Research Article

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Uniqueness on entire functions and their n th order exact differences with two shared values

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Abstract: Let $f(z)$ be an entire function of hyper order strictly less than 1. We prove that if $f(z)$ and its n th exact difference $\Delta_c^n f(z)$ share 0 CM and 1 IM, then $\Delta_c^n f(z) \equiv f(z)$. Our result improves the related results of Zhang and Liao [Sci. China A, 2014] and Gao et al. [Anal. Math., 2019] by using a simple method.

Keywords: entire function, exact difference, uniqueness, shared values

MSC 2010: 30D35, 39A10

1 Introduction and main results

We assume the reader is familiar with the fundamental results and standard notations of Nevanlinna's theory, as found in [1,2], such as the characteristic function $T(r, f)$ of a meromorphic function $f(z)$. Notation $S(r, f)$ means any quantity such that $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ outside of a possible set of finite logarithmic measures. Moreover, the order $\rho(f)$ and hyper order $\rho_2(f)$ of $f(z)$ are defined as usual as follows:

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r} \quad \text{and} \quad \rho_2(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ T(r, f)}{\log r}.$$

For a value $a \in \mathbb{C} \cup \{\infty\}$, we say that two meromorphic functions $f(z)$ and $g(z)$ share a CM (IM) provided that f and g have the same a -points counting multiplicities (ignoring multiplicities).

About 10 years ago, Halburd and Korhonen [3,4] and Chiang and Feng [5] established the difference analogue of Nevanlinna's theory for finite-order meromorphic functions, independently. Later, Halburd et al. [6] showed in 2014 that it is still valid for meromorphic functions of hyper-order strictly less than 1. So far, it has been a most useful tool to study the uniqueness problems between meromorphic functions $f(z)$ and their shifts $f(z+c)$ or n th exact differences $\Delta_c^n f(z)$ ($n \geq 1$). For some related results in this topic, we refer the reader to [7–12] and so on.

In 2013, Chen and Yi first proved a uniqueness theorem for a meromorphic function $f(z)$ and its first order exact difference $\Delta_c f(z)$ with three distinct shared values CM in [13], which had been improved by Zhang and Liao [14] in 2014 as follows.

Theorem A. [14] *Let f be a transcendental entire function of finite order, and a, b be two distinct constants. If $\Delta_{c=1} f(z) \not\equiv 0$ and f share a, b CM, then $\Delta_{c=1} f \equiv f$. Furthermore, $f(z)$ must be of the following form $f(z) = 2^z h(z)$, where $h(z)$ is a periodic entire function with period 1.*

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Theorem A had been improved by Lü and Lü [15] from “entire function” to “meromorphic function” in 2016. More recently, Gao et al. [16] obtained the following uniqueness theorem concerning the n th exact difference.

Theorem B. [16] *Let f be a transcendental meromorphic function of hyper order strictly less than 1 such that $\Delta_{c=1}^n f(z) \not\equiv 0$. If $f(z)$ and $\Delta_{c=1}^n f(z)$ share three distinct periodic functions $a, b, c \in \hat{S}(f)$ with period 1 CM, then $\Delta_{c=1}^n f(z) \equiv f(z)$.*

Here, the notation $\hat{S}(f)$ means $S(f) \cup \{\infty\}$, where $S(f)$ is the set of all meromorphic functions $\alpha(z)$ such that $T(r, \alpha) = S(r, f)$. It is obvious that both Theorems A and B require three shared values CM. So, a nature question is: could those conditions of sharing values be weakened?

In this study, we shall prove a uniqueness theorem for entire functions that share two finite values “1 CM + 1 IM” with their n th exact differences, by using a simple method which is very differential to the proof of Theorems A and B. In fact, we obtain the following result.

Theorem 1.1. *Let f be a transcendental entire function of hyper order $\rho_2(f) < 1$, and let $c \in \mathbb{C} \setminus \{0\}$ such that $\Delta_c^n f(z) \not\equiv 0$. If f and $\Delta_c^n f$ share 0 CM and 1 IM, then $\Delta_c^n f(z) \equiv f(z)$.*

Remark 1. There exist many entire functions satisfying Theorem 1.1, which are arranged in Section 4. Here, we shall only give an example to illustrate it as follows.

