Research Article

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Self-injectivity of semigroup algebras

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Abstract: It is proved that for an IC abundant semigroup (a primitive abundant semigroup; a primitively semisimple semigroup) S and a field K , if $K_0[S]$ is right (left) self-injective, then S is a finite regular semigroup. This extends and enriches the related results of Okniński on self-injective algebras of regular semigroups, and affirmatively answers Okniński's problem: does that a semigroup algebra $K[S]$ is a right (respectively, left) self-injective imply that S is finite? (*Semigroup Algebras*, Marcel Dekker, 1990), for IC abundant semigroups (primitively semisimple semigroups; primitive abundant semigroups). Moreover, we determine the structure of $K_0[S]$ being right (left) self-injective when $K_0[S]$ has a unity. As their applications, we determine some sufficient and necessary conditions for the algebra of an IC abundant semigroup (a primitively semisimple semigroup; a primitive abundant semigroup) over a field to be semisimple.

Keywords: (IC) abundant semigroup, regular semigroup, semigroup algebra, left (right) self-injective algebra

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1 Introduction

Recall that an algebra (possibly without unity) R is right self-injective if R is an injective right R -module. Dually, left self-injective algebra is defined. Equivalently, then R is right self-injective if and only if the right R^1 -module R satisfies the Baer condition, where R^1 is the standard extension of R to an algebra with unity (see [1, Chap. 1]). In this case, R has a left unity. (Left; right) Self-injective algebras are known as the generalizations of Frobenius algebras. These classes of algebras play an important role and have become a central topic in algebras.

For group algebras, it is well known that the group algebra $K[G]$ of a group G over the field K is right self-injective if and only if the group G is finite; if and only if $K[G]$ is Frobenius (for detail, see [2, Theorem 3.2.8]). Along this direction, self-injective and Frobenius semigroup algebras of finite semigroups have been investigated by many authors (for references, see [3,4]). In particular, it was proved that in some cases, the finiteness of the semigroup is a necessary condition for the semigroup algebra to be right (respectively, left) self-injective; for example, the semigroup is an inverse semigroup, a countable semigroup, a regular semigroup, etc. (cf. [3–10]). So, Okniński raised a problem: does that $K[S]$ is a right (respectively, left) self-injective imply that S is finite? (see [11, Problem 6, p. 328]).

A semigroup S is called right principal projective (rpp), if for any $a \in S$, the right principal ideal aS^1 , regarded as a right S^1 -system, is projective. We can dually define left principal projective semigroup (lpp semigroup). As in [12], an abundant semigroup is defined as a semigroup being both rpp and lpp. Moreover, El Qallali and Fountain [13] defined idempotent-connected (IC) abundant semigroups. Indeed, IC abundant semigroups

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become a large class of semigroups including the class of regular semigroups and that of cancellative monoids as its proper subclasses. These three classes of semigroups have relationships as follows:

$$\{\text{Regular semigroups}\} \subset \{\text{IC abundant semigroups}\} \subset \{\text{Abundant semigroups}\}.$$

As known, rpp semigroups come from rpp ring. In precise, a ring R is (lpp; rpp) pp if and only if the multiplicative semigroup of R is (lpp; rpp) pp.

Recently, Guo and Shum [14] proved that the semigroup $K[S]$ of an ample semigroup S is right self-injective; if and only if $K[S]$ is left self-injective; if and only if $K[S]$ is quasi-Frobenius; if and only if $K[S]$ is Frobenius; if and only if S is a finite inverse semigroup. These results show that the “distance” between the class of finite inverse semigroups and that of ample semigroups is the right (left) self-injectivity of semigroup algebras. So-called an ample semigroup is an IC abundant semigroup whose set of regular elements forms an inverse subsemigroup. The class of ample semigroups contains properly the class of inverse semigroups. Indeed, for the self-injectivity, the symmetry of $\mathcal{L}^*$ – $\mathcal{R}^*$ in semigroups need not be so important. For semigroup algebras of finite ample semigroups, see ref. [15]. Guo and Guo [16] pointed out that the mentioned results as above in [14] is valid when the ample semigroup is weakened into a strict RA semigroup or a strict LA semigroup, especially, a right (left) ample monoid.

By inspiring the result of Okniński in [7]: for a regular semigroup S , if $K[S]$ is right (left) self-injective then S is finite, we have a natural problem: whether the Okniński problem is valid for IC abundant semigroups? This is the main aim of this paper. It is worthy to record here that $K[S]$ is right (respectively, left) self-injective if and only if so is $K_0[S]$ (see [11, p. 188]). We shall prove the following result:

Theorem. *Let K be a field and S be in one of the following cases:*

- (a) *primitive abundant semigroups;*
- (b) *IC abundant semigroups; and*
- (c) *primitively semisimple semigroups.*

If $K_0[S]$ is right (left) self-injective, then S is a finite regular semigroup.

As its applications, we determine when the algebra of IC abundant semigroups (respectively, primitively semisimple semigroups; primitive abundant semigroups) is semisimple (Theorem 6.2).

2 Preliminaries

Throughout this paper, we shall use the notions and notations of the monographs of Okniński [11] and Kelarev [17]. For semigroups, the readers can be referred to the textbooks of Clifford [18] and Howie [19]. Let S be a semigroup; we denote the set of idempotents of S as $E(S)$, and the semigroup obtained from S by adjoining an identity if S does not have one by S^1 .

2.1 IC abundant semigroups

The Green’s relations on a semigroup are well known; see for example [19, Chapter II]. As generalizations of Green’s $\mathcal{L}$ - and $\mathcal{R}$ -relations, we have $\mathcal{L}^*$ - and $\mathcal{R}^*$ -relations defined by

$$a\mathcal{L}^*b \quad \text{if } (ax = ay \Leftrightarrow bx = by \quad \text{for all } x, y \in S^1),$$

$$a\mathcal{R}^*b \quad \text{if } (xa = ya \Leftrightarrow xb = yb \quad \text{for all } x, y \in S^1).$$

It is well known that $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence. In general, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. And, if a, b are regular, then $a\mathcal{L}(\mathcal{R})b$ if and only if $a\mathcal{L}^*(\mathcal{R}^*)b$. For the relations $\mathcal{L}^*$ and $\mathcal{R}^*$, the reader can refer to [12].

A left ideal L of S is a left $\ast$ -ideal of S if $L = \sqcup_{x \in L} L_x^*$, where L_x^* is the $\mathcal{L}^*$ -class of S containing x . Dually, right $\ast$ -ideal is defined. Moreover, an ideal of S is a $\ast$ -ideal of S if it is both a left $\ast$ -ideal and a right $\ast$ -ideal. For $a \in S$, we denote by $J^*(a)$ the smallest $\ast$ -ideal of S containing a . Following Fountain [12], we define $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$ and $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$. Also, we define: for $a, b \in S$

$$a\mathcal{J}^*b \quad \text{if } J^*(a) = J^*(b).$$

Evidently, $\mathcal{J}^*$ is an equivalence on S . It is verified that $\mathcal{D}^* \subseteq \mathcal{J}^*$. Denote

$$I^*(a) = \{x \in J^*(a) : (a, x) \notin \mathcal{J}^*\}.$$

It is well known that $I^*(a)$ is a $\ast$ -ideal of S . And, it is clear that $J^*(a) = J_a^* \sqcup I^*(a)$ where J_a^* is the $\mathcal{J}^*$ -class of S containing a . If $\mathcal{K}$ is one of $\mathcal{H}^*, \mathcal{L}^*, \mathcal{R}^*$ and $\mathcal{D}^*$, we shall use K_a to denote the $\mathcal{K}$ -class of S containing a .

The following observation is used in the sequel.

Observation. ($\ast$): If S is a semigroup with zero θ , then

$$H_\theta^* = L_\theta^* = R_\theta^* = D_\theta^* = J_\theta^* = \{\theta\}.$$

Indeed, let $a \in S$.

(i) If $a\mathcal{L}^*\theta$, then by $\theta\theta = \theta$, we get $\theta = a\theta = a$, whereby $L_\theta^* = \{\theta\}$. Dually, we have $R_\theta^* = \{\theta\}$. Hence $H_\theta^* = \{\theta\}$ since $H_\theta^* = L_\theta^* \cap R_\theta^*$.

(ii) Suppose that $a\mathcal{D}^*\theta$. Notice that $\mathcal{D}^*$ is the smallest equivalence containing $\mathcal{L}^*$ and $\mathcal{R}^*$. Then by [19, Proposition 5.14, p. 28], there are $x_1, x_2, \dots, x_{2n-1} \in S$ such that

$$(a, x_1) \in \mathcal{L}^*, (x_1, x_2) \in \mathcal{R}^*, \dots, (x_{2n-1}, \theta) \in \mathcal{R}^*.$$

Now by the foregoing proof, $x_{2n-1} = \theta$, and so $x_{2n-2} = \theta, \dots, x_1 = \theta$, thus, $a = \theta$. Therefore, $D_\theta^* = \{\theta\}$.

(iii) Assume that $a\mathcal{J}^*\theta$. By definition, $J^*(a) = J^*(\theta)$. But by the foregoing proof, $\{\theta\}$ is a $\ast$ -ideal of S , so $J^*(\theta) = \{\theta\}$. It follows that $a = \theta$ since $a \in J^*(a)$. Thus, $J_\theta^* = \{\theta\}$.

A semigroup S is *abundant* if each $\mathcal{L}^*$ -class and $\mathcal{R}^*$ -class of S contains at least one idempotent. Moreover, an abundant semigroup S is *idempotent-connected* (for short, IC) if for any $a \in S$, there exist idempotents e, f satisfying the conditions:

- (i) $e\mathcal{R}^*a\mathcal{L}^*f$;
- (ii) for each $x \in \langle e \rangle$, there exists $y \in \langle f \rangle$ such that $xa = ay$, where for $g \in E(S)$, $\langle g \rangle$ is the subsemigroup of S generated by the set $E(gSg)$.

