Research Article

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Two types of hypergeometric degenerate Cauchy numbers

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Abstract: In 1985, Howard introduced degenerate Cauchy polynomials together with degenerate Bernoulli polynomials. His degenerate Bernoulli polynomials have been studied by many authors, but his degenerate Cauchy polynomials have been forgotten. In this paper, we introduce some kinds of hypergeometric degenerate Cauchy numbers and polynomials from the different viewpoints. By studying the properties of the first one, we give their expressions and determine the coefficients. Concerning the second one, called H-degenerate Cauchy polynomials, we show several identities and study zeta functions interpolating these polynomials.

Keywords: hypergeometric Cauchy numbers, degenerate Cauchy numbers, hypergeometric functions, zeta functions, determinants

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1 Introduction

In 1985, Howard [1, (3.6)] introduced the polynomial $\beta_n^{(k)}(\lambda, x)$, defined by means of the generating function

$$\left(\frac{t}{(1 + \lambda t)^{1/\lambda} - 1} \right)^k (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \beta_n^{(k)}(\lambda, x) \frac{t^n}{n!}. \quad (1)$$

When $k = 1$, $\beta_n(\lambda, x) = \beta_n^{(1)}(\lambda, x)$ are the degenerate Bernoulli polynomials, introduced by Carlitz [2] and defined by

$$\left(\frac{t}{(1 + \lambda t)^{1/\lambda} - 1} \right) (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!}. \quad (2)$$

Nowadays, the polynomials in (1) are called the degenerate Bernoulli polynomials of higher order. When $x = 0$ in (3), $\beta_n(\lambda) = \beta_n(\lambda, 0)$ are the degenerate Bernoulli numbers, defined by

$$\frac{t}{(1 + \lambda t)^{1/\lambda} - 1} = \sum_{n=0}^{\infty} \beta_n(\lambda) \frac{t^n}{n!}. \quad (3)$$

When $\lambda \rightarrow 0$ in (2), $B_n(x) = \beta_n(0, x)$ are the ordinary Bernoulli polynomials because

$$\lim_{\lambda \rightarrow 0} \left(\frac{t}{(1 + \lambda t)^{1/\lambda} - 1} \right) (1 + \lambda t)^{x/\lambda} = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

When $\lambda \rightarrow 0$ and $x = 0$ in (2), $B_n = \beta_n(0, 0)$ are the classical Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \quad (4)$$

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The degenerate Bernoulli polynomials in λ and x have rational coefficients. When $x = 0$, $\beta_n(\lambda) = \beta(\lambda, 0)$ are called degenerate Bernoulli numbers. In [3], explicit formulas for the coefficients of the polynomial $\beta_n(\lambda)$ are found. In [4], a general symmetric identity involving the degenerate Bernoulli polynomials and the sums of generalized falling factorials are proved. In [5], a kind of generalization of degenerate Bernoulli numbers, called hypergeometric degenerate Bernoulli numbers, is introduced and its properties are studied.

The classical Bernoulli numbers may be called Bernoulli numbers of the first kind too. It is because there exists another kind of Bernoulli numbers, that is, the Bernoulli numbers of the second kind b_n , defined by

$$\frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} b_n x^n. \quad (5)$$

Bernoulli numbers of the second kind b_n are often studied as Cauchy numbers (of the first kind) c_n . As seen in [3, Lemma 2.1], the following relation holds.

$$c_n = n! b_n. \quad (6)$$

It is natural that Cauchy numbers are recognized by t being replaced by $\log(1+t)$ and taking its reciprocal in the generating function in (4):

$$\frac{e^{\log(1+t)} - 1}{\log(1+t)} = \sum_{n=0}^{\infty} \lambda_n \frac{t^n}{n!}.$$

On the contrary, Bernoulli numbers can be recognized by t being replaced by $e^t - 1$ and taking its reciprocal in the generating function of Cauchy numbers. In fact, $\log(1+t)$ and $e^t - 1$ are inverse functions of each other. Therefore, it is natural to introduce the degenerate Cauchy numbers $\gamma_n(\lambda)$ by replacing t by $\log(1+t)$ and taking its reciprocal in (2). In fact, in the same paper, as the counterpart of degenerate Bernoulli polynomial, Howard [1, (3.7)] introduces another polynomial $\gamma_n^{(k)}(\lambda, x)$, defined by means of the generating function

$$\left(\frac{\lambda t}{(1+t)^\lambda - 1} \right)^k \frac{1}{(1+t)^{x-\lambda}} = \sum_{n=0}^{\infty} \gamma_n^{(k)}(x|\lambda) \frac{t^n}{n!}. \quad (7)$$

This should be called the degenerate Cauchy polynomials. When $k = 1$, the degenerate Cauchy polynomials $\gamma_n(x|\lambda) = \gamma_n^{(1)}(x|\lambda)$ are defined by

$$\frac{\lambda t}{((1+t)^\lambda - 1)(1+t)^{x-\lambda}} = \sum_{n=0}^{\infty} \gamma_n(x|\lambda) \frac{t^n}{n!}. \quad (8)$$

When $x = \lambda$ in (8), degenerate Cauchy numbers $\gamma_n(\lambda) = \gamma_n(\lambda|\lambda)$ can be naturally defined by

$$\frac{\lambda t}{(1+t)^\lambda - 1} = \sum_{n=0}^{\infty} \gamma_n(\lambda) \frac{t^n}{n!} \quad (9)$$

(see also [6, (2.13)]). This matches our idea about the relation between Bernoulli and Cauchy polynomials by replacing t by $\log(1+t)$ and taking its reciprocal in (2).

It seems that this type of degenerate Cauchy polynomials has not been well studied but forgotten. Unfortunately, there seem to exist another definition of degenerate Cauchy polynomials without any historical background or motivation.