Example 1. Let $f(z) = e^{az} e^{iz}$, where $a = \frac{\log(-2)}{\pi}$ and $c = \pi$. Then, the n th exact difference of $f(z)$ is as follows:

$$\begin{aligned} \Delta_c^n f(z) &\equiv \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(z + kc) \equiv \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} e^{az+akc} e^{iz+ikc} \\ &\equiv \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (-2)^k e^{az} (-1)^k e^{iz} \equiv \left[\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} 2^k \right] e^{az} e^{iz} \equiv f(z). \end{aligned}$$

Remark 2. It is obvious that Theorem 1.1 is invalid for polynomials $f(z)$. Actually, if f and $\Delta_c^n f$ share 0 CM, then we know that the degrees of f and $\Delta_c^n f$ are the same. But, on the other hand, the degree of $\Delta_c^n f$ is strictly less than the degree of f in this case for $n \geq 1$.

Remark 3. As per Theorems A and B, we all hope that the restriction on the growth of f can be dropped. But it seems not to be easy. However, we can also find out many entire functions satisfying the difference equation $\Delta_c^n f(z) = f(z)$ with hyper order greater than 1. Those discussions are arranged in Section 4.

2 Some lemmas

To prove our result, we need the following auxiliary results.

Lemma 2.1. [3,6] *Let $f(z)$ be a nonconstant meromorphic function of hyper order $\rho_2(f) < 1$ and $c \in \mathbb{C} \setminus \{0\}$, n is a positive integer. Then, for any $a \in \mathbb{C}$, we have*

$$m\left(r, \frac{\Delta_c^n f(z)}{f(z) - a}\right) = S(r, f).$$

Lemma 2.2. [2, Theorem 1.38] Suppose that $f(z)$ is a meromorphic function in the complex plane, and a_1, a_2, a_3 are three distinct small functions of $f(z)$. Then,

$$T(r, f) \leq \sum_{j=1}^3 \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f).$$

To estimate $N(r, f(z + c))$ and $T(r, f(z + c))$, we need the next result.

Lemma 2.3. [6] Let $T: [0, +\infty) \rightarrow [0, +\infty)$ be a non-decreasing continuous function and let $s \in (0, \infty)$. If the hyper order of T is strictly less than one, i.e.,

$$\limsup_{r \rightarrow \infty} \frac{\log \log T(r)}{\log r} = \rho_2 < 1,$$

and $\delta \in (0, 1 - \rho_2)$, then

$$T(r + s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right),$$

where r runs to infinity outside of a set of finite logarithmic measures.

Lemma 2.4. [2, Theorem 1.45] Suppose $h(z)$ is a nonconstant entire function and $f(z) = e^{h(z)}$, then $\rho_2(f) = \rho(h)$.

3 Proof of Theorem 1.1

As

$$\Delta_c^n f(z) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(z + kc),$$

we get from Lemma 2.3 that

$$T(r, \Delta_c^n f) \leq \sum_{k=0}^n T(r, f(z + kc)) + S(r, f) \leq (n + 1)T(r, f) + S(r, f). \quad (3.1)$$

On the other hand, by the assumptions of Theorem 1.1, we know that

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r, f) = \overline{N}\left(r, \frac{1}{\Delta_c^n f}\right) + \overline{N}\left(r, \frac{1}{\Delta_c^n f - 1}\right) + S(r, f) \\ &\leq 2T(r, \Delta_c^n f) + S(r, f). \end{aligned} \quad (3.2)$$