Regular semigroups are IC abundant semigroups [13]. Also, an abundant semigroup is adequate if all of its idempotents commute; that is, all of its idempotents form a semilattice. It is proved by El Qallali and Fountain in ref. [13] that ample semigroups are adequate IC abundant semigroups, and *vice versa*.

Lemma 2.1. (i) [20, Lemma 3.5] *Let S be an abundant semigroup and U a $\ast$ -ideal of S . For any $a, b \in S/U$, $(a, b) \in \mathcal{L}_S^*(\mathcal{R}_S^*)$ if and only if $(a, b) \in \mathcal{L}_{S/U}^*(\mathcal{R}_{S/U}^*)$;*

(ii) [20, Lemma 2.2] *Let S be an (IC) abundant semigroup and I a $\ast$ -ideal of S . Then the Rees quotient S/I is (IC) abundant.*

Given an abundant semigroup S , we define: for any $a, b \in S$,

$$a \leq b \quad \text{if there exist } e, f \in E(S) \text{ such that } a = eb = bf.$$

It is verified that $\leq$ is a partial order on S ; see for example [21]. It is pointed out that if $a \leq b$, then $b \in E(S)$ whenever $a \in E(S)$ [22]. A nonzero idempotent e of S is primitive if for any $f \in S$, $f \leq e$ can imply that $f = e$ or $f = \theta$ if S has zero θ . And, S is primitive if each nonzero idempotent of S is primitive.

Lemma 2.2. *Let S be an IC abundant semigroup and e, f be primitive idempotents of S .*

(i) $J^*(e)$ is a primitive abundant subsemigroup of S .

(ii) $J^*(e) = J^*(f)$ or $J^*(e) \cdot J^*(f) = \{\theta\}$ if S has zero θ .

Proof. (i). By definition,

$$\mathcal{L}_S^* \cap (J^*(e) \times J^*(e)) \subseteq \mathcal{L}_{J^*(e)}^* \quad \text{and} \quad \mathcal{R}_S^* \cap (J^*(e) \times J^*(e)) \subseteq \mathcal{R}_{J^*(e)}^*.$$

It follows that $J^*(e)$ is an abundant subsemigroup of S . For any nonzero idempotent $f, g \in J^*(e)$ and $f \leq g$, since $f \in J^*(e)$ and by [21, Lemma 3.7], there exists a nonzero idempotent $g \in D_f^*$ such that $g \leq e$. By e is primitive, it follows from Observation (*) that $g = e$. So that $f \in D_e^*$; similarly, $g \in D_e^*$. Thus, $f, g \in D_e^*$. Now by [21, Lemma 4.3], $f = g$, since e is primitive so that e has no infinite chains under $\leq$. This shows that any nonzero idempotent of $J^*(e)$ is not comparable for $\leq$. Therefore, any nonzero idempotent in $J^*(e)$ is primitive. So, $J^*(e)$ is primitive.

(ii) Suppose that $J^*(e) \neq J^*(f)$. For any $a \in J^*(e)$, $b \in J^*(f)$, it is clear that $ab \in J^*(e) \cap J^*(f)$. If $ab \neq \theta$, then since $ab \in J^*(e)$ and by [21, Lemma 3.7], we have $h \in E(S)$ such that $h \leq e$ and $h\mathcal{D}^*ab$. From Observation (*), it follows that $h \neq \theta$. But e is primitive, so $e = h$. It follows that $e\mathcal{D}^*ab$. Thus, $e \in J^*(f)$, and $J^*(e) \subseteq J^*(f)$; and similarly, $J^*(f) \subseteq J^*(e)$. Therefore, $J^*(e) = J^*(f)$, contrary to our hypothesis. Consequently, $ab = \theta$ and $J^*(e) \cdot J^*(f) = \theta$. $\square$

2.2 Primitive abundant semigroups

Let I, Λ be nonempty sets and let Γ be a nonempty set indexing partitions $P(I) = \{I_\alpha; \alpha \in \Gamma\}$, $P(\Lambda) = \{\Lambda_\alpha; \alpha \in \Gamma\}$ of I and Λ , respectively. For each pair $(\alpha, \beta) \in \Gamma \times \Gamma$, let $M_{\alpha\beta}$ be a set such that for each $\alpha \in \Gamma$, $T_\alpha := M_{\alpha\alpha}$ is a monoid and for $\alpha \neq \beta$, either $M_{\alpha\beta} = \emptyset$ or $M_{\alpha\beta}$ is a (T_α, T_β) -bisystem. Let 0 be a symbol not in any $M_{\alpha\beta}$. By the (α, β) -block of an $I \times \Lambda$ matrix we mean those (i, λ) positions with $i \in I_\alpha$, $\lambda \in \Lambda_\beta$. The (α, α) -blocks are called the diagonal blocks of the matrix. Following the usual convention, we use $(a)_{i\lambda}$ to denote the $I \times \Lambda$ matrix with entry a in the (i, λ) position and zeros elsewhere, and denote by 0 the $I \times \Lambda$ matrix all of whose entries are 0 . Let $P = (p_{\lambda i})$ be an $\Lambda \times I$ sandwich matrix where a non-zero entry in the (α, β) -block is a member of $M_{\alpha\beta}$. Suppose that the following conditions are satisfied:

(M) For all $\alpha, \beta, \gamma \in \Gamma$, if $M_{\alpha\beta}, M_{\alpha\gamma}$ are both non-empty, then $M_{\alpha\gamma}$ is non-empty and there is a (T_α, T_γ) -homomorphism $\phi_{\alpha\beta\gamma}: M_{\alpha\beta} \otimes M_{\beta\gamma} \rightarrow M_{\alpha\gamma}$ such that if $\alpha = \beta$ or $\beta = \gamma$, then $\phi_{\alpha\beta\gamma}$ is a canonical isomorphism such that the square

$$\begin{array}{ccc} M_{\alpha\beta} \otimes M_{\beta\gamma} \otimes M_{\gamma\delta} & \xrightarrow{id_{\alpha\beta} \otimes \phi_{\beta\gamma\delta}} & M_{\alpha\beta} \otimes M_{\beta\delta} \\ \phi_{\alpha\beta} \otimes id_{\beta\delta} \downarrow & & \downarrow \phi_{\alpha\beta\delta} \\ M_{\alpha\gamma} \otimes M_{\gamma\delta} & \xrightarrow{\phi_{\alpha\gamma\delta}} & M_{\alpha\delta} \end{array}$$

is commutative, where $id_{\alpha\beta}$ is the identity mapping on $M_{\alpha\beta}$.

(C) (In what follows, we simply denote $(a \otimes b)\phi_{\alpha\beta\gamma}$ by ab , for $a \in M_{\alpha\beta}$, $b \in M_{\beta\gamma}$). If $a, a_1, a_2 \in M_{\alpha\beta}$, $b, b_1, b_2 \in M_{\alpha\gamma}$, then $ab_1 = ab_2$ implies $b_1 = b_2$; $a_1b = a_2b$ implies $a_1 = a_2$. Clearly, each T_α is a cancellative monoid.

(U) For each $\alpha \in \Gamma$ and each $\lambda \in \Lambda_\alpha$ ($i \in I_\alpha$), there is a member i of I_α (λ of Λ_α) such that $p_{\lambda i}$ is a unit of T_α .

(R) If $M_{\alpha\beta}$, $M_{\beta\alpha}$ are both non-empty where $\alpha \neq \beta$, then $aba \neq a$ for all $a \in M_{\alpha\beta}$, $b \in M_{\beta\alpha}$.

Now, let

$$S = \{(a)_{i\lambda} : a \in M_{\alpha\beta}, i \in I_\alpha, \lambda \in \Lambda_\beta, \alpha, \beta \in \Gamma\} \sqcup \{0\}$$

and $A, B \in S$. If $A = 0$ or $B = 0$, then $APB = 0$. Assume that $A = (a)_{i\lambda}$ and $B = (b)_{j\mu}$ are non-zero. If $p_{\lambda j} = 0$, then $(ap_{\lambda j})b = 0 = a(p_{\lambda j}b)$ so that $APB = 0$. Assume that $p_{\lambda j} \neq 0$ and let $(i, \lambda) \in I_\alpha \times \Lambda_\beta$, $(j, \mu) \in I_\gamma \times \Lambda_\delta$. Then, $a \in M_{\alpha\beta}$, $b \in M_{\gamma\delta}$ and $p_{\lambda j} \in M_{\beta\gamma}$ so that by Condition (M), we see that $(ap_{\lambda j})b = a(p_{\lambda j}b)$ is a well-defined member of $M_{\alpha\delta}$ and $(ap_{\lambda j}b)_{i\mu} \in S$. Thus, we have a product $\circ$ defined on S by $A \circ B = APB$. We can easily check that $(S, \circ)$ is an abundant semigroup which is primitive, and called the *PA blocked Rees matrix semigroup* with the sandwich matrix P . For the sake of convenience, we denote this semigroup by $\mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$. It was pointed out by Fountain that a semigroup is a primitive abundant semigroup if and only if it is isomorphic to some $\mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ [12]. Moreover, by [12, Proposition 2.4 (7)], the number of nonzero regular $\mathcal{D}$ -classes of $\mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ is equal to $|\Gamma|$.