2 Degenerate Cauchy numbers

Carlitz [2] has defined the degenerate Stirling numbers of the first and second kinds, $\bar{s}(n, r|\lambda)$ and $S(n, r|\lambda)$ by

$$\left(\frac{1 - (1-t)^\lambda}{\lambda} \right)^r = r! \sum_{n=r}^{\infty} \bar{s}(n, r|\lambda) \frac{t^n}{n!} \quad (10)$$

and

$$((1 + \lambda t)^{1/\lambda} - 1)^r = r! \sum_{n=r}^{\infty} S(n, r|\lambda) \frac{t^n}{n!}, \quad (11)$$

respectively. When $\lambda \rightarrow 0$, $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right] = (-1)^{n-r} \bar{s}(n, r|0)$ are the ordinary (unsigned) Stirling numbers of the first kind and $\left\{ \begin{smallmatrix} n \\ r \end{smallmatrix} \right\} = S(n, r|0)$ are the ordinary Stirling numbers of the second kind. Their generating functions are given by

$$\frac{(-\log(1-t))^k}{k!} = \sum_{n=k}^{\infty} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \frac{t^n}{n!} \quad (12)$$

and

$$\frac{(e^t - 1)^k}{k!} = \sum_{n=k}^{\infty} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \frac{t^n}{n!}, \quad (13)$$

respectively.

In a similar relation between (2) and (10) with (12), from (11) with (13) the degenerate Cauchy numbers $\gamma_n(\lambda)$ can be naturally defined in (9).

Howard [3, Theorem 3.1] determined all the coefficients of $\beta_n(\lambda)$ in (3) as

$$\beta_n(\lambda) = c_n \lambda^n + \sum_{k=1}^n \frac{n}{k} (-1)^{n-k} \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right] B_k \lambda^{n-k} \quad (n \geq 1).$$

In particular, the leading coefficient is the classical Cauchy numbers c_n and the constant term is the classical Bernoulli numbers B_n . The list of some degenerate Bernoulli numbers is in Appendix.

The coefficients of $\gamma_n(\lambda)$ can also be determined. In fact, the coefficients appear in the reverse order of those of $\beta_n(\lambda)$. The list of some degenerate Cauchy numbers is in Appendix.

Theorem 1. For $n \geq 1$,

$$\gamma_n(\lambda) = c_n + \sum_{k=1}^n \frac{n}{k} (-1)^{n-k} \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right] B_k \lambda^k.$$

Proof. Put $f = \lambda \log(1+t)$. Then, by (12)

$$\begin{aligned} \sum_{n=0}^{\infty} \gamma_n(\lambda) \frac{t^n}{n!} &= \frac{\lambda t}{(1+t)^\lambda - 1} \\ &= \frac{\lambda t}{f} \frac{f}{e^f - 1} \\ &= \frac{\lambda t}{f} + \frac{\lambda t}{f} \sum_{k=1}^{\infty} B_k \frac{f^k}{k!} \\ &= \frac{t}{\log(1+t)} + \lambda t \sum_{k=1}^{\infty} \frac{B_k}{k!} \lambda^{k-1} (\log(1+t))^{k-1} \\ &= \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} + t \sum_{k=1}^{\infty} \frac{B_k}{k} \lambda^k \sum_{n=k-1}^{\infty} (-1)^{n-k+1} \left[\begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right] \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} + \sum_{k=1}^{\infty} \frac{B_k}{k} \lambda^k \sum_{n=k}^{\infty} (-1)^{n-k} \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right] \frac{t^n}{(n-1)!} \\ &= \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^n (-1)^{n-k} \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right] B_k \right) \lambda^k \frac{t^n}{n!}. \end{aligned}$$

□

3 Hypergeometric degenerate Cauchy numbers

In [7,8], the hypergeometric Cauchy numbers $c_{N,n}$ ($N \geq 1$, $n \geq 0$) are introduced as

$$\frac{1}{{}_2F_1(1, N; N+1; -t)} = \frac{(-1)^{N-1}t^N/N}{\log(1+t) - \sum_{n=1}^{N-1} (-1)^{n-1}t^n/n} := \sum_{n=0}^{\infty} c_{N,n} \frac{t^n}{n!}, \quad (14)$$

where

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}} \frac{z^n}{n!}$$

is the Gauss hypergeometric function with the rising factorial $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ ($n \geq 1$) and $(x)^{(0)} = 1$. In this paper, $(x)_n = x(x-1)\cdots(x-n+1)$ ($n \geq 1$) is the falling factorial with $(x)_0 = 1$. Similar hypergeometric numbers are hypergeometric Bernoulli numbers [9–11] and hypergeometric Euler numbers [12,13].

Define the hypergeometric degenerate Cauchy numbers $\gamma_{N,n}(\lambda)$ by the generating function

$$\frac{1}{{}_2F_1(1, N-\lambda; N+1; -t)} = \frac{(\lambda-1)_{N-1}t^N/N!}{\frac{(1+t)^\lambda-1}{\lambda} - \sum_{n=1}^{N-1} (\lambda-1)_{n-1}t^n/n!} := \sum_{n=0}^{\infty} \gamma_{N,n}(\lambda) \frac{t^n}{n!}. \quad (15)$$

We list some initial values of $\gamma_{N,n}(\lambda)$ ($0 \leq n \leq 5$) in Appendix. When $\lambda \rightarrow 0$ in (15), $c_{N,n} = \gamma_{N,n}(0)$ are the hypergeometric Cauchy numbers in (14). When $N = 1$ in (15), $\gamma_n(\lambda) = \gamma_{1,n}(\lambda)$ are the degenerate Cauchy numbers in (9).

Note that

$${}_2F_1(1, N-\lambda; N+1; -t) = \sum_{n=0}^{\infty} \frac{(N-\lambda)^{(n)}}{(N+1)^{(n)}} (-t)^n = \sum_{n=0}^{\infty} \frac{(\lambda-N)_n N!}{(N+n)!} t^n = \left(\sum_{n=N}^{\infty} \frac{(\lambda-1)_{n-1}}{n!} t^n \right) \left(\frac{(\lambda-1)_{N-1}}{N!} t^N \right)^{-1}. \quad (16)$$

First, we shall show the recurrence relation of hypergeometric degenerate Cauchy numbers.

Proposition 1. For $n \geq 1$ and $N \geq 1$, we have

$$\gamma_{N,n}(\lambda) = -n! \sum_{k=0}^{n-1} \frac{(\lambda-N)_{n-k} N!}{(N+n-k)! k!} \gamma_{N,k}(\lambda)$$

with $\gamma_{N,0}(\lambda) = 1$.