It follows from (3.1) and (3.2) that

$$S(r, \Delta_c^n f) = S(r, f) = S(r)$$

and

$$\rho_2(\Delta_c^n f) = \rho_2(f) < 1. \quad (3.3)$$

Since f and $\Delta_c^n f$ share 0 CM, we have

$$\frac{\Delta_c^n f}{f} = e^h, \quad (3.4)$$

where h is some entire function. In addition, by using Lemma 2.1, we have

$$T(r, e^h) = S(r). \quad (3.5)$$

Now, we suppose on the contrary that the assertion of Theorem 1.1 is not true, i.e., $\Delta_c^n f \neq f$. Hence,

$$e^h \neq 1. \quad (3.6)$$

Next, by the assumption that f and $\Delta_c^n f$ share 1 IM, we can deduce from (3.4) and (3.5) that

$$\bar{N}\left(r, \frac{1}{f-1}\right) = \bar{N}\left(r, \frac{1}{\Delta_c^n f - 1}\right) \leq N\left(r, \frac{1}{e^h - 1}\right) \leq S(r). \quad (3.7)$$

Rewrite formula (3.4) as

$$\Delta_c^n f - 1 = e^h(f - e^{-h}). \quad (3.8)$$

Together (3.8) with (3.7), we have

$$\bar{N}\left(r, \frac{1}{f - e^{-h}}\right) = S(r). \quad (3.9)$$

Finally, by using the second main theorem for three small functions (Lemma 2.2), we deduce from (3.6), (3.7) and (3.9) that

$$T(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f-1}\right) + \bar{N}\left(r, \frac{1}{f - e^{-h}}\right) + S(r) = S(r),$$

which is impossible. And this completes the proof of Theorem 1.1.

4 Examples and discussions

To construct the proper examples for Theorem 1.1, we recall a result obtained by Ozawa [17]. That is, for an arbitrary number $\sigma \in [1, \infty)$, there exists a periodic entire function $D(z)$ with period $c \neq 0$ such that $\rho(D) = \sigma$. Throughout this section, the notation $D(z)$ always means such an entire function.

Example 2. Let $g(z) = e^{i\pi \frac{z}{c}} D(z)$, where $D(z+c) \equiv D(z)$. Then, we have $g(z+kc) = -g(z)$ if k is odd, and $f(z+kc) = f(z)$ if k is even. And let $f(z) = e^{az}g(z)$ where $a = \frac{\log(-2)}{\pi}$. Thus,

$$\begin{aligned} \Delta_c^n f(z) &\equiv \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(z+kc) \equiv \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} e^{az+akc} g(z+kc) \\ &\equiv \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (-2)^k e^{az} (-1)^k g(z) \equiv \left[\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} 2^k \right] e^{az} g(z) \equiv f(z). \end{aligned}$$

It is clear that there exist many entire functions satisfying Theorem 1.1 from Example 2.

Next, we shall show that there also exist many entire functions satisfying the difference equation $\Delta_c^n f(z) \equiv f(z)$ with hyper order greater than or equal to 1.

Example 3. Let $g(z) = e^{\sin z} - e^{\sin z}$ and $c = \pi$. Then, we also have $g(z + kc) = -g(z)$ if k is odd, and $g(z + kc) = g(z)$ if k is even. And let $f(z) = e^{az}g(z)$ where $a = \frac{\log(-2)}{\pi}$. Thus, the n th exact difference of $f(z)$ is as follows:

$$\Delta_c^n f(z) \equiv \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(z + kc) \equiv \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} e^{az+akc} g(z + kc) \equiv \left[\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} 2^k \right] e^{az} g(z) \equiv f(z).$$

In general, we have the following example.

Example 4. Let $g(z) = e^{D(z)\sin(\frac{\pi z}{c})} - e^{-D(z)\sin(\frac{\pi z}{c})}$, where $D(z+c) \equiv D(z)$. It is clear that the hyper order of g is greater than or equal to 1 and that $g(z+c) = -g(z)$. Set $f(z) = e^{az}g(z)$ where $a = \frac{\log(-2)}{\pi}$. Then,

$$\Delta_c^n f(z) \equiv \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} e^{az+akc} g(z + kc) \equiv f(z).$$

Inspired by the above example, we raise the following open problem.

Problem. If $f(z)$ is a transcendental entire function solution of the difference equation $\Delta_c^n f(z) \equiv f(z)$, then $f(z)$ must be of the form $f(z) = e^{az}g(z)$, where $a = \frac{\log(-2)}{\pi}$ and g satisfies $g(z+c) = -g(z)$?

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