Let $Q = (V, E)$ be a quiver (a directed graph) with set V of vertex and set E of edges. Then, a vertex α is a source of Q if no edges end at α ; α is a sink if no edges begin at α . Assume that $\mathcal{M} := \mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ is a primitive abundant semigroup. From $\mathcal{M}$, we construct a quiver $Q(\mathcal{M})$ whose set of vertex is Γ and in which there is an edge beginning at α and ending at β if $\alpha \neq \beta$ and $M_{\alpha\beta} \neq \emptyset$. By Condition (M), it is easy to see that if $M_{\alpha\beta}$ and $M_{\beta\gamma}$ are both not empty, then $M_{\alpha\gamma}$ must not be empty. So, in $Q(\mathcal{M})$, there is a path beginning at α and ending at β if and only if $M_{\alpha\beta} \neq \emptyset$.

Lemma 2.3. *Let $\mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ satisfy the following conditions:*

- (F1) $|T_\alpha| < \infty$ for all α ; that is, T_α is a finite group;
- (F2) $|\Gamma| = n$.

Then

- (i) $Q(\mathcal{M})$ is acyclic;
- (ii) $Q(\mathcal{M})$ has sources and sinks;
- (iii) The vertex set Γ of $Q(\mathcal{M})$ can be labeled $\Gamma = \{1, 2, \dots, n\}$ in such a way that $(i, j) \in E$ implies $i < j$.

Proof. We first prove Claim (*): For any $\alpha, \beta \in \Gamma$ such that $\alpha \neq \beta$, at most one of $M_{\alpha\beta}$ and $M_{\beta\alpha}$ is not empty. If not, take some $u \in M_{\alpha\beta}$, some $v \in M_{\beta\alpha}$, some $k \in I_\alpha$, and some $\lambda \in \Lambda_\beta$. By the definition of blocked Rees matrix semigroup and Condition (U), there exist $\mu \in \Lambda_\beta$, $l \in I_\beta$ such that the entry $p_{\mu l}$ of the sandwich matrix P is in T_β . By $(up_{\mu l}v)_{k\lambda} = (u)_{k\mu}(v)_{l\lambda}$, we have $up_{\mu l}v \in M_{\alpha\beta}T_\beta M_{\beta\alpha} \subseteq T_\alpha$. But T_α is a group, so there exists $a \in T_\alpha$ such that $up_{\mu l}va = 1_\alpha$. It follows that $u \cdot p_{\mu l}va \cdot u = 1_\alpha u = u$, contrary to Condition (R). We prove Claim (*).

Let us consider the quiver $Q(\mathcal{M})$. By Claim (*), we know that $Q(\mathcal{M})$ is acyclic and by [23, Lemma, p. 142], $Q(\mathcal{M})$ has sources and sinks; by [23, Corollary, p. 143], we have (iii). $\square$

3 Primitive abundant semigroup algebras

In this section, we determine when a primitive abundant semigroup algebra is right (left) self-injective. We first recall some known results.

Lemma 3.1. *Let S be a semigroup and K a field. If $K_0[S]$ is a right self-injective K -algebra, then*

- (i) [7, 8] *There exist ideals S_i , $i = 0, 1, \dots, n$, such that $\theta = S_0 \sqsubset S_1 \sqsubset \dots \sqsubset S_n = S$ and the Rees quotients S_i/S_{i+1} are completely 0-simple or T -nilpotent.*
- (ii) [11, Lemmas 9 and 10, p. 192] *S satisfies the descending chain condition on principal left ideals and has no infinite subgroups.*
- (iii) [11, Lemma 1, p. 187] *$K_0[S]$ has a left identity.*

Lemma 3.2. Let $\mathcal{A}$ be a right self-injective K -algebra. Then

- (i) [11, Lemma 3, p. 188] Let J be an ideal of $\mathcal{A}$. If $J^2 = 0$ and $\mathcal{A}/J$ is finite dimensional, then $\mathcal{A}$ is finite dimensional.
- (ii) [11, Lemma 1, pp. 187–188] $\text{ann}_\ell(\text{ann}_r(\mathcal{W})) \subseteq \mathcal{W} + \text{ann}_\ell(\mathcal{A})$ for any finitely generated left ideal $\mathcal{W}$ of $\mathcal{A}$, where $\text{ann}_r(X)(\text{ann}_\ell(X))$ stands for the right (left) annihilators of a subset X of $\mathcal{A}$ in $\mathcal{A}$.

Lemma 3.3. Let $\mathcal{M} = \mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ be a primitive abundant semigroup and K be a field. If $K_0[\mathcal{M}]$ is right self-injective, then

- (i) $|I| < \infty$ and $|\Gamma| < \infty$.
- (ii) T_α is a finite group for any α .

Proof. (i) By Lemma 3.1, $K_0[\mathcal{M}]$ has a left identity ε . Denote

$$I_\varepsilon = \{i \in I: (a)_{i\lambda} \in \text{supp}(\varepsilon)\} \quad \text{and} \quad \Lambda_\varepsilon = \{\lambda \in \Lambda: (b)_{i\lambda} \in \text{supp}(\varepsilon)\}.$$

For any $(u)_{i\lambda} \in \mathcal{M}$, since $\varepsilon(u)_{i\lambda} = (u)_{i\lambda}$, there is $(a)_{k\mu} \in \text{supp}(\varepsilon)$ such that $(a)_{k\mu}(u)_{i\lambda} = (u)_{i\lambda}$. It follows that $k = i$. So, $i \in I_\varepsilon$ and $i \in I_\varepsilon$. Thus, $|I| < \infty$ since $|I_\varepsilon| < \infty$. Note that, by definition, $\sqcup_{\alpha \in \Gamma} I_\alpha$ is a partition of I . We can observe that $|I| \leq |I| < \infty$.

(ii) Let $x \in T_\alpha$ and $j \in I_\alpha$. By Condition (U), there exists $\xi \in \Lambda_\alpha$ such that the entry $p_{\xi j}$ of the sandwich matrix P is a unit of T_α . Because

$$\mathcal{M}(x)_{j\xi} \supseteq \mathcal{M}(x)_{j\xi}^2 = \mathcal{M}(xp_{\xi j}x)_{j\xi} \supseteq \cdots \mathcal{M}(x)_{j\xi}^n = \mathcal{M}((xp_{\xi j})^{n-1}x)_{j\xi} \supseteq \cdots,$$

there exists n such that

$$\mathcal{M}((xp_{\xi j})^{n-1}x)_{j\xi} = \mathcal{M}(x)_{j\xi}^n = \mathcal{M}(x)_{j\xi}^{n+1} = \mathcal{M}(xp_{\xi j})_{j\xi}^n$$

by Lemma 3.1. Thus, there exists $(v)_{k\gamma} \in \mathcal{M}$ such that $((xp_{\xi j})^{n-1}x)_{j\xi} = (v)_{k\gamma}((xp_{\xi j})^n x)_{j\xi}$. It follows that

$$\begin{aligned} ((xp_{\xi j})^{n-1}x)_{j\xi} &= (p_{\xi j}^{-1})_{j\xi} ((xp_{\xi j})^{n-1}x)_{j\xi} \\ &= (p_{\xi j}^{-1})_{j\xi} (v)_{k\gamma} ((xp_{\xi j})^n x)_{j\xi} \\ &= (p_{\xi j}^{-1} p_{\xi k} v p_{\gamma j} (xp_{\xi j})^n x)_{j\xi} \end{aligned}$$

and $(xp_{\xi j})^{n-1}x = p_{\xi j}^{-1} p_{\xi k} v p_{\gamma j} (xp_{\xi j})^n x$. Therefore, by Condition (C), $1_\alpha = p_{\xi j}^{-1} p_{\xi k} v p_{\gamma j} \cdot xp_{\xi j}$ since T_α is a cancellative monoid and

$$x, (xp_{\xi j})^{n-1}, (p_{\xi j}^{-1} p_{\xi k} v p_{\gamma j} p_{\xi j}^{-1}) \in T_\alpha.$$

So, $x p_{\xi j}$ is a unit in T_α , whereby x is invertible in T_α since $p_{\xi j}$ is a unit in T_α . Consequently, T_α is a group and by Lemma 3.1, T_α is finite. $\square$

Lemma 3.4. Let $\mathcal{M} = \mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ be a primitive abundant semigroup and K be a field. If $K_0[\mathcal{M}]$ is right self-injective, then $|M_{\alpha\beta}| < \infty$ for any $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$.

Proof. We first verify

Fact (+). If β is a sink of $Q(\mathcal{M})$ and $M_{\alpha\beta} \neq \emptyset$, then $|M_{\alpha\beta}| = |T_\beta| < \infty$.

Pick $a \in M_{\alpha\beta}$. We shall prove that $M_{\alpha\beta} = aT_\beta$. If not, there exists $b \in M_{\alpha\beta}/aT_\beta$. Obviously, $bT_\beta \subseteq M_{\alpha\beta}$ and $aT_\beta \cap bT_\beta = \emptyset$ since T_β is a group. Take some $i_0 \in I_\alpha$ and set

$$\mathcal{A} = \{(x)_{i_0\lambda} : \lambda \in \Lambda_\beta, x \in aT_\beta\}, \quad \text{and} \quad \mathcal{B} = \{(y)_{i_0\mu} : \mu \in \Lambda_\beta, y \in bT_\beta\}.$$

Again by β is a sink of $Q(\mathcal{M})$, we know that $M_{\beta\gamma} \neq \emptyset$ for any $\gamma \neq \beta$. By definition, we have that $p_{ij} \in M_{\beta\gamma} \sqcup \{0\}$ for all $j \in I_\gamma$, so that

Fact (#). *The entry $p_{\lambda j}$ of P is equal to 0 whenever $\lambda \in \Lambda_\beta, j \in I_\gamma$.*

By Fact (#), a routine check shows that $C := \mathcal{A} \sqcup \mathcal{B} \sqcup \{0\}$ is a right ideal of $\mathcal{M}$. Moreover, $K_0[C]$ is a right ideal of $K_0[\mathcal{M}]$. By Fact (#), a routine computation shows that the mapping φ is defined by the linear span of the mapping:

$$x \mapsto \begin{cases} x & \text{if } x \in \mathcal{A}; \\ 0 & \text{if } x \in \mathcal{B} \sqcup \{0\} \end{cases}$$

is a homomorphism of $K_0[C]$ into $K_0[\mathcal{M}]$. But $K_0[\mathcal{M}]$ is injective, so by Baer condition, there exists $z \in K_0[\mathcal{M}]$ such that

$$\varphi(x) = zx \quad \text{for all } x \in K_0[C]. \quad (1)$$

By Condition (U), there exists $\eta \in \Lambda_\alpha$ such that $p_{\eta i_0}$ (the entry of the sandwich matrix P) is a unit of T_α . It is easy to see that

- (a) $(p_{\eta i_0}^{-1})_{i_0\eta} x = x$ for any $x \in \mathcal{A} \sqcup \mathcal{B}$.
- (b) $(p_{\eta i_0}^{-1})_{i_0\eta} z (p_{\eta i_0}^{-1})_{i_0\eta} = \sum_{w \in J} k_w (w)_{i_0\eta} k$ where $J \subseteq T_\alpha$.