Proof. From the definition in (15),

$$1 = \left(1 + \sum_{l=1}^{\infty} \frac{(\lambda-N)_l N!}{(N+l)!} t^l \right) \left(\sum_{n=0}^{\infty} \gamma_{N,n}(\lambda) \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} \gamma_{N,n}(\lambda) \frac{t^n}{n!} + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(\lambda-N)_{n-k} N!}{(N+n-k)!} \frac{\gamma_{N,k}(\lambda)}{k!} t^n.$$

Comparing the coefficients on both sides, we get the desired result. $\square$

The hypergeometric degenerate Cauchy numbers can be expressed in terms of determinants.

Theorem 2. For $n, N \geq 1$,

$$\gamma_{N,n}(\lambda) = (-1)^n n! \begin{vmatrix} \frac{(\lambda-N)N!}{(N+1)!} & 1 & & 0 \\ \frac{(\lambda-N)_2 N!}{(N+2)!} & \frac{(\lambda-N)N!}{(N+1)!} & & \\ \vdots & \vdots & \ddots & 1 \\ \frac{(\lambda-N)_{n-1} N!}{(N+n-1)!} & \frac{(\lambda-N)_{n-2} N!}{(N+n-2)!} & \cdots & \frac{(\lambda-N)N!}{(N+1)!} & 1 \\ \frac{(\lambda-N)_n N!}{(N+n)!} & \frac{(\lambda-N)_{n-1} N!}{(N+n-1)!} & \cdots & \frac{(\lambda-N)_2 N!}{(N+2)!} & \frac{(\lambda-N)N!}{(N+1)!} \end{vmatrix}. \quad (17)$$

Remark. When $\lambda \rightarrow 0$, Theorem 2 is reduced to [7, Theorem 1]. When $\lambda \rightarrow 0$ and $N = 1$, we have a determinant expression of Cauchy numbers by Glaisher [14, p. 50].

Proof. The proof is obtained by induction on n . When $n = 1$, the result is trivial because $y_{N,1}(\lambda) = -\frac{(\lambda-N)N!}{(N+1)!}$. Assume that the result is valid up to $n-1$. Then, the determinant of the right-hand side (RHS) of (17) is expanded along the first row.

$$\begin{aligned}
 & \frac{(\lambda-N)N!}{(N+1)!} \frac{(-1)^{n-1} y_{N,n-1}(\lambda)}{(n-1)!} - \begin{vmatrix} \frac{(\lambda-N)_2 N!}{(N+2)!} & 1 & & 0 \\ \frac{(\lambda-N)_3 N!}{(N+3)!} & \frac{(\lambda-N)N!}{(N+1)!} & & \\ \vdots & \vdots & \ddots & 1 \\ \frac{(\lambda-N)_{n-1} N!}{(N+n-1)!} & \frac{(\lambda-N)_{n-3} N!}{(N+n-3)!} & \dots & \frac{(\lambda-N)N!}{(N+1)!} & 1 \\ \frac{(\lambda-N)_n N!}{(N+n)!} & \frac{(\lambda-N)_{n-2} N!}{(N+n-2)!} & \dots & \frac{(\lambda-N)_2 N!}{(N+2)!} & \frac{(\lambda-N)N!}{(N+1)!} \end{vmatrix} \\
 &= \dots \\
 &= \frac{(\lambda-N)N!}{(N+1)!} \frac{(-1)^{n-1} y_{N,n-1}(\lambda)}{(n-1)!} - \frac{(\lambda-N)_2 N!}{(N+2)!} \frac{(-1)^{n-2} y_{N,n-2}(\lambda)}{(n-2)!} + \dots + \begin{vmatrix} \frac{(N\lambda)_{n-1} N!}{(N+n-1)!} & 1 \\ \frac{(\lambda-N)_n N!}{(N+n)!} & \frac{(\lambda-N)N!}{(N+1)!} \end{vmatrix} \\
 &= (-1)^{n-1} \sum_{m=0}^{n-1} \frac{(\lambda-N)_{n-m} N!}{(N+n-m)!} \frac{y_{N,m}(\lambda)}{m!} = \frac{(-1)^n y_{N,n}(\lambda)}{n!}.
 \end{aligned}$$

Here, we used the recurrence relation in Proposition 1. $\square$

The hypergeometric degenerate Cauchy numbers can be expressed explicitly.

Theorem 3. For $n, N \geq 1$, we have

$$y_{N,n}(\lambda) = n! \sum_{k=1}^n (-N!)^k \sum_{\substack{i_1 + \dots + i_k = n \\ i_1, \dots, i_k \geq 1}} \frac{(\lambda-N)_{i_1}}{(N+i_1)!} \dots \frac{(\lambda-N)_{i_k}}{(N+i_k)!}.$$

Proof. The proof is obtained by induction on n . When $n = 1$, the result is valid because $y_{N,1}(\lambda) = -\frac{(\lambda-N)N!}{(N+1)!}$. Assume that the result is valid up to $n-1$. Then, by using the recurrence relation in Proposition 1, we have

$$\begin{aligned}
 \frac{y_{N,n}(\lambda)}{n!} &= - \sum_{l=0}^{n-1} \frac{(\lambda-N)_{n-l} N!}{(N+n-l)!} \frac{y_{N,l}(\lambda)}{l!} \\
 &= - \sum_{l=1}^{n-1} \frac{(\lambda-N)_{n-l} N!}{(N+n-l)!} \sum_{k=1}^l (-N!)^k \sum_{\substack{i_1 + \dots + i_k = l \\ i_1, \dots, i_k \geq 1}} \frac{(\lambda-N)_{i_1}}{(N+i_1)!} \dots \frac{(\lambda-N)_{i_k}}{(N+i_k)!} - \frac{(\lambda-N)_n N!}{(N+n)!} \\
 &= - \sum_{k=1}^{n-1} (-N!)^k \sum_{l=k}^{n-1} \frac{(\lambda-N)_{n-l} N!}{(N+n-l)!} \sum_{\substack{i_1 + \dots + i_k = l \\ i_1, \dots, i_k \geq 1}} \frac{(\lambda-N)_{i_1}}{(N+i_1)!} \dots \frac{(\lambda-N)_{i_k}}{(N+i_k)!} - \frac{(\lambda-N)_n N!}{(N+n)!} \\
 &= - \sum_{k=2}^n (-N!)^{k-1} \sum_{l=k-1}^{n-1} \frac{(\lambda-N)_{n-l} N!}{(N+n-l)!} \sum_{\substack{i_1 + \dots + i_{k-1} = l \\ i_1, \dots, i_{k-1} \geq 1}} \frac{(\lambda-N)_{i_1}}{(N+i_1)!} \dots \frac{(\lambda-N)_{i_{k-1}}}{(N+i_{k-1})!} - \frac{(\lambda-N)_n N!}{(N+n)!} \\
 &= \sum_{k=2}^n (-N!)^k \sum_{\substack{i_1 + \dots + i_k = n \\ i_1, \dots, i_k \geq 1}} \frac{(\lambda-N)_{i_1}}{(N+i_1)!} \dots \frac{(\lambda-N)_{i_k}}{(N+i_k)!} \quad (i_k = n-l \geq 1) - \frac{(\lambda-N)_n N!}{(N+n)!} \\
 &= \sum_{k=1}^n (-N!)^k \sum_{\substack{i_1 + \dots + i_k = n \\ i_1, \dots, i_k \geq 1}} \frac{(\lambda-N)_{i_1}}{(N+i_1)!} \dots \frac{(\lambda-N)_{i_k}}{(N+i_k)!}.
 \end{aligned}$$

