\lambda_{n_v}}{\log^+[E_{n_v-1}(F, \beta) \exp\{-\beta \lambda_{n_v}\}]}\right)^\rho} = \frac{(\rho + 1)^{\rho+1}}{\rho^\rho} T, \quad \lim_{v \rightarrow \infty} \frac{\lambda_{n_v}}{\lambda_{n_v-1}} = 1. \quad (1.11)$$

Corollary 1.2. If $F(s) \in L_0$ is of finite order ρ ($0 < \rho < \infty$), then

$$\lim_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\left(-\frac{1}{\sigma}\right)^\rho} = T \Leftrightarrow (i) \quad \limsup_{n \rightarrow +\infty} \frac{\log^+ A_n^*}{\left(\frac{\lambda_n}{\log^+ A_n^*}\right)^\rho} = \frac{(\rho + 1)^{\rho+1}}{\rho^\rho} T;$$

(ii) There exists a non-decreasing positive integer sequence $\{n_v\}$ satisfying

$$\limsup_{v \rightarrow +\infty} \frac{\log^+ A_{n_v}^*}{\left(\frac{\lambda_{n_v}}{\log^+ A_{n_v}^*}\right)^\rho} = \frac{(\rho + 1)^{\rho+1}}{\rho^\rho} T, \quad \lim_{v \rightarrow \infty} \frac{\lambda_{n_v}}{\lambda_{n_{v+1}}} = 1.$$

Remark 1.4. In view of Corollaries 1.1 and 1.2, this shows that our results are some generalizations and improvements of Theorem 1.1.

2 Some lemmas

To prove our results, we also need to give the following lemmas.

Lemma 2.1. [17, Lemma 2.2] If Laplace-Stieltjes transform $F(s) \in L_0$, for any σ ($-\infty < \sigma < 0$) and ε (> 0), we have

$$\frac{1}{p} \mu(\sigma, F) \leq M_u(\sigma, F) \leq C \mu((1 - \varepsilon)\sigma, F) \frac{-1}{\sigma},$$

where $p > 2$ and C ($\neq 0$) are constants.

Lemma 2.2. Let $\phi \in \Xi_0$ and $0 < T_\phi < +\infty$, then the conclusion that $\log \mu(\sigma, F) \leq T_\phi \phi(\sigma)$ for any $\sigma \in (-\infty, 0)$ holds if and only if $\log A_n^* \leq -\lambda_n \psi\left(\varphi\left(\frac{\lambda_n}{T_\phi}\right)\right)$ for all $n \geq 0$.

Proof. Suppose that $\log A_n^* \leq -\lambda_n \psi\left(\varphi\left(\frac{\lambda_n}{T_\phi}\right)\right)$ for all $n \geq 0$. Thus, for any $\sigma < 0$ and $x < 0$, we have

$$(\sigma - x)\phi'(x) \leq \int_x^\sigma \phi'(t) dt = \phi(\sigma) - \phi(x).$$

Hence, it follows

$$\begin{aligned} \log \mu(\sigma, F) &\leq \left\{ \max_{n \geq 0} -\lambda_n \psi\left(\varphi\left(\frac{\lambda_n}{T_\phi}\right)\right) + \lambda_n \sigma : n \geq 0 \right\} \leq \max\{T_\phi(-t\psi(\varphi(t)) + t\sigma) : t \geq 0\} \\ &= T_\phi \max\{-\phi'(x)\psi(x) + \sigma + \phi'(x) : x > -\infty\} = T_\phi \max\{(\sigma - x)\phi'(x) + \phi(x) : x > -\infty\} \\ &= T_\phi \phi(\sigma). \end{aligned}$$

On the other hand, assume that $\log \mu(\sigma, F) \leq T_\phi \phi(\sigma)$ for any $\sigma \in (-\infty, 0)$, then it yields $\log A_n^* \leq T_\phi \phi(\sigma) - \sigma \lambda_n$ for all $n > 0$ and $\sigma \in (-\infty, 0)$. Let $\sigma = \varphi\left(\frac{\lambda_n}{T_\phi}\right)$, and by considering with φ being the inverse function of ϕ' , thus for all $n \geq 0$, it leads to

$$\log A_n^* \leq T_\phi \phi\left(\varphi\left(\frac{\lambda_n}{T_\phi}\right)\right) - \lambda_n \varphi\left(\frac{\lambda_n}{T_\phi}\right) = -\lambda_n \left(\varphi\left(\frac{\lambda_n}{T_\phi}\right) - \frac{\phi\left(\varphi\left(\frac{\lambda_n}{T_\phi}\right)\right)}{\phi'\left(\varphi\left(\frac{\lambda_n}{T_\phi}\right)\right)} \right) = -\lambda_n \psi\left(\varphi\left(\frac{\lambda_n}{T_\phi}\right)\right).$$

Therefore, we complete the proof of Lemma 2.2. $\square$

Lemma 2.3. For $0 < a < b < +\infty$ and $q > 0$, we have

$$G_1(a, b, q, \phi) < G_2(a, b, q, \phi).$$

Proof. First, denote $G(x) = G_1(a, x, q, \phi) - G_2(a, x, q, \phi)$, for $x \in (a, +\infty)$. Thus, it is easy to get that $G(x) \rightarrow 0$ as $x \rightarrow a^+$. Here, we only prove that $G(x)$ is a decreasing function on $(a, +\infty)$. In view of the definitions G_1 and G_2 , it follows