For any $(a)_{i_0\lambda} \in \mathcal{A}$, since, by Eq. (1),

$$\begin{aligned} (a)_{i_0\lambda} &= (p_{\eta i_0}^{-1})_{i_0\eta} \circ (a)_{i_0\lambda} = (p_{\eta i_0}^{-1})_{i_0\eta} \circ \varphi(a)_{i_0\lambda} \\ &= (p_{\eta i_0}^{-1})_{i_0\eta} z (a)_{i_0\lambda} = (p_{\eta i_0}^{-1})_{i_0\eta} z (p_{\eta i_0}^{-1})_{i_0\eta} \circ (a)_{i_0\lambda} \\ &= \left(\sum_{w \in J} k_w (w)_{i_0\eta} \right) \circ (a)_{i_0\lambda} \\ &= \sum_{w \in J} k_w (wp_{\eta i_0} a)_{i_0\lambda} \end{aligned} \quad (2)$$

and, by Condition (C), $w_1 p_{\eta i_0} a \neq w_2 p_{\eta i_0} a$ for any $w_1, w_2 \in T_\alpha$ and $w_1 \neq w_2$, we get that $wp_{\eta i_0} a = a$ for any $w \in J$. So, $w = p_{\eta i_0}^{-1}$ for any $w \in J$. Again by Eq. (2), $(a)_{i_0\lambda} = (\sum_{w \in J} k_w) \cdot (a)_{i_0\lambda}$ and $\sum_{w \in J} k_w = 1$ (the unity of K). Therefore, $(p_{\eta i_0}^{-1})_{i_0\eta} z (p_{\eta i_0}^{-1})_{i_0\eta} = (p_{\eta i_0}^{-1})_{i_0\eta}$. Let $y \in \mathcal{B}$. Then, by Eq. (1),

$$0 = (p_{\eta i_0}^{-1})_{i_0\eta} \circ \varphi(y) = (p_{\eta i_0}^{-1})_{i_0\eta} \circ zy = (p_{\eta i_0}^{-1})_{i_0\eta} z (p_{\eta i_0}^{-1})_{i_0\eta} \circ y = (p_{\eta i_0}^{-1})_{i_0\eta} \circ y = y$$

and $\mathcal{B} = 0$, contrary to the definition of $\mathcal{B}$. Thus, $M_{\alpha\beta} = aT_\beta$ and we prove Fact (+).

Now, let $M_{\pi\zeta} \neq \emptyset$. If ζ is not a sink of $Q(\mathcal{M})$, then as $Q(\mathcal{M})$ is acyclic and has only finite vertices, there exists a path $\zeta = \alpha_0 \rightarrow \alpha_2 \rightarrow \dots \rightarrow \alpha_n$ such that α_n is a sink of $Q(\mathcal{M})$. So, by Condition (M),

$$M_{\pi\zeta} M_{\alpha_0\alpha_1} M_{\alpha_1\alpha_2} \dots M_{\alpha_{n-1}\alpha_n} \subseteq M_{\pi\zeta} M_{\alpha_0\alpha_n} \subseteq M_{\pi\alpha_n}.$$

For $d \in M_{\alpha_0\alpha_n}$, since $M_{\pi\zeta} d \subseteq M_{\pi\alpha_n}$, we have $|M_{\pi\zeta} d| \leq |M_{\pi\alpha_n}|$. By Fact (+), $|M_{\pi\alpha_n}| < \infty$ and $|M_{\pi\zeta} d| < \infty$. But, by Condition (C), $|M_{\pi\zeta}| = |M_{\pi\zeta} d|$, now $|M_{\pi\zeta}| < \infty$. This proves the lemma. $\square$

Lemma 3.5. *Let $\mathcal{M} = \mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ be a primitive abundant semigroup and K a field. If $K_0[\mathcal{M}]$ is right self-injective, then $\mathcal{M}$ is finite.*

Proof. Let ε be a left identity, and I_ε and Λ_ε have the same meaning as in Lemma 3.3. Set $\mathcal{U} = K_0[\mathcal{M}] \cdot \varepsilon$ and $\mathcal{V} = \{x - x \cdot \varepsilon : x \in K_0[\mathcal{M}]\}$. It is easy to know that

- (a) $K_0[\mathcal{M}] = \mathcal{U} \oplus \mathcal{V}$ (regarded as K -vector spaces);
- (b) $\mathcal{V}^2 = 0$;
- (c) $\mathcal{V}$ is an ideal of $K_0[\mathcal{M}]$.

So, $\dim_K(K_0[\mathcal{M}]/\mathcal{V}) = \dim_K(\mathcal{U})$. Take $X = \{(a)_{i\lambda} \in \mathcal{M} : i \in I_\varepsilon, \lambda \in \Lambda_\varepsilon\}$. Notice that $|I_\varepsilon| < \infty$ and $|\Lambda_\varepsilon| < \infty$. By Lemma 3.4, we get $|X| < \infty$, it follows that $\dim_K(X) < \infty$ where X is a K -space with a basis X . On the other hand, for any $u \in \mathcal{U}$, we know that $\varepsilon u \varepsilon = u$, and further for any $(v)_{kl} \in \text{supp}(u)$, there exist $(x)_{i\lambda}, (y)_{j\mu} \in \text{supp}(\varepsilon), (z)_{mn} \in \text{supp}(u)$ such that $(v)_{kl} = (x)_{i\lambda}(z)_{mn}(y)_{j\mu} = (xp_{\lambda m}zp_{nj}y)_{i\mu}$. Hence, $k = i \in I_\varepsilon$ and $l = \mu \in \Lambda_\varepsilon$. Thus, $(v)_{kl} \in X$ and so $u \in X$. It follows that $\mathcal{U} \subseteq X$. This means that $\mathcal{U}$ is a subspace of X . Therefore, $\dim_K(\mathcal{U}) \leq \dim_K(X)$ and $\mathcal{U}$ is finite dimensional. Now by Lemma 3.2 (i), $K_0[\mathcal{M}]$ is finite dimensional. Consequently, $\mathcal{M}$ is a finite semigroup. $\square$

Theorem 3.6. Let $\mathcal{M} = \mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ be a primitive abundant semigroup and K a field. If $K_0[\mathcal{M}]$ is right self-injective, then $\mathcal{M}$ is a finite primitive regular semigroup.

Proof. We need to only verify that $\mathcal{M}$ is a regular semigroup. By Lemmas 2.3 and 3.3, we may assume that

- (a) $\Gamma = \{1, 2, \dots, n\}$;
- (b) For any $1 \leq i, j \leq n$ and $i \neq j$, whenever $M_{ij} \neq \emptyset$, we have $i < j$;
- (c) T_i is a finite group, for any $1 \leq i \leq n$. Moreover, we let $|I_i| = p_i$ and $|\Lambda_i| = q_i$ for any i . By computation,

$$K_0[\mathcal{M}] = \begin{pmatrix} M_{p_1 \times q_1}(K[T_1]) & M_{p_1 \times q_2}(K[M_{12}]) & \cdots & M_{p_1 \times q_n}(K[M_{1n}]) \\ 0 & M_{p_2 \times q_2}(K[M_{22}]) & \cdots & M_{p_2 \times q_n}(K[M_{2n}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{p_n \times q_n}(K[T_n]) \end{pmatrix},$$

where $M_{p_i \times q_j}(K[M_{ij}])$ is the set consisting of all $p_i \times q_j$ matrices over $K[M_{ij}]$, and

$$P = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ 0 & P_{22} & \cdots & P_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & P_{nn} \end{pmatrix},$$

where $P_{ij} \in M_{q_i \times p_j}(M_{ij} \sqcup \{0\})$. By multiplying with $\mathcal{M}$, we easily know that the contracted semigroup algebra $K_0[\mathcal{M}]$ is an algebra with the usual matrix addition and the multiplication is defined by: for $A, B \in K_0[\mathcal{M}]$

$$A \circ B = APB,$$

where the right side is the usual matrix multiplication of A, P and B .