$\square$

There exists another form of the hypergeometric degenerate Cauchy numbers.

Theorem 4. For $n, N \geq 1$, we have

$$\gamma_{N,n}(\lambda) = n! \sum_{k=1}^n (-N!)^k \binom{n+1}{k+1} \sum_{\substack{i_1+\dots+i_k=n \\ i_1, \dots, i_k \geq 0}} \frac{(\lambda-N)_{i_1}}{(N+i_1)!} \cdots \frac{(\lambda-N)_{i_k}}{(N+i_k)!}.$$

Proof. The proof can be obtained similar to that of Theorem 3. However, we give a different proof here. Put $1+w = {}_2F_1(1, N-\lambda; N+1; -t)$. By the definition in (15),

$$\begin{aligned} \gamma_{N,n}(\lambda) &= \frac{d^n}{dt^n} (1+w)^{-1} \Big|_{t=0} = \frac{d^n}{dt^n} \left(\sum_{l=0}^{\infty} (-w)^l \right) \Big|_{t=0} = \sum_{l=0}^{\infty} \frac{d^n}{dt^n} (-w)^l \Big|_{t=0} \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^l (-1)^k \binom{l}{k} \frac{d^n}{dt^n} ({}_2F_1(1, N-\lambda; N+1; -t))^k \Big|_{t=0}. \end{aligned}$$

By relation (16), we know that

$$\frac{d^n}{dt^n} ({}_2F_1(1, N-\lambda; N+1; -t))^k \Big|_{t=0} = \frac{d^n}{dt^n} \left(\sum_{j=0}^{\infty} \frac{(\lambda-N)_j N!}{(N+j)!} t^j \right)^k \Big|_{t=0} = n! R_k(n),$$

where

$$R_k(n) = \sum_{\substack{i_1+\dots+i_k=n \\ i_1, \dots, i_k \geq 0}} \frac{(\lambda-N)_{i_1} N!}{(N+i_1)!} \cdots \frac{(\lambda-N)_{i_k} N!}{(N+i_k)!}.$$

Since $R_k(n) = 0$, if $k = 0$, we have

$$\begin{aligned} \gamma_{N,n}(\lambda) &= \sum_{l=0}^n \sum_{k=0}^l (-1)^k \binom{l}{k} n! R_k(n) = n! \sum_{k=0}^n (-1)^k \sum_{l=k}^n \binom{l}{k} R_k(n) \\ &= n! \sum_{k=1}^n (-1)^k R_k(n) \sum_{l=k}^n \binom{l}{k} = n! \sum_{k=1}^n (-1)^k \binom{n+1}{k+1} R_k(n). \end{aligned}$$

□

3.1 Coefficients of the hypergeometric degenerate Cauchy numbers

Some coefficients of $\gamma_{N,n}(\lambda)$ can also be explicitly described. In particular, the leading coefficient is the hypergeometric Bernoulli numbers $B_{N,n}$ and the constant is the hypergeometric Cauchy numbers $c_{N,n}$. When $N = 1$, this result is reduced to the coefficients given in Theorem 1.

Theorem 5. The leading coefficient of $\gamma_{N,n}(\lambda)$ is equal to $B_{N,n}$ and its constant term is equal to $c_{N,n}$.

Proof. From Theorem 3, we get the leading coefficient of $\gamma_{N,n}(\lambda)$ with λ^n as

$$n! \sum_{k=1}^n (-N!)^k \sum_{\substack{i_1+\dots+i_k=n \\ i_1, \dots, i_k \geq 1}} \frac{\lambda^{i_1}}{(N+i_1)!} \cdots \frac{\lambda^{i_k}}{(N+i_k)!} = n! \sum_{k=1}^n (-N!)^k \sum_{\substack{i_1+\dots+i_k=n \\ i_1, \dots, i_k \geq 1}} \frac{1}{(N+i_1)! \cdots (N+i_k)!} \lambda^n,$$

which is equal to the expression of $B_{N,n} \lambda^n$ in [15, Proposition 2]. From Theorem 3 again, we get the constant term of $\gamma_{N,n}(\lambda)$ as

$$n! \sum_{k=1}^n (-N!)^k \sum_{\substack{i_1 + \dots + i_k = n \\ i_1, \dots, i_k \geq 1}} \frac{(-N)_{i_1}}{(N + i_1)!} \dots \frac{(-N)_{i_k}}{(N + i_k)!} = n! \sum_{k=1}^n (-1)^{n-k} \sum_{\substack{i_1 + \dots + i_k = n \\ i_1, \dots, i_k \geq 1}} \frac{N^k}{(N + i_1) \dots (N + i_k)},$$

which is equal to the expression of $c_{N,n}$ in [7, Lemma 1]. $\square$

Nevertheless, it seems difficult to determine other coefficients in any explicit form.