$$\begin{aligned} \frac{d}{dx} G_1(a, x, q, \phi) &= \frac{a}{(x-a)^2} \left\{ \phi(\varphi(qx)) - \frac{a}{x} \phi(\varphi(qx)) - a \int_a^x \frac{\phi(\varphi(qt))}{t^2} dt \right\} \\ &= \frac{a}{(x-a)^2} \left\{ \phi(\varphi(qx)) - \frac{a}{x} \phi(\varphi(qx)) + a \left(\phi(\varphi(qx)) \frac{1}{x} - \phi(\varphi(qx)) \frac{1}{a} \right. \right. \\ &\quad \left. \left. - \int_a^x q \frac{\phi'(\varphi(qt)) \varphi'(qt)}{t} dt \right) \right\} \\ &= \frac{aq}{(x-a)^2} \int_a^x (t-a) d\varphi(qt) \\ &= \frac{aq}{(x-a)^2} \left((x-a) \varphi(qx) - \int_a^x \varphi(qt) dt \right) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \frac{d}{dx}G_2(a, x, q, \phi) &= \phi' \left(\frac{1}{x-a} \int_a^x \varphi(qt) dt \right) \left(-\frac{1}{(x-a)^2} \int_a^x \varphi(qt) dt + \frac{1}{x-a} \varphi(qx) \right) \\ &= \phi' \left(\frac{1}{x-a} \int_a^x \varphi(qt) dt \right) \frac{1}{(x-a)^2} \left((x-a) \varphi(qx) - \int_a^x \varphi(qt) dt \right). \end{aligned} \quad (2.2)$$

Since φ is an increasing function, then for $x > a$, it follows

$$(x-a) \varphi(qx) - \int_a^x \varphi(qt) dt > 0, \quad \phi' \left(\frac{1}{x-a} \int_a^x \varphi(qt) dt \right) > qa. \quad (2.3)$$

In view of (2.1)–(2.3), it yields that

$$\begin{aligned} \frac{d}{dx}G(x) &= \frac{d}{dx}G_1(a, x, q, \phi) - \frac{d}{dx}G_2(a, x, q, \phi) \\ &= \frac{1}{(x-a)^2} \left((x-a) \varphi(qx) - \int_a^x \varphi(qt) dt \right) \left(aq - \phi' \left(\frac{1}{x-a} \int_a^x \varphi(qt) dt \right) \right) < 0, \end{aligned} \quad (2.4)$$

which means that $G(x)$ is a decreasing function on $(a, +\infty)$. Thus, $G(x) < G(a) \rightarrow 0$ as $x \rightarrow a^+$, that is, $G(x) < 0$ for all $x > a$. Hence, it follows that $G_1(a, b, q, \phi) < G_2(a, b, q, \phi)$. Therefore, this completes the proof of Lemma 2.3. $\square$

Lemma 2.4. Suppose that $F(s) \in L_0$, $T_\phi \in (0, +\infty)$, and if for some positive integer sequence $\{n_k\}_1^\infty$ increasing to $+\infty$,

$$\log A_{n_k}^* \geq -\lambda_{n_k} \psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right), \quad (2.5)$$

then for all $k \geq k_0$ and all $\sigma \in \left[\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right), \varphi \left(\frac{\lambda_{n_{k+1}}}{T_\phi} \right) \right]$, we have

$$\log \mu(\sigma, F) \geq T_\phi \phi(\sigma) \frac{G_1(\lambda_{n_k}, \lambda_{n_{k+1}}, T_\phi^{-1}, \phi)}{G_2(\lambda_{n_k}, \lambda_{n_{k+1}}, T_\phi^{-1}, \phi)}. \quad (2.6)$$

Proof. Set

$$H(\sigma) = \sup \left\{ -\lambda_{n_k} \psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right) + \sigma \lambda_{n_k}; \quad k \geq 1 \right\}$$

and

$$\eta_k = \frac{\lambda_{n_{k+1}} \psi \left(\varphi \left(\frac{\lambda_{n_{k+1}}}{T_\phi} \right) \right) - \lambda_{n_k} \psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right)}{\lambda_{n_{k+1}} - \lambda_{n_k}}.$$

Then, in view of (2.5), it follows that $H(\sigma) \leq \log \mu(\sigma, F)$ for all $\sigma < 0$. Since $(t\psi(\varphi(t)))' = (t\varphi(t) - \phi(\varphi(t)))' = \varphi(t)$, it leads to

$$\eta_k = \frac{1}{\lambda_{n_{k+1}} - \lambda_{n_k}} \int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}} \varphi \left(\frac{t}{T_\phi} \right) dt. \quad (2.7)$$

Since φ is continuous on $(0, +\infty)$ and increases to 0, thus in view of (2.6), it yields that $\eta_k \rightarrow 0$ as $k \rightarrow +\infty$ and $\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) < \eta_k < \varphi \left(\frac{\lambda_{n_{k+1}}}{T_\phi} \right)$.

For $\eta_{k-1} \leq \sigma \leq \eta_k$, if $j < k$, it follows that

$$\begin{aligned}
 & -\lambda_{n_k} \psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right) + \sigma \lambda_{n_k} - \left(-\lambda_{n_j} \psi \left(\varphi \left(\frac{\lambda_{n_j}}{T} \right) \right) + \sigma \lambda_{n_j} \right) \\
 &= - \sum_{p=j+1}^k \left(\lambda_{n_p} \psi \left(\varphi \left(\frac{\lambda_{n_p}}{T_\phi} \right) \right) - \lambda_{n_{p-1}} \psi \left(\varphi \left(\frac{\lambda_{n_{p-1}}}{T_\phi} \right) \right) \right) + \sigma (\lambda_{n_k} - \lambda_{n_j}) \\
 &= - \sum_{p=j+1}^k \eta_{p-1} (\lambda_{n_p} - \lambda_{n_{p-1}}) + \sigma (\lambda_{n_k} - \lambda_{n_j}) \\
 &= -\eta_{k-1} (\lambda_{n_k} - \lambda_{n_j}) + \sigma (\lambda_{n_k} - \lambda_{n_j}) \geq 0.
 \end{aligned} \tag{2.8}$$