Let

$$\varepsilon = \begin{pmatrix} \varepsilon_1 & \varepsilon_{12} & \cdots & \varepsilon_{1n} \\ 0 & \varepsilon_2 & \cdots & \varepsilon_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}$$

be a left identity of $K_0[\mathcal{M}]$. Since,

$$\begin{aligned}
\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A \end{pmatrix} &= \begin{pmatrix} \varepsilon_1 & \varepsilon_{12} & \cdots & \varepsilon_{1n} \\ 0 & \varepsilon_2 & \cdots & \varepsilon_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A \end{pmatrix} \\
&= \begin{pmatrix} \varepsilon_1 & \varepsilon_{12} & \cdots & \varepsilon_{1n} \\ 0 & \varepsilon_2 & \cdots & \varepsilon_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} P \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A \end{pmatrix} \\
&= \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n P_{nn} A \end{pmatrix}
\end{aligned} \tag{3}$$

we have

$$\varepsilon_n P_{nn} A = A \quad \text{where } A \in M_{p_n \times q_n}(K[M_{nn}]), \tag{4}$$

especially, $\varepsilon_n P_{nn} \varepsilon_n = \varepsilon_n \neq 0$. So,

$$\begin{aligned}
\begin{pmatrix} 0 & 0 & \cdots & A_1 \\ 0 & 0 & \cdots & A_2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \circ \varepsilon &= \begin{pmatrix} 0 & 0 & \cdots & A_1 \\ 0 & 0 & \cdots & A_2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} P \varepsilon \\
&= \begin{pmatrix} 0 & 0 & \cdots & A_1 P_{nn} \varepsilon_n \\ 0 & 0 & \cdots & A_2 P_{nn} \varepsilon_n \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n P_{nn} \varepsilon_n \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \cdots & A_1 P_{nn} \varepsilon_n \\ 0 & 0 & \cdots & A_2 P_{nn} \varepsilon_n \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}
\end{aligned}$$

and

$$\begin{pmatrix} 0 & 0 & \cdots & A_1 \\ 0 & 0 & \cdots & A_2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \notin \text{ann}_\ell(K_0[\mathcal{M}]). \tag{5}$$

We next verify that

(*) All of M_{in} with $1 \leq i \leq n-1$ are empty sets.

We assume, on the contrary, that not all M_{in} with $1 \leq i \leq n-1$ are empty sets. Without loss of generality, assume that $M_{1n} \neq \emptyset$ and $u \in M_{1n}$. By a routine check,

$$\mathcal{I}_n := \begin{pmatrix} 0 & 0 & \cdots & M_{p_1 \times q_n}(K[M_{1n}]) \\ 0 & 0 & \cdots & M_{p_2 \times q_n}(K[M_{2n}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

is a left ideal of $K_0[\mathcal{M}]$ generated by the finite set

$$\begin{pmatrix} 0 & 0 & \cdots & M_{p_1 \times q_n}(M_{1n} \sqcup \{0\}) \\ 0 & 0 & \cdots & M_{p_2 \times q_n}(M_{2n} \sqcup \{0\}) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{p_{n-1} \times q_n}(M_{n-1,n} \sqcup \{0\}) \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

(since all M_{ij} are finite). If

$$\begin{pmatrix} 0 & 0 & \cdots & U \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \circ \begin{pmatrix} X_1 & X_{12} & \cdots & X_{1n} \\ 0 & X_2 & \cdots & X_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & X_n \end{pmatrix} = 0,$$

i.e.,

$$\begin{pmatrix} 0 & 0 & \cdots & UP_{nn}X_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & U \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} P \begin{pmatrix} X_1 & X_{12} & \cdots & X_{1n} \\ 0 & X_2 & \cdots & X_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & X_n \end{pmatrix} = 0,$$

then $UP_{nn}X_n = 0$. It is not difficult to know that $P_{nn}X_n \in M_{q_n \times q_n}(K[T_n])$. Let $U = (u)_{11}$ and $u \in M_{12}$. Let

$$P_{nn}X_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q_n} \\ a_{21} & a_{22} & \cdots & a_{2q_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{q_n1} & a_{q_n2} & \cdots & a_{q_nq_n} \end{pmatrix}.$$

We have

$$0 = UP_{nn}X_n = \begin{pmatrix} ua_{11} & ua_{12} & \cdots & ua_{1q_n} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and $ua_{ij} = 0$ for any $1 \leq j \leq q_n$. Now let $J = \text{supp}(a_{ij})$ and $a_{ij} = \sum_{v_l \in J} r_l v_l$ ($r_l \in K$). Note that by Condition (C) $ux \neq uy$ for any $x \neq y \in \text{supp}(a_{ij})$. So, $\sum_{l \in J} r_l (uv_l) = ua_{ij} = 0$ implies that $r_l = 0$. Thus, $a_{ij} = 0$. If $U = (u)_{i1}$, then by applying a similar argument as above, we may obtain that $a_{ij} = 0$ for any $1 \leq i, j \leq q_n$. So, $P_{nn}X_n = 0$. Now, by Eq. (4), $X_n = \varepsilon_n P_{nn}X_n = 0$. It follows that

$$\text{ann}_r(I_n) \subseteq \begin{pmatrix} M_{p_1 \times q_1}(K[T_1]) & M_{p_1 \times q_2}(K[M_{12}]) & \cdots & M_{p_1 \times q_n}(K[M_{1n}]) \\ 0 & M_{p_2 \times q_2}(K[M_{22}]) & \cdots & M_{p_2 \times q_n}(K[M_{2n}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{p_{n-1} \times q_n}(K[M_{n-1,n}]) \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and a routine check shows the reverse inclusion. Thus,

$$\text{ann}_r(I_n) = \begin{pmatrix} M_{p_1 \times q_1}(K[T_1]) & M_{p_1 \times q_2}(K[M_{12}]) & \cdots & M_{p_1 \times q_n}(K[M_{1n}]) \\ 0 & M_{p_2 \times q_2}(K[M_{22}]) & \cdots & M_{p_2 \times q_n}(K[M_{2n}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{p_{n-1} \times q_n}(K[M_{n-1,n}]) \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (6)$$

On the other hand, by computation and Lemma 3.2, we have

$$\begin{pmatrix} 0 & 0 & \cdots & M_{p_1 \times q_n}(K[M_{1n}]) \\ 0 & 0 & \cdots & M_{p_2 \times q_n}(K[M_{2n}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{p_n \times q_n}(K[M_{nn}]) \end{pmatrix} \subseteq \text{ann}_\ell(\text{ann}_r(\mathcal{I}_n)) \subseteq \mathcal{I}_n + \text{ann}_\ell(K_0[\mathcal{M}]). \quad (7)$$

Obviously,

$$V_n := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \in \begin{pmatrix} 0 & 0 & \cdots & M_{p_1 \times q_n}(K[M_{1n}]) \\ 0 & 0 & \cdots & M_{p_2 \times q_n}(K[M_{2n}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{p_n \times q_n}(K[M_{nn}]) \end{pmatrix}.$$

By Eq. (5), $V_n \notin \mathcal{I}_n + \text{ann}_\ell(K_0[\mathcal{M}])$, contrary to Eq. (7). Thus, $M_{in} = \emptyset$ for any $i < n$. Moreover, $P_{in} = 0$ in the sandwich matrix P , for any $i < n$.

By applying a similar argument as above to the set

$$\mathcal{I}_{n-1} := \begin{pmatrix} 0 & 0 & \cdots & M_{p_1 \times q_{n-1}}(K[M_{1,n-1}]) & 0 \\ 0 & 0 & \cdots & M_{p_2 \times q_{n-1}}(K[M_{2,n-1}]) & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{p_{n-2} \times q_{n-1}}(K[M_{n-2,n-1}]) & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$V_{n-1} := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

we can obtain that $M_{j,n-1} = \emptyset$ for any $1 \leq j < n-1$. Continuing this process, we can prove that $M_{ji} = \emptyset$ for $j = 1, 2, \dots, i-1$, $i = 2, 3, \dots, n-2$. Thus, $M_{ji} = \emptyset$ for any $i = j$. Now by [12, Proposition 2.4 (7)], $\mathcal{M}$ is regular. We complete the proof. $\square$

4 Algebras of IC abundant semigroups

In this section, we shall research the self-injectivity of algebras of IC abundant semigroups.

Lemma 4.1. *Let S be a semigroup and K a field. If $K_0[S]$ is right self-injective, then*

- (i) *S has only finite regular $\mathcal{J}$ -classes.*
- (ii) *S has a primitive idempotent.*

Proof. (i) By Lemma 3.1, there exist ideals S_i , $i = 0, 1, \dots, n$, of S such that $\theta = S_0 \sqsubset S_1 \sqsubset \cdots \sqsubset S_n = S$ and the Rees quotients S_i/S_{i+1} are completely 0-simple or T -nilpotent. Note that $S = \sqcup_{i=1}^n S_i \setminus S_{i-1}$. So, any regular $\mathcal{J}$ -class of S must be in some Rees quotient S_i/S_{i-1} being completely 0-simple. It follows that the number of regular $\mathcal{J}$ -classes of S is smaller than n .

(ii) We claim: *under $\leq$, S has a minimal nonzero idempotent e_0* ; for, if no, S has a chain of nonzero idempotents: $e_1 > e_2 > \dots > e_n > \dots$, so $\dots Se_n \subset \dots \subset Se_2 \subset Se_1$, contrary to Lemma 3.1 (ii). It is not difficult to know that e_0 is primitive. $\square$

Lemma 4.2. [11, Lemma 5, p. 189] *Assume that $K_0[S]$ is right self-injective. Then there is no infinite sequence of elements $a_1, a_2, \dots$ of $K_0[S]$ such that the principal right algebra ideals generated by the a_i are independent and $\dim_K(K_0[S]a_i) = \infty$ for all $i = 1, 2, \dots$*

Lemma 4.3. *Assume that $J = \mathcal{M}(M_{\alpha\beta}, I, \Lambda; \Gamma)$ is a primitive abundant ideal of an IC abundant semigroup S . If $K_0[S]$ is right self-injective, then*

- (i) $|I| < \infty$;
- (ii) $K_0[J]$ has a left identity;
- (iii) all $T_\alpha = M_{\alpha\alpha}$ are finite groups.