4 A different kind of hypergeometric degenerate Cauchy numbers

The Cauchy numbers c_n are defined by the generating function

$$G_C(x) := \frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}.$$

Notice that Cauchy numbers are strongly related to Bernoulli numbers B_n via the relation

$$G_C(x) = \frac{1}{G_B(\log(1+x))}, \quad (18)$$

where $G_B(x)$ is the generating function of Bernoulli numbers

$$G_B(x) := \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

One kind of generalization of Bernoulli numbers is called hypergeometric Bernoulli numbers $B_{N,n}$ ([9–11]), defined by

$$G_{HB,N}(x) := \frac{1}{{}_1F_1(1; N+1; x)} = \sum_{n=0}^{\infty} B_{N,n} \frac{x^n}{n!},$$

where

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}} \frac{z^n}{n!}$$

with the rising factorial $(x)^{(n)} = x(x+1) \dots (x+n-1)$ ($n \geq 1$) and $(x)^{(0)} = 1$ is the confluent hypergeometric function. When $N = 1$, we have $B_n = B_{1,n}$. One kind of generalization of Cauchy numbers is called hypergeometric Cauchy numbers $c_{N,n}$ ([8]). When $N = 1$, we have $c_n = c_{1,n}$. Nevertheless, in [16], another type of hypergeometric numbers is introduced with respect to (18). For $n \geq 0$ and $N \geq 1$, H-Cauchy numbers are defined by

$$G_{HC,N}(x) := \frac{1}{N! G_{HB,N}(\log(1+x))} = \frac{{}_1F_1(1; N+1; \log(1+x))}{N!} = \sum_{n=0}^{\infty} c_n^{(N)} \frac{x^n}{n!}.$$

In this section, similar to H-Cauchy numbers, we shall introduce a different kind of hypergeometric degenerate Cauchy numbers with respect to (18). The H-degenerate Cauchy numbers $C_{N,n}(\lambda)$ are defined by the generating function

$$G_{HDC,N}(x) := \frac{1}{N! G_{HDB,N}(\log(1+x))} = \frac{{}_2F_1(1, N-1/\lambda; N+1; -\lambda \log(1+x))}{N!} = \sum_{n=0}^{\infty} C_{N,n}(\lambda) \frac{x^n}{n!}, \quad (19)$$

where the hypergeometric degenerate Bernoulli numbers $\beta_{N,n}(\lambda)$ ([17]) are defined by

$$G_{HDB,N}(x) := \frac{1}{{}_2F_1(1, N-1/\lambda; N+1; -\lambda x)} = \sum_{n=0}^{\infty} \beta_{N,n}(\lambda) \frac{x^n}{n!}.$$

In this case, another type of degenerate Cauchy numbers $C_{1,n}(\lambda)$ are given by

$$\frac{(1 + \lambda \log(1 + x))^{1/\lambda} - 1}{\log(1 + x)} = \sum_{n=0}^{\infty} C_{1,n}(\lambda) \frac{x^n}{n!}. \quad (20)$$

When $\lambda \rightarrow 0$, the degenerate Cauchy numbers in (20) are reduced to the classical Cauchy numbers c_n . We list some initial values of $C_{N,n}(\lambda)$ ($0 \leq n \leq 5$) in Appendix.

H-degenerate Cauchy numbers can be expressed explicitly in terms of the (unsigned) Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$. Here, $(a|b)_n = a(a-b)\cdots(a-(n-1)b$ ($n \geq 1$) is the generalized falling factorial with $(a|b)_0 = 1$. Note that when $b = 1$, $(a)_n = (a|1)_n$ is the original falling factorial.

Theorem 6. For $N, n \geq 0$, we have

$$C_{N,n}(\lambda) = \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] \frac{(-1)^{n-m} (1 - N\lambda| \lambda)_m m!}{(N+m)!}.$$

Remark. When $\lambda \rightarrow 0$ in Theorem 6, we have

$$C_{N,n}(0) = \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] \frac{(-1)^{n-m} m!}{(N+m)!},$$

that is [16, Theorem 3.1] (see also [18, Theorem 2.2]). When $N = 1$ in Theorem 6, we have

$$C_{1,n}(\lambda) = \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] \frac{(-1)^{n-m} (1 - \lambda| \lambda)_m}{m+1},$$

which shows an explicit expression of another type of degenerate Cauchy numbers. When $\lambda \rightarrow 0$ and $N = 1$ in Theorem 6, we have

$$C_{1,n}(0) = \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] \frac{(-1)^{n-m}}{m+1} = c_n,$$

that is, an explicit expression of the classical Cauchy numbers ([19, Ch. VII], [20, p. 1908]).

Proof of Theorem 6. By the definition in (19),

$$\begin{aligned} \sum_{n=0}^{\infty} C_{N,n}(\lambda) \frac{x^n}{n!} &= \frac{1}{N!} \sum_{m=0}^{\infty} \frac{(1 - N\lambda| \lambda)_m}{(N+m)_m} (\log(1+x))^m = \sum_{m=0}^{\infty} \frac{(1 - N\lambda| \lambda)_m m!}{(N+m)!} \frac{(\log(1+x))^m}{m!} \\ &= \sum_{m=0}^{\infty} \frac{(1 - N\lambda| \lambda)_m m!}{(N+m)!} \sum_{n=m}^{\infty} (-1)^{n-m} \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] \frac{(-1)^{n-m} (1 - N\lambda| \lambda)_m m!}{(N+m)!} \frac{x^n}{n!}. \end{aligned}$$

Comparing the coefficients on both sides, we get the desired result. $\square$

By Theorem 6, we obtain an expression of H-degenerate Cauchy numbers for negative indices, where the definition in (19) can be interpreted as

$$\sum_{n=0}^{\infty} C_{-N,n}(\lambda) \frac{x^n}{n!} = \sum_{n=N}^{\infty} \frac{(N-1/\lambda)^{(n)}}{(n-N)!} (-\lambda \log(1+x))^n.$$

Corollary 1. For $N, n \geq 0$, we have

$$C_{-N,n}(\lambda) = \sum_{m=N}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] \frac{(-1)^{n-m} (1 + N\lambda| \lambda)_m m!}{(m-N)!}.$$

Theorem 7.

$$\sum_{n=0}^{\infty} \sum_{N=0}^{\infty} C_{-N,n}(\lambda) \frac{x^n y^N}{n! N!} = \frac{(1 + \lambda \log(1+x))^{1/\lambda}}{(1 - \lambda y \log(1+x))^{1+1/\lambda}}.$$

Remark. When $\lambda \rightarrow 0$,

$$\text{RHS} \rightarrow \frac{e^{\log(1+x)}}{1 \cdot e^{-y \log(1+x)}} = (1+x)^y,$$

that is, the result in [16, Theorem 3.4].