Similarly, if $j > k$, we get

$$\begin{aligned}
 & -\lambda_{n_k} \psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right) + \sigma \lambda_{n_k} - \left(-\lambda_{n_j} \psi \left(\varphi \left(\frac{\lambda_{n_j}}{T_\phi} \right) \right) + \sigma \lambda_{n_j} \right) \\
 &= - \sum_{p=k+1}^j \left(\lambda_{n_p} \psi \left(\varphi \left(\frac{\lambda_{n_p}}{T_\phi} \right) \right) - \lambda_{n_{p-1}} \psi \left(\varphi \left(\frac{\lambda_{n_{p-1}}}{T_\phi} \right) \right) \right) + \sigma (\lambda_{n_j} - \lambda_{n_k}) \\
 &= - \sum_{p=k+1}^j \eta_{p-1} (\lambda_{n_p} - \lambda_{n_{p-1}}) + \sigma (\lambda_{n_j} - \lambda_{n_k}) \\
 &= -\eta_k (\lambda_{n_j} - \lambda_{n_k}) + \sigma (\lambda_{n_j} - \lambda_{n_k}) \geq 0.
 \end{aligned} \tag{2.9}$$

Hence, for $\eta_{k-1} \leq \sigma \leq \eta_k$, it yields

$$H(\sigma) = -\lambda_{n_k} \psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right) + \sigma \lambda_{n_k}. \tag{2.10}$$

Thus, for $\eta_{k-1} \leq \varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \leq \sigma \leq \eta_k$, we can deduce

$$\left(\frac{H(\sigma)}{\phi(\sigma)} \right)' = \frac{\lambda_{n_k} \phi(\sigma) - \phi'(\sigma) \left[\sigma \lambda_{n_k} - \lambda_{n_k} \psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right) \right]}{\phi^2(\sigma)} = \frac{\lambda_{n_k} \phi'(\sigma) \left[\psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right) - \psi(\sigma) \right]}{\phi^2(\sigma)} \leq 0, \tag{2.11}$$

which implies

$$\begin{aligned}
 \frac{H(\sigma)}{\phi(\sigma)} &\geq \frac{H(\eta_k)}{\phi(\eta_k)} = \frac{-\lambda_{n_k} \psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right) + \eta_k \lambda_{n_k}}{G_2(\lambda_{n_k}, \lambda_{n_{k+1}}, T_\phi^{-1}, \phi)} \\
 &= \frac{\lambda_{n_k} \left(\frac{\lambda_{n_{k+1}} \psi \left(\varphi \left(\frac{\lambda_{n_{k+1}}}{T_\phi} \right) \right) - \lambda_{n_k} \psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right)}{\lambda_{n_{k+1}} - \lambda_{n_k}} - \psi \left(\varphi \left(\frac{\lambda_{n_k}}{T_\phi} \right) \right) \right)}{G_2(\lambda_{n_k}, \lambda_{n_{k+1}}, T_\phi^{-1}, \phi)} \\
 &= \frac{\frac{\lambda_{n_k} \lambda_{n_{k+1}}}{\lambda_{n_{k+1}} - \lambda_{n_k}} \int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}} T_\phi^{-1} \left(\psi \left(\varphi \left(\frac{t}{T_\phi} \right) \right) \right)' dt}{G_2(\lambda_{n_k}, \lambda_{n_{k+1}}, T_\phi^{-1}, \phi)} \\
 &= \frac{T_\phi \frac{\lambda_{n_k} \lambda_{n_{k+1}}}{\lambda_{n_{k+1}} - \lambda_{n_k}} \int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}} \frac{\phi \left(\varphi \left(\frac{t}{T_\phi} \right) \right)}{t^2} dt}{G_2(\lambda_{n_k}, \lambda_{n_{k+1}}, T_\phi^{-1}, \phi)} = \frac{T_\phi G_1(\lambda_{n_k}, \lambda_{n_{k+1}}, T_\phi^{-1}, \phi)}{G_2(\lambda_{n_k}, \lambda_{n_{k+1}}, T_\phi^{-1}, \phi)}.
 \end{aligned} \tag{2.12}$$

Let $\eta_k \leq \sigma \leq \varphi\left(\frac{\lambda_{n_{k+1}}}{T_\phi}\right)$, then it follows that $H(\sigma) = -\lambda_{n_{k+1}}\psi\left(\varphi\left(\frac{\lambda_{n_{k+1}}}{T_\phi}\right)\right) + \sigma\lambda_{n_{k+1}}$ and

$$\left(\frac{H(\sigma)}{\phi(\sigma)}\right)' = \frac{\lambda_{n_{k+1}}\phi(\sigma) - \phi'(\sigma)\left(\sigma\lambda_{n_{k+1}} - \lambda_{n_{k+1}}\psi\left(\varphi\left(\frac{\lambda_{n_{k+1}}}{T_\phi}\right)\right)\right)}{\phi^2(\sigma)} = \frac{\lambda_{n_{k+1}}\phi'(\sigma)\left(\psi\left(\varphi\left(\frac{\lambda_{n_{k+1}}}{T_\phi}\right)\right) - \psi(\sigma)\right)}{\phi^2(\sigma)} \geq 0, \quad (2.13)$$

which shows that $\frac{H(\sigma)}{\phi(\sigma)}$ is nondecreasing on $\left[\eta_k, \varphi\left(\frac{\lambda_{n_{k+1}}}{T_\phi}\right)\right]$. By combining with (2.12), for all $k \geq k_0$ and all $\sigma \in \left[\varphi\left(\frac{\lambda_{n_k}}{T_\phi}\right), \varphi\left(\frac{\lambda_{n_{k+1}}}{T_\phi}\right)\right]$, it yields that

$$\frac{H(\sigma)}{\phi(\sigma)} \geq \frac{H(\eta_k)}{\phi(\eta_k)} = T_\phi \frac{G_1(\lambda_{n_k}, \lambda_{n_{k+1}}, T_\phi^{-1}, \phi)}{G_2(\lambda_{n_k}, \lambda_{n_{k+1}}, T_\phi^{-1}, \phi)}.$$