Proof. We first prove that for any $\alpha \in \Gamma$, $|I_\alpha| < \infty$. Indeed, if I_α is infinite and choose elements $i_1, i_2, \dots$ from I_α , then by Condition (U), there exist elements $\lambda_1, \lambda_2, \dots$ of Λ_α such that $p_{\lambda_j i_j}$ (the entry of the sandwich matrix P) is a unity of T_α , for $j = 1, 2, \dots$. By a routine check, every $(p_{\lambda_j i_j}^{-1})_{i_j \lambda_j}$ is an idempotent in J . Moreover, we have

$$(a) \quad K_0 \left[(p_{\lambda_j i_j}^{-1})_{i_j \lambda_j} S \right] = K_0 \left[(p_{\lambda_j i_j}^{-1})_{i_j \lambda_j} \circ (p_{\lambda_j i_j}^{-1})_{i_j \lambda_j} S \right] = K_0 \left[(p_{\lambda_j i_j}^{-1})_{i_j \lambda_j} J \right];$$

$$(b) \quad K_0 \left[(p_{\lambda_1 i_1}^{-1})_{i_1 \lambda_1} J \right], K_0 \left[(p_{\lambda_2 i_2}^{-1})_{i_2 \lambda_2} J \right], \dots \text{ are independent};$$

(c) By computation,

$$\left\{ (a)_{i \lambda_j} \in J : a \in T_\alpha, i \in I_\alpha \right\} \subseteq S (p_{\lambda_j i_j}^{-1})_{i_j \lambda_j} = \left(S (p_{\lambda_j i_j}^{-1})_{i_j \lambda_j} \right) \circ (p_{\lambda_j i_j}^{-1})_{i_j \lambda_j} = J (p_{\lambda_j i_j}^{-1})_{i_j \lambda_j}$$

and so is infinite. It follows that $\dim_K \left(K_0 \left[S (p_{\lambda_j i_j}^{-1})_{i_j \lambda_j} \right] \right) = \infty$.

This is contrary to Lemma 4.2. Thus, $|I_\alpha| < \infty$ for any α . On the other hand, by Lemma 4.1, S has finite regular $\mathcal{J}$ -classes. So, J has finite regular $\mathcal{J}$ -classes. But, by [[12], Proposition 2.6 (6) and Proposition 4.1], the number of regular $\mathcal{J}$ -classes of J is equal to $|\Gamma| + 1$, now $|\Gamma| < \infty$. Thus, $|I| < \infty$ since $I = \sqcup_{\alpha \in \Gamma} I_\alpha$.

For any $i \in I$, by Condition (U), there exists $\lambda_i \in \Lambda$ such that $p_{\lambda_i i}$ (the entry of the sandwich matrix P) is a unit of some T_α . So, $(p_{\lambda_i i}^{-1})_{i \lambda_i}$ is an idempotent of J and

$$K_0[J] = \sum_{i \in I} K_0 \left[(p_{\lambda_i i}^{-1})_{i \lambda_i} J \right] = \sum_{i \in I} K_0 \left[(p_{\lambda_i i}^{-1})_{i \lambda_i} S \right] = \oplus_{i \in I} K_0 \left[(p_{\lambda_i i}^{-1})_{i \lambda_i} S \right].$$

Now by [11, Lemma 1 (iv), pp. 187–188], $K_0[J] = eK_0[S]$ for some $e = e^2 \in K_0[J]$. It follows that $K_0[J]$ has a left identity.

In addition, by the same reason as Lemma 3.3 (ii), we can prove (iii). We omit the detail. $\square$

The following lemma is a key result to research the self-injective algebras of IC abundant semigroups, which may be proved by revising the proof of Theorem 3.6. For the completeness, we give the proof.

Lemma 4.4. *With notations in Lemma 4.3, if J is a proper ideal of S , then J is a regular subsemigroup of S .*

Proof. Suppose that J is a proper ideal of S . By Lemmas 4.3 and 2.3, we may assume that $J = \mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ in which

- (i) $\Gamma = \{1, 2, \dots, n\}$;
- (ii) For any $1 \leq i, j \leq n$ and $i \neq j$, whenever $M_{ij} \neq \emptyset$, we have $i < j$;
- (iii) T_i is a finite group, for any $1 \leq i \leq n$.

Moreover, we let $|I_i| = p_i$ and $|A_i| < q_i$ for any i . We shall use the notations in the proof of Theorem 3.6. By (5),

$$\begin{pmatrix} 0 & 0 & \cdots & A_1 \\ 0 & 0 & \cdots & A_2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \notin \text{ann}_\ell(K_0[J]) \quad (8)$$

and of course, not in $\text{ann}_\ell(K_0[S])$. Notice that by (4)

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}$$

and $K_0[J]$ is an ideal of $K_0[S]$, we have

$$\begin{aligned} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} K_0[S] &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} K_0[S] \\ &\subseteq \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \circ K_0[J] \\ &\subseteq \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{p_n \times q_n}(K_0[M_{nn}]) \end{pmatrix}. \end{aligned} \quad (9)$$

We next prove that $M_{in} = \emptyset$ for $i = 1, 2, \dots, n-1$. Suppose, on the contrary, that not all of M_{in} are empty sets. Obviously,

$$\mathcal{I}_n := \begin{pmatrix} 0 & 0 & \cdots & M_{p_1 \times q_n}(K[M_{1n}]) \\ 0 & 0 & \cdots & M_{p_2 \times q_n}(K[M_{2n}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \neq 0.$$

For any $X \in \text{ann}_r(\mathcal{I}_n)$,

$$\begin{aligned} \mathcal{I}_n \circ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} X &\subseteq \begin{pmatrix} 0 & 0 & \cdots & M_{p_1 \times q_n}(K[M_{1n}]) \\ 0 & 0 & \cdots & M_{p_2 \times q_n}(K[M_{2n}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} P \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} X \\ &\subseteq \begin{pmatrix} 0 & 0 & \cdots & M_{p_1 \times q_n}(K[M_{1n}]) \\ 0 & 0 & \cdots & M_{p_2 \times q_n}(K[M_{2n}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} X = \mathcal{I}_n X = 0. \end{aligned}$$

Hence,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} X \in \text{ann}_r(\mathcal{I}_n) \cap K_0[J] = \text{ann}_r^J(\mathcal{I}_n),$$

where $\text{ann}_r^J(\mathcal{I}_n)$ is the right annihilators of $\mathcal{I}_n$ in $K_0[J]$. From (6), it follows that

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} X \in \begin{pmatrix} M_{p_1 \times q_1}(K[T_1]) & M_{p_1 \times q_2}(K[M_{12}]) & \cdots & M_{p_1 \times q_n}(K[M_{1n}]) \\ 0 & M_{p_2 \times q_2}(K[M_{22}]) & \cdots & M_{p_2 \times q_n}(K[M_{2n}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{p_{n-1} \times q_n}(K[M_{n-1,n}]) \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and so

$$\begin{aligned} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} X &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \circ \left(\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} X \right) \\ &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} P \left(\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} X \right) \\ &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n P_{nn} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ 0 & B_{22} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\ &= 0, \end{aligned}$$

where,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} X = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ 0 & B_{22} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \in \text{ann}_\ell(\text{ann}_r(\mathcal{I}_n)).$$

Now, by Lemma 3.2 (2),

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \in \mathcal{I}_n + \text{ann}_\ell(K_0[S])$$

it follows that $\text{ann}_\ell(K_0[S])$ has an element of the form:

$$\begin{pmatrix} 0 & 0 & \cdots & A_1 \\ 0 & 0 & \cdots & A_2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix},$$

in contradiction to (5). We have now proved that $M_{in} = \emptyset$ for $i = 1, 2, \dots, n-1$.

By applying the similar arguments as above to

$$I_{n-1} := \begin{pmatrix} 0 & 0 & \cdots & M_{p_1 \times q_{n-1}}(K[M_{1,n-1}]) & 0 \\ 0 & 0 & \cdots & M_{p_2 \times q_{n-1}}(K[M_{2,n-1}]) & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{p_{n-2} \times q_{n-1}}(K[M_{n-2,n-1}]) & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varepsilon_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

we may verify that $M_{i,n-1} = \emptyset$ for $i = 1, 2, \dots, n-2$. Continuing this process, we can show that $M_{ij} = \emptyset$ for $j = 1, 2, \dots, i-1$, $i = 2, \dots, n-2$. Thus, $M_{ij} = \emptyset$ whenever $i = j$. By [12, Proposition 2.4 (7)], J is a regular subsemigroup of S . $\square$

Lemma 4.5. *Let I be a proper ideal of an algebra $\mathcal{A}$. If*

- (i) *I has a left identity e ; and*
 - (ii) *$\mathcal{A}$ is right self-injective,*
- then $\mathcal{A}/I$ is right self-injective.*

Proof. Assume that $\mathcal{A} = I$. As pointed out in the Introduction, by hypothesis that $\mathcal{A}$ is right self-injective, $\mathcal{A}$ has a left identity and let ε be a left identity of $\mathcal{A}$. Then, $\mathcal{A} = e\mathcal{A} \oplus (\varepsilon - e)\mathcal{A}$ as right $\mathcal{A}^1$ -modules. Thus, $(\varepsilon - e)\mathcal{A}$ is an injective right $\mathcal{A}^1$ -module. Hence, $\mathcal{A}/I \cong (\varepsilon - e)\mathcal{A}$ and is an injective right $\mathcal{A}^1$ -module.