Proof of Theorem 7. By Corollary 1 with the proof of Theorem 6, we have

$$\begin{aligned} \text{LHS} &= \sum_{N=0}^{\infty} \sum_{m=N}^{\infty} \frac{(1 + N\lambda|\lambda)_m}{(m-N)!} (\log(1+x))^m \frac{y^N}{N!} \\ &= \sum_{N=0}^{\infty} \sum_{m=0}^{\infty} \frac{(1 + N\lambda|\lambda)_{m+N}}{m!} (\log(1+x))^{m+N} \frac{y^N}{N!} \\ &= \sum_{m=0}^{\infty} \frac{(1/\lambda)_m (\lambda \log(1+x))^m}{m!} \sum_{N=0}^{\infty} \frac{(N + 1/\lambda)_N (\lambda y \log(1+x))^N}{N!} \\ &= \sum_{m=0}^{\infty} \binom{1/\lambda}{m} (\lambda \log(1+x))^m \sum_{N=0}^{\infty} \binom{N + 1/\lambda}{N} (\lambda y \log(1+x))^N \\ &= \sum_{m=0}^{\infty} \binom{1/\lambda}{m} (\lambda \log(1+x))^m \sum_{N=0}^{\infty} \binom{-1 - 1/\lambda}{N} (-\lambda y \log(1+x))^N \\ &= (1 + \lambda \log(1+x))^{1/\lambda} (1 - \lambda y \log(1+x))^{-1-1/\lambda} \\ &= \text{RHS}. \end{aligned}$$

□

5 Hypergeometric degenerate Cauchy polynomials

The H-degenerate Cauchy polynomials $C_{N,n}(\lambda, w; z)$ are defined by the generating function

$$\begin{aligned} (1+x)^{wz} \frac{{}_2F_1(1, N-1/\lambda; N+1; w\lambda \log(1+x))}{N!} &= (1+x)^{wz} \sum_{n=0}^{\infty} \frac{(1 - N\lambda|\lambda)_n}{(N+n)!} (-w \log(1+x))^n \\ &= \sum_{n=0}^{\infty} C_{N,n}(\lambda, w; z) \frac{x^n}{n!}. \end{aligned} \quad (21)$$

When $w = -1$, $C_{N,n}(\lambda, -1; z)$ are the H-degenerate Cauchy polynomials of the first kind. When $w = 1$, $C_{N,n}(\lambda, 1; z)$ are the H-degenerate Cauchy polynomials of the second kind. When $\lambda \rightarrow 0$, $C_{N,n}(0, -1; z)$ and $C_{N,n}(0, 1; z)$ are the H-Cauchy polynomials of the first kind and those of the second kind, respectively [16]. Furthermore, when $N = 1$ and $z = 0$, $c_n = C_{1,n}(0, -1; 0)$ and $\hat{c}_n = C_{1,n}(0, 1; 0)$ are the Cauchy polynomials of the first kind and those of the second kind, respectively [19,20]. Note that

$$\frac{x}{(1+x)\log(1+x)} = \sum_{n=0}^{\infty} \hat{c}_n \frac{x^n}{n!}.$$

First, we shall show an explicit expression of the H-degenerate Cauchy polynomials.

Theorem 8. For $N, n \geq 0$, we have

$$C_{N,n}(\lambda, w; z) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} (-w)^m \sum_{j=0}^m \binom{m}{j} \frac{(1 - N\lambda|\lambda)_{m-j} (m-j)!}{(N+m-j)!} (-z)^j.$$

Proof. By the definition in (21) with the proof of Theorem 6, we have

$$\begin{aligned}\sum_{n=0}^{\infty} C_{N,n}(\lambda, w; z) \frac{x^n}{n!} &= \sum_{j=0}^{\infty} \frac{(wz \log(1+x))^j}{j!} \sum_{l=0}^{\infty} \frac{(1-N\lambda|\lambda)_l (-w \log(1+x))^l}{(N+l)!} \\ &= \sum_{m=0}^{\infty} \frac{(w \log(1+x))^m}{m!} \sum_{j=0}^m \frac{(-1)^{m-j} z^j (1-N\lambda|\lambda)_{m-j} m!}{j! (N+m-j)!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} (-w)^m \sum_{j=0}^m \binom{m}{j} \frac{(1-N\lambda|\lambda)_{m-j} (m-j)!}{(N+m-j)!} (-z)^j \frac{x^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides, we get the desired result. $\square$

If one takes the summation in terms of the Stirling numbers of the second kind, we have the following identity.

Theorem 9. For $N, n \geq 1$, we have

$$\sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} C_{N,n}(\lambda, w; z) = (-w)^m \sum_{j=0}^n \binom{n}{j} \frac{(1-N\lambda|\lambda)_{n-j} (n-j)!}{(N+n-j)!} (-z)^j.$$

Proof. We use the orthogonal relation between two kinds of Stirling numbers:

$$\sum_{m=l}^n (-1)^{m-l} \begin{bmatrix} m \\ l \end{bmatrix} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \begin{cases} 1 & (n=l), \\ 0 & (n \neq l). \end{cases}$$

By Theorem 8, we have

$$\begin{aligned}\sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} C_{N,n}(\lambda, w; z) &= \sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \sum_{l=0}^m \begin{bmatrix} m \\ l \end{bmatrix} (-1)^{m-l} (-w)^l \sum_{j=0}^l \binom{l}{j} \frac{(1-N\lambda|\lambda)_{l-j} (l-j)!}{(N+l-j)!} (-z)^j \\ &= \sum_{l=0}^n \sum_{m=l}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \begin{bmatrix} m \\ l \end{bmatrix} (-1)^{m-l} (-w)^l \sum_{j=0}^l \binom{l}{j} \frac{(1-N\lambda|\lambda)_{l-j} (l-j)!}{(N+l-j)!} (-z)^j \\ &= (-w)^n \sum_{j=0}^n \binom{n}{j} \frac{(1-N\lambda|\lambda)_{n-j} (n-j)!}{(N+n-j)!} (-z)^j.\end{aligned}$$

$\square$

5.1 Relations between two kinds of H-degenerate Cauchy polynomials

There are two kinds of H-degenerate Cauchy polynomials $C_{N,n}(\lambda, w; z)$ and $C_{N,n}(\lambda, -w; z)$. We shall show a relation between two kinds.