Hence, it is easy to get (2.6). $\square$

Lemma 2.5. [15, Lemma 2.6]. If $F(s) \in L_0$ and γ is any real number, then for $\sigma (< 0)$ sufficiently reaching 0, we have $\mu(\sigma, F) \leq 4M_u(\sigma, F)$ and

$$\left| \int_{\lambda_k}^{\infty} \exp\{(\gamma + it)y\} d\alpha(y) \right| \leq 2 \sum_{n=k}^{+\infty} A_n^* \exp\{\gamma\lambda_{n+1}\}.$$

3 Proofs of Theorems 1.2 and 1.3

3.1 The proof of Theorem 1.2

First of all, we prove the necessity of Theorem 1.2. Suppose that

$$\lim_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\phi(\sigma)} = T_\phi, \quad (3.1)$$

then for any sufficiently small $\varepsilon (> 0)$, there exists $\sigma_0 < 0$ such that

$$(T - \varepsilon)\phi(\sigma) \leq \log^+ M_u(\sigma, F) \leq (T_\phi + \varepsilon)\phi(\sigma), \quad \text{as } \sigma_0 < \sigma < 0.$$

In view of (1.7) and Lemma 2.1, it follows that

$$(T_\phi - \varepsilon)\phi(\sigma) \leq \log^+ \mu(\sigma, F) \leq \log^+ M_u(\sigma, F) \leq (T_\phi + \varepsilon)\phi(\sigma), \quad \text{as } \sigma_0 < \sigma < 0. \quad (3.2)$$

Since $F(s) \in L_0$, thus it follows $F(s) \in \bar{L}_\beta$ for any $\beta (-\infty < \beta < 0)$. And for $\beta < \sigma < 0$, we have

$$\begin{aligned} E_n(F, \beta) &\leq \|F - p_n\|_\beta \leq |F(\beta + it) - p_n(\beta + it)| \\ &\leq \left| \int_0^{+\infty} \exp\{(\beta + it)y\} d\alpha(y) - \int_0^{\lambda_n} \exp\{(\beta + it)y\} d\alpha(y) \right| \\ &= \left| \int_{\lambda_n}^{\infty} \exp\{(\beta + it)y\} d\alpha(y) \right|. \end{aligned} \quad (3.3)$$

In view of Lemma 2.6, it yields that $A_n^* \leq 4M_u(\sigma, F)e^{-\sigma\lambda_n}$ for any $\sigma < 0$. By combining with this and (3.3), we obtain that

$$E_n(F, \beta) \leq 2 \sum_{k=n+1}^{\infty} A_{k-1}^* \exp\{\beta\lambda_k\} \leq 8M_u(\sigma, F) \sum_{k=n+1}^{\infty} \exp\{(\beta - \sigma)\lambda_k\}. \quad (3.4)$$

In view of (1.4), we can choose $h'(0 < h' < h)$ such that $(\lambda_{n+1} - \lambda_n) \geq h'$ for $n \geq 0$. Then for $\sigma \geq \frac{\beta}{2}$, in view of (3.4), we can deduce that

$$\begin{aligned} E_n(F, \beta) &\leq 8M_u(\sigma, F) \exp\{\lambda_{n+1}(\beta - \sigma)\} \sum_{k=n+1}^{\infty} \exp\{(\lambda_k - \lambda_{n+1})(\beta - \sigma)\} \\ &\leq 8M_u(\sigma, F) \exp\{\lambda_{n+1}(\beta - \sigma)\} \exp\left\{-\frac{\beta}{2}h'(n+1)\right\} \sum_{k=n+1}^{\infty} \left(\exp\left\{\frac{\alpha}{2}h'k\right\}\right) \\ &= 8M_u(\sigma, F) \exp\{\lambda_{n+1}(\beta - \sigma)\} \left(1 - \exp\left\{\frac{\beta}{2}h'\right\}\right)^{-1}, \end{aligned}$$

that is,

$$E_{n-1}(F, \beta) \leq KM_u(\sigma, F) \exp\{\lambda_n(\beta - \sigma)\}, \quad (3.5)$$

where K is a constant and only depends on h and β .

Thus, from (3.2) and (3.5), it follows that

$$\log [E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}] \leq (T_\phi + \varepsilon)\phi(\sigma) - \sigma\lambda_n - \log K, \quad (3.6)$$

for all $n > 0$ and $\sigma_0 < \sigma < 0$. Let $\sigma = \varphi\left(\frac{\lambda_n}{T_\phi + \varepsilon}\right)$, and by using the same argument as in Lemma 2.2, it yields

$$\log [E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}] \leq -(1 + o(1))\lambda_n \psi\left(\varphi\left(\frac{\lambda_n}{T_\phi + \varepsilon}\right)\right),$$

which implies

$$\limsup_{n \rightarrow +\infty} \frac{\lambda_n}{\Omega_n(F, \beta, \lambda_n, \phi', \psi^{-1})} \leq T_\phi. \quad (3.7)$$

Now, we prove that the inequality

$$\limsup_{n \rightarrow +\infty} \frac{\lambda_n}{\Omega_n(F, \beta, \lambda_n, \phi', \psi^{-1})} = T' < T_\phi \quad (3.8)$$

does not hold. In view of (3.8), for any sufficiently small $\varepsilon \left(0 < \varepsilon < \frac{T_\phi - T'}{3}\right)$, there exists a positive integer n_1 such that

$$\log [E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}] \leq -\lambda_n \psi\left(\varphi\left(\frac{\lambda_n}{T' + \varepsilon}\right)\right), \quad n > n_1. \quad (3.9)$$