Now, let J be a right ideal of $(\mathcal{A}/I)^1$ and ϕ a $(\mathcal{A}/I)^1$ -module homomorphism of J into $\mathcal{A}/I$. Observe that the $\mathcal{A}^1$ -module and $\mathcal{A}^1/I$ -module structures on $\mathcal{A}/I$ coincide. Notice that $\mathcal{A}^1/I \cong (\mathcal{A}/I)^1$. If we identify $\mathcal{A}^1/I$ with $(\mathcal{A}/I)^1$, then the inclusion mapping $\iota: J \rightarrow (\mathcal{A}/I)^1$ is an injective $\mathcal{A}^1$ -module homomorphism and ϕ also an $\mathcal{A}^1$ -module homomorphism of J into $\mathcal{A}/I$. By $\mathcal{A}/I$ is an injective right $\mathcal{A}^1$ -module, there exists an $\mathcal{A}^1$ -module homomorphism φ of $(\mathcal{A}/I)^1$ into $\mathcal{A}/I$ such that $\phi = \varphi\iota$. On the other hand, for any $x + I \in \mathcal{A}^1/I$, $x + I = (1 + I) \cdot x$, it follows that there is $u + I \in \mathcal{A}/I$ such that $\varphi(x + I) = (u + I) \cdot x$. From $(u + I) \cdot x = (u + I)(x + I)$, it follows that φ is indeed an $(\mathcal{A}/I)^1$ -module homomorphism. Therefore, $\mathcal{A}/I$ is a right injective $(\mathcal{A}/I)^1$ -module, when $\mathcal{A}/I$ is right self-injective. $\square$

Lemma 4.6. *Let S be a semigroup and U an ideal. Then S is a regular semigroup if and only if U and the Rees quotient S/U are both regular.*

Proof. We only verify the sufficiency. To the end, we assume that U and S/U are both regular. For any $a \in S$, if $a \in U$, then a is regular in S ; if $a \in S/U$, then as S/U is regular, there is $b \in S/U$ such that $a \circ b \circ a = a$ in S/U , in this case, by the definition of Rees quotient, $a \circ b \circ a = aba$ and so $aba = a$ in S , it follows that a is regular in S . However, a is regular in S . Thus, S is a regular semigroup. $\square$

We arrive now at the main result of this section, which generalizes the main result of Okniński on right (respectively, left) self-injective algebras of a regular semigroup (see [7, Theorem 2], which answers affirmatively the Okniński's problem mentioned in the Introduction for the IC abundant semigroup case.

Theorem 4.7. *Let S be an IC abundant semigroup and K a field. If $K_0[S]$ is right (respectively, left) self-injective, then S is a finite regular semigroup. In this case, $K_0[S]$ is Artinian.*

Proof. By Lemma 4.1, we pick a primitive idempotent f_1 of S . By Lemma 2.2, $S_1 = J^*(f_1)$ is a primitive abundant ideal of S .

If $S = S_1$, then by Theorem 3.6, S is a finite regular semigroup.

Suppose that S_1 is a proper ideal of S . Then by Lemma 4.4, S_1 is a regular subsemigroup of S ; and by Lemmas 4.3 and 4.5, $K_0[S/S_1]$ is right self-injective.

Case (i). If S/S_1 is primitive, then by Theorem 3.6, S/S_1 is regular and by Lemma 4.6, S is regular.

Case (ii). Assume that $T_1 := S/S_1$ is not primitive. By Lemma 4.6, S is regular if and only if T_1 is regular. On the other hand, by the definition of Green's $\mathcal{J}$ -relation, it is not difficult to see that for any ideal I of S , $J_a \subseteq I$ for all $a \in I$. This shows that $D_r(T_1) < D_r(S)$ where $D_r(T)$ stands for the number of nonzero regular $\mathcal{J}$ -classes of T . By applying the similar argument to T_1 , there exists a primitive abundant ideal S_2 of T_1 such that

- (1) S_2 is a regular semigroup;
- (2) $T_2 = T_1/S_2$ is an IC abundant semigroup (by Lemma 2.1);
- (3) S is regular if and only if T_2 is regular (by Lemma 4.6);
- (4) $K_0[T_2]$ is right self-injective; and
- (5) $D_r(T_2) < D_r(T_1)$.

This proceedings can continue only finite times since $|D_r(S)| < \infty$ (by Lemma 4.1). So, there exists a positive integer r such that

- (a) T_r is a primitive abundant semigroup;
- (b) $K_0[(T_r)]$ is right self-injective;
- (c) S is regular if and only if so is T_r .

Again by Theorem 3.6, T_r is a finite regular semigroup. Therefore, S is a regular semigroup. By the result of [7], for a regular semigroup S , if $K_0[S]$ is right self-injective, then S is finite, we get that S is a finite regular semigroup. We have finished the proof. $\square$

5 Algebras of primitively semisimple semigroups

Following [13], we call an abundant semigroup S to be *primitively semisimple* if for all $a \in S$, the Rees quotient $J^*(a)/I^*(a)$ is primitive. Indeed, by Lemma 2.1, $J^*(a)/I^*(a)$ is a primitive abundant semigroup. Of course, S is a completely semisimple semigroup if and only if S is a primitively semisimple semigroup being regular.

By the definition of Rees quotients, $J^*(a)/I^*(a)$ is a semigroup whose lying set is $J_a^* \sqcup \{\theta\}$ and in which the multiplication is defined as follows: for any $x, y \in J^*(a)/I^*(a)$

$$x \cdot y = \begin{cases} xy & \text{if } x, y, xy \in J_a^*; \\ \theta & \text{otherwise,} \end{cases}$$

where xy is the product of x and y in S . This shows that

- (a) for any $x \in J_a^*$, x is an idempotent in $J^*(a)/I^*(a)$ if and only if so is x in S ;
- (b) for any $f, g \in E(J_a^*)$, $f \leq g$ in $J^*(a)/I^*(a)$ if and only if $f \leq g$ in S .

It follows that for an abundant semigroup S , $J^*(a)/I^*(a)$ is primitive if and only if any two nonzero idempotents in J_a^* are not comparable under $\leq$. Based on this argument, the following lemma is immediate.

Lemma 5.1. *Let S be an abundant semigroup. Then S is primitively semisimple if and only if any two nonzero idempotents related by $\mathcal{J}^*$ are not comparable under $\leq$.*

Lemma 5.2. *Let S be an abundant semigroup and U a $*$ -ideal of S . If $a, b \in S/U$, then*

- (i) $a\mathcal{D}_S^*b$ if and only if $a\mathcal{D}_{S/U}^*b$.
- (ii) $a\mathcal{J}_S^*b$ if and only if $a\mathcal{J}_{S/U}^*b$.

Proof. (i). Suppose that $(a, b) \in \mathcal{D}_S^*$. Because $\mathcal{D}^*$ is the smallest equivalence containing $\mathcal{L}^*$ and $\mathcal{R}^*$, it follows from [19, Proposition 5.14, p. 28] that there exist $x_1, x_2, \dots, x_{2n-1} \in S$ such that

$$(a, x_1) \in \mathcal{L}_S^*, (x_1, x_2) \in \mathcal{R}_S^*, \dots, (x_{2n-1}, b) \in \mathcal{R}_S^*.$$

By the definition of $*$ -ideal, $a \notin U$ implies that $x_1 \notin U$, whereby $x_2 \notin U, \dots$ and $x_{2n-1} \notin U$. From Lemma 2.1 (i), it follows that

$$(a, x_1) \in \mathcal{L}_{S/U}^*, (x_1, x_2) \in \mathcal{R}_{S/U}^*, \dots, (x_{2n-1}, b) \in \mathcal{R}_{S/U}^*;$$

that is, $(a, b) \in \mathcal{D}_{S/U}^*$. By interchanging the roles of $\mathcal{D}_S^*$ and $\mathcal{D}_{S/U}^*$, we may equally show well the sufficiency.

(ii). By the definition of the relation $\mathcal{J}^*$, $a\mathcal{J}^*b$ if and only if $a \in J^*(b)$ and $b \in J^*(a)$. Suppose now that $(a, b) \in \mathcal{J}_S^*$. Then, $a \in J_S^*(b)$ and $b \in J_S^*(a)$. It follows from [12, Lemma 1.7 (3)] and $b \in J^*(a)$ that there are elements $a_0, a_1, \dots, a_n \in S$, $x_1, \dots, x_n, y_1, \dots, y_n \in S^1$ such that $a = a_0$, $b = a_n$ and $(a_i, x_i a_{i-1} y_i) \in \mathcal{D}_S^*$ for $i = 1, \dots, n$. Notice that $U = \sqcup_{x \in U} J_x^*$ (by the definition of $\mathcal{J}^*$) and $\mathcal{D}^* \subseteq \mathcal{J}^*$. We observe that if $(x, y) \in \mathcal{D}_S^*$, then $x \in U$ if and only if $y \in U$. So, $b \notin U$ can imply that $x_n a_{n-1} y_n \notin U$, whereby $x_n, y_n, a_{n-1} \notin U$, moreover by the same reason, we may show that $x_{n-1}, y_{n-1}, a_{n-2}, \dots, x_1, y_1, a_0 = a \notin U$. Hence, $a_0, a_1, \dots, a_n \in S/U$, $x_1, \dots, x_n, y_1, \dots, y_n \in (S/U)^1$ and $(a_i, x_i a_{i-1} y_i) \in \mathcal{D}_{S/U}^*$ for $i = 1, \dots, n$. Now, by [12, Lemma 1.7 (3)], $b \in J_{S/U}^*(a)$. Similarly, $a \in J_{S/U}^*(b)$ can imply that $a \in J_{S/U}^*(b)$. Thus, $(a, b) \in \mathcal{J}_{S/U}^*$. By interchanging the roles of $\mathcal{J}_S^*$ and $\mathcal{J}_{S/U}^*$, we may equally well show the sufficiency. $\square$

Lemma 5.3. *Let S be a primitively semisimple semigroup and U a $*$ -ideal of S . Then S/U is a primitively semisimple semigroup.*

Proof. For any idempotents $e, f \in S/U$, by the arguments before Lemma 5.1, we know that $e \leq f$ in the semigroup S if and only if $e \leq f$ in the Rees quotient S/U . The remainder of the proof is immediate from Lemmas 5.1 and 5.2. $\square$

Until now, we does not know whether any primitively semisimple semigroup is an IC abundant semigroup in the literature. But for algebras of primitively semisimple semigroups, we have the following theorem.