Theorem 10.

$$(-1)^n \frac{C_{N,n}(\lambda, w; z)}{n!} = \sum_{m=0}^n \binom{n-1}{m-1} \frac{C_{N,n}(\lambda, -w; z)}{m!}.$$

Proof. First,

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n \frac{C_{N,n}(\lambda, w; z)}{n!} x^n &= (1-x)^{wz} \sum_{m=0}^{\infty} \frac{(1-N\lambda|\lambda)_m}{(N+m)!} (-w \log(1-x))^m \\ &= \left(1 + \frac{x}{1-x}\right)^{-wz} \sum_{m=0}^{\infty} \frac{(1-N\lambda|\lambda)_m}{(N+m)!} \left(w \log\left(1 + \frac{x}{1-x}\right)\right)^m \\ &= \sum_{m=0}^{\infty} C_{N,n}(\lambda, -w; z) \frac{\left(\frac{x}{1-x}\right)^m}{m!}.\end{aligned}$$

Second,

$$\sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n-1}{m-1} \frac{C_{N,n}(\lambda, -w; z)}{m!} x^n = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \binom{n-1}{m-1} x^n \frac{C_{N,m}(\lambda, -w; z)}{m!} = \sum_{m=0}^{\infty} \left(\frac{x}{1-x} \right)^m \frac{C_{N,m}(\lambda, -w; z)}{m!}.$$

□

5.2 H-degenerate Cauchy polynomials with negative indices

By Theorem 8, we obtain an expression of H-degenerate Cauchy numbers with negative indices, where the definition in (21) can be interpreted as

$$\sum_{n=0}^{\infty} C_{-N,n}(\lambda, w; z) \frac{x^n}{n!} = (1+x)^{wz} \sum_{n=N}^{\infty} \frac{(1-N\lambda|\lambda)_n}{(n-N)!} (-w \log(1+x))^n.$$

Proposition 2. For $N, n \geq 0$, we have

$$C_{-N,n}(\lambda, w; z) = \sum_{m=0}^n \left[\begin{matrix} n \\ m \end{matrix} \right] (-1)^{n-m} (-w)^m \sum_{j=0}^m \binom{m}{j} \frac{(1+N\lambda|\lambda)_{m-j} (m-j)!}{(m-j-N)!} (-z)^j.$$

By Proposition 2, we have the identity for the double summations.

Theorem 11.

$$\sum_{n=0}^{\infty} \sum_{N=0}^{\infty} C_{-N,n}(\lambda) \frac{x^n}{n!} \frac{y^N}{N!} = \frac{(1+x)^{wz} (1-w\lambda \log(1+x))^{1/\lambda}}{(1+w\lambda y \log(1+x))^{1+1/\lambda}}.$$

Remark. When $\lambda \rightarrow 0$, $\text{RHS} \rightarrow (1+x)^{(z-y-1)w}$. This is reduced to Theorem 6.2 in [16].

Proof of Theorem 11. Similar to the proof of Theorem 7, by Proposition 2,

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{N=0}^{\infty} C_{-N,n}(\lambda) \frac{x^n}{n!} \frac{y^N}{N!} &= (1+x)^{wz} \sum_{N=0}^{\infty} \sum_{m=N}^{\infty} \frac{(1+N\lambda|\lambda)_m}{(m-N)!} (-w \log(1+x))^m \frac{y^N}{N!} \\ &= (1+x)^{wz} \sum_{m=0}^{\infty} \frac{(1/\lambda)_m (-w\lambda \log(1+x))^m}{m!} \sum_{N=0}^{\infty} \frac{(N+1/\lambda)_N (-w\lambda y \log(1+x))^N}{N!} \\ &= (1+x)^{wz} (1-w\lambda \log(1+x))^{1/\lambda} (1+w\lambda y \log(1+x))^{-1-1/\lambda}. \end{aligned}$$

□

5.3 Zeta functions interpolating H-degenerate Cauchy polynomials

Let $0 < c < 1$. For a positive integer N , define the function $Z_N(s, \lambda, w, z)$ ($s, w, z \in \mathbb{C}$) with $\Re s > 0$ and $|w| < 1/c$ as

$$Z_N(s, \lambda, w, z) = \frac{1}{\Gamma(s)} \int_0^{1-e^{-c/\lambda}} \frac{(1-t)^{wz}}{N!} {}_2F_1(1, N-1/\lambda; N+1; w\lambda \log(1-t)) t^{s-1} dt, \quad (22)$$

or by the change of the variables $t = 1 - e^{-u}$,

$$Z_N(s, \lambda, w, z) = \frac{1}{\Gamma(s)} \int_0^{c/\lambda} \frac{(1-e^{-u})^{s-1}}{N!} {}_2F_1(1, N-1/\lambda; N+1; -w\lambda u) e^{-(1+wz)u} du. \quad (23)$$

Since ${}_2F_1$ in the integrand of (23) is continuous for $|w\lambda u| \leq |w|c < 1$, (23) is integrable.

Notice that when $\lambda \rightarrow 0$, (22) and (23) are reduced to the zeta functions [16].

$$Z_N(s, \lambda, w, z) = \frac{1}{\Gamma(s)} \int_0^1 \frac{(1-t)^{wz}}{N!} {}_2F_1(1, N; N+1; -w\lambda \log(1-t)) t^{s-1} dt$$

and

$$Z_N(s, \lambda, w, z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{(1-e^{-u})^{s-1}}{N!} {}_2F_1(1, N; N+1; w\lambda u) e^{-(1+wz)u} du,$$

respectively.