And since

$$\begin{aligned} A_n^* \exp\{\beta\lambda_n\} &= \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^x \exp\{ity\} d\alpha(y) \right| \exp\{\beta\lambda_n\} \\ &\leq \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^x \exp\{(\beta + it)y\} d\alpha(y) \right| \\ &\leq \sup_{-\infty < t < +\infty} \left| \int_{\lambda_n}^{\infty} \exp\{(\beta + it)y\} d\alpha(y) \right|, \end{aligned}$$

thus for any $p \in \Pi_{n-1}$ and $\beta < 0$, it yields

$$A_n^* \exp\{\beta\lambda_n\} \leq |F(\beta + it) - p(\beta + it)| \leq \|F - p\|_\beta, \quad (3.10)$$

and there exists $p_1 \in \Pi_{n-1}$ such that

$$\|F - p_1\| \leq 2E_{n-1}(F, \beta). \quad (3.11)$$

Hence, we can conclude in view of (3.10) and (3.11) that

$$A_n^* \exp\{\beta\lambda_n\} \leq 2E_{n-1}(F, \beta), \quad (3.12)$$

for any $\beta < 0$ and $F(s) \in L_0$. Then from (3.9) and (3.12), it follows

$$\log A_n^* \leq -(1 + o(1))\lambda_n \psi\left(\varphi\left(\frac{\lambda_n}{T' + \varepsilon}\right)\right). \quad (3.13)$$

By Lemma 2.2, it yields

$$\log \mu(\sigma, F) \leq (T' + \varepsilon)\phi(\sigma).$$

Thus, in view of Lemma 2.1, (1.7) and $0 < \varepsilon < \frac{T_\phi - T'}{3}$, we obtain

$$\limsup_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\phi(\sigma)} \leq T' + 2\varepsilon < T_\phi,$$

which is a contradiction with (3.1). Therefore, we have

$$\limsup_{n \rightarrow +\infty} \frac{\lambda_n}{\Omega_n(F, \beta, \lambda_n, \phi', \psi^{-1})} = T_\phi. \quad (3.14)$$

To prove (1.8), in view of (3.14), we only need to prove

$$\liminf_{v \rightarrow +\infty} \frac{\lambda_{n_v}}{\Omega_{n_v}(F, \beta, \lambda_{n_v}, \phi', \psi^{-1})} = T_\phi.$$

In view of (3.2), we can conclude that there exists a positive integer subsequence $\{n_v\}$ such that

$$\log [E_{n_v-1}(F, \beta) \exp\{-\beta\lambda_{n_v}\}] \geq -\lambda_{n_v} \psi\left(\varphi\left(\frac{\lambda_{n_v}}{T_\phi - \varepsilon}\right)\right) \quad (3.15)$$

and

$$(T_\phi + \varepsilon)G_1^*(\lambda_{n'_v}, \lambda_{n''_v}, (T_\phi + \varepsilon)^{-1}, \phi) \geq (T_\phi - \varepsilon)G_2^*(\lambda_{n'_v}, \lambda_{n''_v}, (T_\phi + \varepsilon)^{-1}, \phi), \quad (3.16)$$

where

$$G_1^*(\lambda_{n'_v}, \lambda_{n''_v}, (T_\phi + \varepsilon)^{-1}, \phi) = \frac{\lambda_{n'_v} \lambda_{n''_v}}{\lambda_{n''_v} - \lambda_{n'_v}} \int_{\lambda_{n'_v}}^{\lambda_{n''_v}} \frac{\phi\left(\varphi\left(\frac{t}{T_\phi + \varepsilon}\right)\right)}{t^2} dt,$$

$$G_2^*(\lambda_{n'_v}, \lambda_{n''_v}, (T_\phi + \varepsilon)^{-1}, \phi) = \phi\left(\frac{1}{\lambda_{n''_v} - \lambda_{n'_v}} \int_{\lambda_{n'_v}}^{\lambda_{n''_v}} \varphi\left(\frac{t}{T_\phi + \varepsilon}\right) dt\right).$$

Indeed, we suppose that such sequence $\{n_v\}$ does not exist, that is, there exist two sequences $\{n'_v\}, \{n''_v\}$ increasing to $+\infty$ such that

$$\log [E_{p-1}(F, \beta) \exp\{-\beta\lambda_p\}] < -\lambda_p \psi\left(\varphi\left(\frac{\lambda_p}{T_\phi - \varepsilon}\right)\right), \quad \text{for } n'_v \leq p \leq n''_v, \quad (3.17)$$

and

$$(T_\phi + \varepsilon)G_1^*(\lambda_{n'_v}, \lambda_{n''_v}, (T_\phi + \varepsilon)^{-1}, \phi) < (T_\phi - \varepsilon)G_2^*(\lambda_{n'_v}, \lambda_{n''_v}, (T_\phi + \varepsilon)^{-1}, \phi). \quad (3.18)$$

Assume that $\lambda_{n'_{v+1}} > \lambda_{n''_v}$. Denote

$$\eta_v^1 = \frac{\lambda_{n''_v} \psi\left(\varphi\left(\frac{\lambda_{n''_v}}{T_\phi + \varepsilon}\right)\right) - \lambda_{n'_v} \psi\left(\varphi\left(\frac{\lambda_{n'_v}}{T_\phi + \varepsilon}\right)\right)}{\lambda_{n''_v} - \lambda_{n'_v}}.$$

Thus, by Lemma 2.4, it follows that

$$\eta_v^1 = \frac{1}{\lambda_{n_v''} - \lambda_{n_v'}} \int_{\lambda_{n_v'}}^{\lambda_{n_v''}} \varphi\left(\frac{t}{T_\phi + \varepsilon}\right) dt, \quad (3.19)$$

and $\eta_v^1 \rightarrow 0$ as $k \rightarrow +\infty$, $\varphi\left(\frac{\lambda_{n_v'}}{T_\phi + \varepsilon}\right) < \eta_v^1 < \varphi\left(\frac{\lambda_{n_v''}}{T_\phi + \varepsilon}\right)$.