Theorem 5.4. *Let S be a primitively semisimple semigroup and K a field. If $K_0[S]$ is a right (resp. left) self-injective algebra, then S is a finite regular semigroup; that is, S is a finite completely semisimple semigroup.*

Proof. By Theorem 4.7, we need to only prove that S is regular since any regular semigroup is always an IC abundant semigroup. On the set $S/\mathcal{J}^* = \{J_a^*: a \in S\}$, define

$$J_a^* \leq_J J_b^* \quad \text{if } J^*(a) \subseteq J^*(b).$$

It is a routine check that $\leq_J$ is a partial order on $S/\mathcal{J}^*$. (In what follows, we use $J_a^* <_J J_b^*$ to denote $J_a^* \leq_J J_b^*$ but $J_a^* = J_b^*$.) Because S is abundant, we know that any nonzero $\mathcal{J}^*$ -class of S contains at least one nonzero regular $\mathcal{J}$ -class. It follows from Lemma 4.1 (i) that $S/\mathcal{J}^*$ is a finite set. So, $S/\mathcal{J}^*$ exists a minimum nonzero $\mathcal{J}^*$ -class $J_{a_0}^*$ under $\leq_J$.

Lemma 5.5. $I^*(a_0) = \emptyset$ or $I^*(a_0) = \{\theta\}$ if S has zero θ .

Proof. Assume that $I^*(a_0) \neq \emptyset$ and S has zero θ . For any $x \in I^*(a_0)$, since $I^*(a_0)$ is a $*$ -ideal of S , we get $J^*(x) \subseteq I^*(a_0)$. But, by definition, $I^*(a_0) \subset J^*(a_0)$, now $J^*(x) \subseteq I^*(a_0) \subset J^*(a_0)$. So then, $J_x^* <_J J_{a_0}^*$. By the minimality of $J_{a_0}^*$, $J_x^* = \theta$ so that $x = 0$ by Observation (*), whereby $I^*(a_0) = \{\theta\}$. $\square$

However, $J^*(a_0)$ is isomorphic to $J^*(a_0)/I^*(a_0)$, so then $J^*(a_0)$ is a primitive abundant semigroup.

Case (i). If $S = J^*(a_0)$, then by Theorem 3.6, S is a finite regular semigroup.

Case (ii). If $S \neq J^*(a_0)$, then by Lemma 4.4, $S_0 := J^*(a_0)$ is regular and by Lemmas 4.3 and 4.5, $K_0[S_1] \cong K_0[S]/K_0[S_0]$ and is right self-injective where $S_1 = S/J^*(a_0)$. Also, $D_r(S_1) < D_r(S)$. By Lemma 5.3, S_1 is a primitively semisimple semigroup. To verify that S is regular, it suffices to show that S_1 is regular by Lemma 4.6. By applying the foregoing proof to $K_0[S_1]$, there exists a primitively semisimple semigroup S_2 such that

- (i) S is regular if and only if S_2 is regular;
- (ii) $K_0[S_2]$ is right self-injective; and
- (iii) $D_r(S_2) < D_r(S_1)$.

This proceedings can continue only finite times since $|D_r(S)| < \infty$ (by Lemma 4.1). So, there exists a positive integer r such that

- (a) S_r is a primitive abundant semigroup;
- (b) S is regular if and only if S_r is regular; and
- (c) $K_0[S_r]$ is right self-injective.

By Theorem 3.6, S_r is regular. Consequently, S is a regular semigroup.

However, S is a regular semigroup and further a finite regular semigroup by Theorem 4.7 and hypothesis that $K_0[S]$ is right self-injective. $\square$

Recall that a semigroup is said to be semisimple if all its principal factors are 0-simple. Obviously, any regular semigroup is semisimple. By the proof of (ii) $\Rightarrow$ (iii) in Theorem [11, Theorem 17, p. 196], each G_i in Theorem [11, Theorem 17, p. 196] (iv) comes from the completely 0-simple semigroup S_i/S_{i-1} such that S_i/S_{i-1} is isomorphic to $M^0(G_i, I_i, \Lambda_i; P_i)$, hence G_i is isomorphic to a maximum subgroup of S_i/S_{i-1} , so that G_i may be chosen as a maximum subgroup of S , thus by [18, Theorem 2.20, p. 61], all G_i are just non-isomorphic nontrivial maximum subgroups of S . By Theorems 3.6, 4.7 and 5.4, and [11, Theorem 17, p. 196], the following theorem is immediate and extends the main result of Guo and Shum in [14].

Theorem 5.6. Let S be an IC abundant semigroup (respectively, a primitively semisimple semigroup; a primitive abundant semigroup) and K a field. If $K_0[S]$ has an identity, then the following statements are equivalent:

- (i) $K_0[S]$ is a left self-injective algebra;
- (ii) $K_0[S]$ is a right self-injective algebra;
- (iii) $K_0[S]$ is a quasi-Frobenius algebra;
- (iv) $K_0[S] \cong M_{n_1}(K[G_1]) \oplus M_{n_2}(K[G_2]) \oplus \cdots \oplus M_{n_r}(K[G_r])$, where
 - (a) $r \geq 1$, $n_i \geq 1$;
 - (b) all G_i are just all non-isomorphic nontrivial maximum subgroups G_i of S and are finite.

The following example, due to Okniński [7], shows that not all of right self-injective algebras of IC abundant semigroups have identities.

Example 5.7. Let $S = \{g, h\}$ be the semigroup of left zeros, and $\mathbb{Q}$ the field of rational numbers. Obviously, S is a regular semigroup and of course an IC abundant semigroup. Consider the algebra $\mathbb{Q}[S] = \mathbb{Q}_0[S]$ and the standard extension $\mathbb{Q}[S]^1$ of $\mathbb{Q}[S]$ to a $\mathbb{Q}$ -algebra with unity. It may be shown that for any left ideal I of $\mathbb{Q}[S]^1$, any homomorphism of left $\mathbb{Q}[S]^1$ -modules $I \rightarrow \mathbb{Q}[S]$, extends to a homomorphism of $\mathbb{Q}[S]^1$ -modules $I\mathbb{Q}[S]^1 \rightarrow \mathbb{Q}[S]$. Moreover, by computing the right ideals of $\mathbb{Q}[S]^1$, one can easily check that $\mathbb{Q}[S]^1$ satisfies Baer's condition. Hence, $\mathbb{Q}[S]$ satisfies Baer's condition as $\mathbb{Q}[S]^1$ -module, which means that $\mathbb{Q}[S]$ is left self-injective. It is easy to see that $\mathbb{Q}[S]$ has no left identities.

Remark 5.8. Let S be a semisimple semigroup. If $K_0[S]$ is right (left) self-injective, then by [11, Theorem 14, p. 194], S is finite. Note that any finite 0-simple semigroup is a completely 0-simple semigroup (for completely 0-simple semigroups, see [19, p. 60]). So, all principal factors of S are completely 0-simple semigroups, and hence S is regular; that is, S is a completely semisimple semigroup. Based on this view, Theorem 5.6 is indeed a generalization of [11, Theorem 14, p. 194] while Theorem 4.7 is a generalization of [11, Theorem 17, p. 196].

6 An application

Ji [24], and Ji and Luo [25] researched the semisimplicity of orthodox semigroup algebras. We next consider the semisimplicity of algebras of IC abundant semigroups. Obviously, any semisimple algebra is right (respectively, left) self-injective. So, the following is an immediate consequence of Theorem 4.7.

Proposition 6.1. *Let S be an IC abundant semigroup and K a field. If $K[S]$ is semisimple, then S is a finite regular semigroup.*

The following theorem gives a sufficient and necessary condition for an algebra of IC abundant semigroup to be semisimple.

Theorem 6.2. *Let S be an IC abundant semigroup (respectively, a primitively semisimple semigroup; a primitive abundant semigroup) and K a field. Then $K[S]$ is semisimple if and only if $K_0[S] \cong M_{n_1}(K[G_1]) \oplus M_{n_2}(K[G_2]) \oplus \cdots \oplus M_{n_r}(K[G_r])$ where*

- (i) $r \geq 1$, $n_i \geq 1$ for $i = 1, 2, \dots, r$;
- (ii) *each G_i is a maximal subgroup of S and each $K[G_i]$ is semisimple.*

Proof. We need to only verify the necessity. Assume that $K[S]$ is semisimple. Then, $K[S]$ is a right self-injective algebra with unity. The rest of proof follows from Theorem 5.6. $\square$

Based on Theorems 3.6, 4.7, and 5.4, we have

Corollary 6.3. *Let S be an IC abundant semigroup (respectively, a primitively semisimple semigroup; a primitive abundant semigroup) and K a field. Then $K[S]$ is semisimple if and only if*

- (i) $K_0[S]$ has an unity;
- (ii) $K_0[S]$ is left (resp. right) self-injective;
- (iii) *for any maximum subgroup G of S , $K[G]$ is semisimple.*

Notice that any right (left) self-injective algebra has a left (right) identity. By Corollary 5.3, we have the following.

Corollary 6.4. *Let S be an IC abundant semigroup (respectively, a primitively semisimple semigroup; a primitive abundant semigroup) and K a field. Then $K[S]$ is semisimple if and only if*

- (i) $K_O[S]$ is right self-injective;
- (ii) $K_O[S]$ is left self-injective;
- (iii) for any maximum subgroup G of S , $K[G]$ is semisimple.

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