If $\lambda_p < \lambda_{n_v'}$, then we have

$$\left(-\lambda_p \psi\left(\varphi\left(\frac{\lambda_p}{T_\phi + \varepsilon}\right)\right) + \eta_v^1 \lambda_p\right)' = -\varphi\left(\frac{\lambda_p}{T_\phi + \varepsilon}\right) + \eta_v^1 > -\varphi\left(\frac{\lambda_{n_v'}}{T_\phi + \varepsilon}\right) + \eta_v^1 > 0,$$

which means that

$$-\lambda_p \psi\left(\varphi\left(\frac{\lambda_p}{T_\phi + \varepsilon}\right)\right) + \eta_v^1 \lambda_p < -\lambda_{n_v'} \psi\left(\varphi\left(\frac{\lambda_{n_v'}}{T_\phi + \varepsilon}\right)\right) + \eta_v^1 \lambda_{n_v'}. \quad (3.20)$$

If $\lambda_p > \lambda_{n_v''}$, then

$$\left(-\lambda_p \psi\left(\varphi\left(\frac{\lambda_p}{T_\phi + \varepsilon}\right)\right) + \eta_v^1 \lambda_p\right)' = -\varphi\left(\frac{\lambda_p}{T_\phi + \varepsilon}\right) + \eta_v^1 < -\varphi\left(\frac{\lambda_{n_v''}}{T_\phi + \varepsilon}\right) + \eta_v^1 < 0,$$

which means that

$$-\lambda_p \psi\left(\varphi\left(\frac{\lambda_p}{T_\phi + \varepsilon}\right)\right) + \eta_v^1 \lambda_p < -\lambda_{n_v''} \psi\left(\varphi\left(\frac{\lambda_{n_v''}}{T_\phi + \varepsilon}\right)\right) + \eta_v^1 \lambda_{n_v''}. \quad (3.21)$$

Since

$$-\lambda_{n_v'} \psi\left(\varphi\left(\frac{\lambda_{n_v'}}{T_\phi + \varepsilon}\right)\right) + \eta_v^1 \lambda_{n_v'} = -\lambda_{n_v''} \psi\left(\varphi\left(\frac{\lambda_{n_v''}}{T_\phi + \varepsilon}\right)\right) + \eta_v^1 \lambda_{n_v''}, \quad (3.22)$$

thus in view of (3.20)–(3.22), it yields that for any $\lambda_p \notin [n_v', n_v'']$

$$\begin{aligned} -\lambda_p \psi\left(\varphi\left(\frac{\lambda_p}{T_\phi + \varepsilon}\right)\right) + \eta_v^1 \lambda_p &< \frac{\lambda_{n_v'} \lambda_{n_v''}}{\lambda_{n_v''} - \lambda_{n_v'}} \left(\psi\left(\varphi\left(\frac{\lambda_{n_v''}}{T_\phi + \varepsilon}\right)\right) - \psi\left(\varphi\left(\frac{\lambda_{n_v'}}{T_\phi + \varepsilon}\right)\right) \right) \\ &= (T_\phi + \varepsilon) \frac{\lambda_{n_v'} \lambda_{n_v''}}{\lambda_{n_v''} - \lambda_{n_v'}} \int_{\lambda_{n_v'}}^{\lambda_{n_v''}} \frac{\phi\left(\varphi\left(\frac{t}{T_\phi + \varepsilon}\right)\right)}{t^2} dt. \end{aligned} \quad (3.23)$$

In view of (3.5), (3.12), (3.17)–(3.19) and (3.23), we can deduce

$$\begin{aligned} \log \mu(\eta_v^1, F) &\leq \sup\{\log [E_{p-1}(F, \beta) \exp\{-\beta \lambda_p\}] + \eta_v^1 \lambda_p + O(1); p > 0\} \\ &= \sup\{\sup\{\log [E_{p-1}(F, \beta) \exp\{-\beta \lambda_p\}] + \eta_v^1 \lambda_p + O(1) : p \notin [n_v', n_v'']\}, \\ &\quad \sup\{\log [E_{p-1}(F, \beta) \exp\{-\beta \lambda_p\}] + \eta_v^1 \lambda_p + O(1) : p \in [n_v', n_v'']\}\} \\ &< \sup\left\{(T_\phi + \varepsilon) \frac{\lambda_{n_v'} \lambda_{n_v''}}{\lambda_{n_v''} - \lambda_{n_v'}} \int_{\lambda_{n_v'}}^{\lambda_{n_v''}} \frac{\phi\left(\varphi\left(\frac{t}{T_\phi + \varepsilon}\right)\right)}{t^2} dt, \right. \\ &\quad \left. \sup\left\{-(1 + o(1)) \lambda_p \psi\left(\varphi\left(\frac{\lambda_p}{T_\phi - \varepsilon}\right)\right) + \eta_v^1 \lambda_p : p \in [n_v', n_v'']\right\}\right\} \\ &\leq \sup\{(T_\phi + \varepsilon) G_1^*(\lambda_{n_v'}, \lambda_{n_v''), (T_\phi + \varepsilon)^{-1}, \phi), (T_\phi - \varepsilon) \phi(\eta_v^1)\} \leq (T_\phi - \varepsilon) \phi(\eta_v^1), \end{aligned}$$

which is a contradiction with (3.2). Hence, we conclude that there exists a positive integer subsequence $\{n_v\}$ satisfying (3.15) and (3.16). Let $v \rightarrow \infty$ and $\varepsilon \rightarrow 0^+$, and by combining with (3.14) and Lemma 2.3, then we can deduce (1.8) and (1.9).

Thus, the proof of the necessity for Theorem 1.2 is completed.

Now, we prove the sufficiency of Theorem 1.2. Suppose that

$$\limsup_{n \rightarrow +\infty} \frac{\lambda_n}{\Omega_n(F, \beta, \lambda_n, \phi', \psi^{-1})} = T_\phi,$$

then in view of (3.12), for any sufficiently small $\varepsilon(>0)$, there exists a positive integer n_2 such that

$$\log A_n^* \leq -(1 + o(1))\lambda_n \psi \left(\varphi \left(\frac{\lambda_n}{T_\phi + \varepsilon} \right) \right), \quad \text{for } n > n_2. \quad (3.24)$$

Thus, in view of (1.7), (3.24) and Lemmas 2.1 and 2.2, we can easily obtain

$$\limsup_{\sigma \rightarrow 0^-} \frac{\log^+ M_u(\sigma, F)}{\phi(\sigma)} = T_\phi. \quad (3.25)$$

On the other hand, in view of (1.8), it follows that for any sufficiently small $\varepsilon(>0)$, there exists a positive integer n_v such that

$$\log [E_{n_{v-1}}(F, \beta) \exp\{-\beta\lambda_{n_v}\}] \geq -\lambda_{n_v} \psi \left(\varphi \left(\frac{\lambda_{n_v}}{T_\phi - \varepsilon} \right) \right). \quad (3.26)$$

Then, from (1.7), (3.5) and by Lemma 2.1, it yields

$$\log A_{n_v}^* \geq -(1 + o(1))\lambda_{n_v} \psi \left(\varphi \left(\frac{\lambda_{n_v}}{T_\phi - \varepsilon} \right) \right).$$

So, by Lemma 2.4, it follows that for all $v \geq v_0$ and all $\sigma \in \left[\varphi \left(\frac{\lambda_{n_v}}{T_\phi - \varepsilon} \right), \varphi \left(\frac{\lambda_{n_{v+1}}}{T_\phi - \varepsilon} \right) \right]$, we have

$$\log \mu(\sigma, F) \geq (T_\phi - \varepsilon)\phi(\sigma) \frac{G_1(\lambda_{n_v}, \lambda_{n_{v+1}}, (T_\phi - \varepsilon)^{-1}, \phi)}{G_2(\lambda_{n_v}, \lambda_{n_{v+1}}, (T_\phi - \varepsilon)^{-1}, \phi)}.$$

Hence, by combining with (1.9), it leads to

$$\log \mu(\sigma, F) \geq (T_\phi - \varepsilon)\phi(\sigma),$$

which implies

$$\liminf_{\sigma \rightarrow 0^-} \frac{\log^+ \mu(\sigma, F)}{\phi(\sigma)} \geq T_\phi. \quad (3.27)$$

Thus, in view of (3.25) and (3.26), and by applying Lemma 2.1, (3.1) holds. The proof of the sufficiency for Theorem 1.2 is completed.

Therefore, this completes the proof of Theorem 1.2.

3.2 The proof of Theorem 1.3

By using the same argument as in the proof of Theorem 1.3, and combining with the relation between $E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}$ with A_n^* (see (3.5) and (3.12)), it is easy to prove the conclusions of Theorem 1.3.

4 Proofs of Corollaries 1.1 and 1.2

4.1 The proof of Corollary 1.1

Without loss of generalization, assume that $T_\phi = 1$. Set $\phi(\sigma) = (-\sigma)^{-\rho}$, otherwise, let $\phi(\sigma) = T(-\sigma)^{-\rho}$. Thus,

$$\phi'(\sigma) = \rho(-\sigma)^{-\rho-1}, \quad \varphi(x) = -\left(\frac{\rho}{x}\right)^{\frac{1}{\rho+1}}, \quad \psi(\sigma) = \frac{\rho+1}{\rho}\sigma,$$

$$x\psi(\varphi(x)) = -(\rho+1)\left(\frac{x}{\rho}\right)^{\frac{\rho}{\rho+1}}$$

and

$$G_1(a, b, 1, \phi) = (\rho+1)\rho^{-\frac{\rho}{\rho+1}} \frac{ab}{b-a} \left(a^{-\frac{1}{\rho+1}} - b^{-\frac{1}{\rho+1}} \right),$$

$$G_2(a, b, 1, \phi) = (\rho+1)^{-\rho} \rho^{\frac{\rho^2}{\rho+1}} \left(\frac{b^{\frac{\rho}{\rho+1}} - a^{\frac{\rho}{\rho+1}}}{b-a} \right)^{-\rho}.$$

Thus, it yields that

$$\begin{aligned} \Omega_n(F, \beta, \lambda_n, \phi', \psi^{-1}) &= \phi' \left(\psi^{-1} \left(\frac{1}{\lambda_n} \log \frac{1}{E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}} \right) \right) \\ &= \frac{(\rho+1)^{\rho+1}}{\rho^\rho} \left(\frac{\lambda_n}{\log^+[E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}]} \right)^{\rho+1} \end{aligned} \quad (4.1)$$

and

$$\frac{G_1(\lambda_{n_v}, \lambda_{n_{v+1}}, 1, \phi)}{G_2(\lambda_{n_v}, \lambda_{n_{v+1}}, 1, \phi)} = \frac{(\rho+1)^{\rho+1}}{\rho^\rho} \frac{\lambda_{n_v} \lambda_{n_{v+1}}}{\lambda_{n_{v+1}} - \lambda_{n_v}} \left(\frac{1}{\lambda_{n_v}^{\frac{1}{\rho+1}}} - \frac{1}{\lambda_{n_{v+1}}^{\frac{1}{\rho+1}}} \right) \left(\frac{\lambda_{n_{v+1}}^{\frac{\rho}{\rho+1}} - \lambda_{n_v}^{\frac{\rho}{\rho+1}}}{\lambda_{n_{v+1}} - \lambda_{n_v}} \right)^\rho. \quad (4.2)$$

Hence, in view of (4.1), and by combining with the conclusions of Theorem 1.2, we can prove that the conclusion of Corollary 1.1(i) and the first conclusion of Corollary 1.1(ii) hold. Thus, it remains to prove that (4.2) is equivalent to the second formula of (1.11).

Set $\lambda_{n_{v+1}} = (1 + \delta_v)\lambda_{n_v}$. Then (4.2) becomes the following form:

$$f(\delta_v) = \frac{(\rho+1)^{\rho+1}}{\rho^\rho} \frac{1 + \delta_v}{\delta_v} \left(1 - \frac{1}{(1 + \delta_v)^{\frac{1}{\rho+1}}} \right) \left(\frac{(1 + \delta_v)^{\frac{\rho}{\rho+1}} - 1}{\delta_v} \right)^\rho. \quad (4.3)$$

By the aforementioned arguments and in view of Theorem 1.2, it remains to prove that $f(\delta_v) \rightarrow 1$ as $v \rightarrow +\infty$ if and only if $\delta_v \rightarrow 0$ as $v \rightarrow +\infty$. Since $\lambda_{n+1} > \lambda_n$, it follows $\delta_k > 0$. Assume that $f(\delta_v) \rightarrow 1$ as $v \rightarrow +\infty$, we will prove that $\delta_v \rightarrow 0$ as $v \rightarrow +\infty$ as follows.

If $\limsup_{v \rightarrow +\infty} \delta_v = +\infty$, then for some increasing sequence v_j , it follows

$$f(\delta_{v_j}) = \frac{(\rho+1)^{\rho+1}}{\rho^\rho} \frac{1}{\delta_{v_j}^{\frac{\rho}{\rho+1}}} \rightarrow 0, \quad \text{as } j \rightarrow +\infty,$$

which is a contradiction with $f(\delta_v) \rightarrow 1$ as $v \rightarrow +\infty$.

If $\limsup_{v \rightarrow +\infty} \delta_v = \delta > 0$, then for some increasing sequence v_j , it follows

$$f(\delta_{v_j}) = (1 + o(1)) \frac{(\rho + 1)^{\rho+1}}{\rho^\rho} \frac{1 + \delta}{\delta} \left(1 - \frac{1}{(1 + \delta)^{\frac{1}{\rho+1}}} \right) \left(\frac{(1 + \delta)^{\frac{\rho}{\rho+1}} - 1}{\delta} \right)^\rho,$$

as $j \rightarrow +\infty$. Set $x = (1 + \delta)^{\frac{1}{\rho+1}}$, then $x > 1$ and $f(\delta_{v_j}) = (1 + o(1))g(x)$, where

$$g(x) = \frac{(\rho + 1)^{\rho+1}}{\rho^\rho} \frac{x^\rho(x - 1)}{x^{\rho+1} - 1} \left(\frac{x^\rho - 1}{x^{\rho+1} - 1} \right)^\rho.$$

Let

$$g_1(x) = (\rho + 1)^{\rho+1} \frac{x - 1}{x^{\rho+1} - 1} - \rho^\rho \left(\frac{x^{\rho+1} - 1}{x^{\rho+1} - x} \right)^\rho,$$

then

$$\frac{d}{dx} g_1(x) = ((\rho + 1)x^\rho - \rho x^{\rho+1} - 1) \frac{(\rho + 1)^{\rho+1}(x^{\rho+1} - x)^{\rho+1} - \rho^{\rho+1}(x^{\rho+1} - 1)^{\rho+1}}{(x^{\rho+1} - 1)^2(x^{\rho+1} - x)^{\rho+1}}.$$

In view of $(\rho + 1)x^\rho - \rho x^{\rho+1} - 1 < 0$ and for all $x > 1$ $(\rho + 1)(x^{\rho+1} - x) - \rho(x^{\rho+1} - 1) > 0$, then we obtain $\frac{d}{dx} g_1(x) < 0$ for all $x > 1$. Since $\lim_{x \rightarrow 1^+} g_1(x) = (\rho + 1)^\rho - (\rho + 1)^\rho = 0$ and $\lim_{x \rightarrow +\infty} g_1(x) = -\rho^\rho < 0$, thus $g_1(x) < g_1(1) = 0$ for all $x > 1$, that is,

$$(\rho + 1)^{\rho+1} \frac{x - 1}{x^{\rho+1} - 1} - \rho^\rho \left(\frac{x^{\rho+1} - 1}{x^{\rho+1} - x} \right)^\rho < 0.$$

Hence, for all $x > 1$, we can deduce

$$g(x) = \frac{(\rho + 1)^{\rho+1}}{\rho^\rho} \frac{x^\rho(x - 1)}{x^{\rho+1} - 1} \left(\frac{x^\rho - 1}{x^{\rho+1} - 1} \right)^\rho < 1.$$

Thus, for all $\delta > 0$, it yields

$$\frac{(\rho + 1)^{\rho+1}}{\rho^\rho} \frac{1 + \delta}{\delta} \left(1 - \frac{1}{(1 + \delta)^{\frac{1}{\rho+1}}} \right) \left(\frac{(1 + \delta)^{\frac{\rho}{\rho+1}} - 1}{\delta} \right)^\rho < 1,$$

which is a contradiction with $f(\delta_v) \rightarrow 1$ as $v \rightarrow +\infty$.

If $\lim_{v \rightarrow +\infty} \delta_v = 0$, then

$$\begin{aligned} f(\delta_v) &= \frac{(\rho + 1)^{\rho+1}}{\rho^\rho} \frac{1 + \delta_v}{\delta_v} \left(\frac{\delta_v}{\rho + 1} + \frac{(1 + o(1))(\rho + 2)\delta_v^2}{2(\rho + 1)^2} \right) \left(\frac{\rho}{\rho + 1} - \frac{(1 + o(1))\rho\delta_v}{2(\rho + 1)^2} \right)^\rho \\ &= \frac{(\rho + 1)^{\rho+1}}{\rho^\rho} (1 + \delta_v) \left(\frac{1}{\rho + 1} + O(\delta_v) \right) \left(\frac{\rho}{\rho + 1} + O(\delta_v) \right)^\rho, \end{aligned}$$

and $f(\delta_v) \rightarrow 1$ as $v \rightarrow +\infty$.

Therefore, $f(\delta_v) \rightarrow 1$ as $v \rightarrow +\infty$ if and only if $\delta_v \rightarrow 0$ as $v \rightarrow +\infty$. Thus, (1.9) is equivalent to $\lim_{v \rightarrow \infty} \frac{\lambda_{n_v}}{\lambda_{n_v+1}} = 1$.

This completes the proof of Corollary 1.1.

4.2 The proof of Corollary 1.2

By using the same argument as in the proof of Corollary 1.1, it is easy to prove the conclusions of Corollary 1.2.

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