Theorem 12. For $n \geq 0$,

$$Z_N(-n, \lambda, w, z) = C_{N,n}(\lambda, w; z).$$

Proof. Assume that $\Re s > 0$. Rewrite (22) with Hankel contour H_1 , where H_1 starts at $1 - e^{-c/\lambda}$ and goes to $\delta > 0$, and goes around the origin with radius δ , and goes back to $1 - e^{-c/\lambda}$. Since the integrand except for t^{s-1} is holomorphic in the neighborhood of $t = 0$, we have

$$Z_N(s, \lambda, w, z) = \frac{1}{\Gamma(s)(e^{2\pi i s} - 1)} \int_{H_1} \frac{(1-t)^{wz}}{N!} {}_2F_1(1, N - 1/\lambda; N+1; w\lambda \log(1-t)) t^{s-1} dt.$$

Then, the RHS gives the meromorphic continuation on the whole space $\mathbb{C}$ in s . By (21), we have $Z_N(-n, \lambda, w, z) = C_{N,n}(\lambda, w; z)$. $\square$

The function $Z_N(n, \lambda, w, z)$ can be expressed as the following form.

Theorem 13. For positive integers k and n , if $|wz| < \frac{1-|w|c}{1+|w|c} (<1)$ and $\Re \lambda > 0$, then

$$Z_N(n, \lambda, w, z) = \sum_{j=0}^{\infty} \sum_{m=j}^{\infty} \frac{(1-N\lambda|\lambda)_{m-j} m! (-1)^j}{n! j! (N+m-j)!} \sum_{l=1}^n \binom{n}{l} \frac{(-1)^{l-1}}{l^m} w^m z^j P_{m-j}(c(l+wz)/\lambda),$$

where

$$P_m(a) = 1 - e^{-a} \sum_{k=0}^m \frac{a^k}{k!}.$$

Proof. When $|wz| < \frac{1-|w|c}{1+|w|c} (<1)$ and $\Re \lambda > 0$, we get

$$\begin{aligned} Z_N(n, \lambda, w, z) &= \frac{1}{N!(n-1)!} \int_0^{c/\lambda} (1-e^{-u})^{n-1} {}_2F_1(1, N-1/\lambda; N+1; -w\lambda u) e^{-(1+wz)u} du \\ &= \frac{1}{(n-1)!} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \sum_{m=0}^{\infty} \frac{(1-N\lambda|\lambda)_m w^m}{(N+m)!} \int_0^{c/\lambda} e^{-(l+1+wz)u} u^m du \\ &= \frac{1}{(n-1)!} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \sum_{m=0}^{\infty} \frac{(1-N\lambda|\lambda)_m w^m}{(N+m)! (\lambda+1+wz)^{m+1}} \int_0^{c(l+1+wz)/\lambda} e^{-u} u^m du. \end{aligned} \quad (24)$$

Here, the integral and the summation in (24) are interchangeable. It is because by replacing the variable with $u = v/\lambda$,

$$\sum_{m=0}^{\infty} \frac{(1/\lambda - N)_m w^m}{(N+1)^{(m)}} \int_0^c e^{-(l+1+wz)v/\lambda} v^m dv. \quad (25)$$

Its absolute value is bounded by

$$\int_0^c e^{-\Re(l+1+wz)v/\lambda} \sum_{m=0}^{\infty} \frac{(N+1/|\lambda|)^{(m)}}{(N+1)^{(m)}} (|w|v)^m dv. \quad (26)$$

The series in the integrand is ${}_2F_1(1, N+1/|\lambda|; N+1; |w|v)$. Since $|w|v \leq |w|c < 1$, (26) is finite. Hence, the integral and the summation in (24) are interchangeable.

Now, since

$$\int_0^a e^{-u} u^m du = m! P_m(a),$$

we have

$$\begin{aligned} Z_N(n, \lambda, w, z) &= \frac{1}{(n-1)!} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \sum_{m=0}^{\infty} \frac{m! P_m(c(l+1+wz)/\lambda) (1-N\lambda|\lambda|)_m w^m}{(N+m)! (l+1+wz)^{m+1}} \\ &= \frac{1}{(n-1)!} \sum_{m=0}^{\infty} \frac{m! (1-N\lambda|\lambda|)_m w^m}{(N+m)!} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{(l+1)^{m+1}} \\ &\quad \times \sum_{j=0}^{\infty} \binom{m+j}{m} \left(-\frac{wz}{l+1}\right)^j P_m(c(l+1+wz)/\lambda) \\ &= \frac{1}{n!} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{m! (1-N\lambda|\lambda|)_m w^m}{(N+m)!} \binom{m+j}{m} \sum_{l=1}^n \binom{n}{l} \frac{(-1)^{l-1}}{l^{m+j}} (-wz)^j P_m(c(l+wz)/\lambda) \\ &= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(1-N\lambda|\lambda|)_m (m+j)! (-1)^j}{n! j! (N+m)!} \sum_{l=1}^n \binom{n}{l} \frac{(-1)^{l-1}}{l^{m+j}} w^{m+j} z^j P_m(c(l+wz)/\lambda). \end{aligned} \quad (27)$$

Here, the summations in (27) are interchangeable. It is because by

$$\sum_{m=0}^{\infty} \frac{(1/\lambda - N)_m (\lambda w)^m}{(N+1)^{(m)}} \frac{1}{(l+1)^{m+1}} \sum_{j=0}^{\infty} \binom{m+j}{m} \left(-\frac{wz}{l+1}\right)^j P_m(c(l+1+wz)/\lambda),$$

its absolute value is bounded by the following.

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{(N+1/|\lambda|)^{(m)} |\lambda w|^m}{(N+1)^{(m)}} \frac{|P_m(c(l+1+wz)/\lambda)|}{(l+1-|wz|)^{m+1}} &= \sum_{m=0}^{\infty} \frac{(N+1/|\lambda|)^{(m)} |\lambda w|^m}{(N+1)^{(m)}} \frac{1}{(l+1-|wz|)^{m+1}} \\ &\quad \times \left| \left(\frac{l+1+wz}{\lambda} \right)^{m+1} \int_0^c e^{-(l+1+wz)v/\lambda} v^m dv \right| \\ &\leq \frac{1}{|\lambda|} \sum_{m=0}^{\infty} \frac{(N+1/|\lambda|)^{(m)} |w|^m}{(N+1)^{(m)}} \frac{|l+1+wz|^{m+1}}{(l+1-|wz|)^{m+1}} \\ &\quad \times \int_0^c e^{-\Re(l+1+wz)v/\lambda} v^m dv. \end{aligned} \quad (28)$$

When $|wz| < \frac{1-|w|c}{1+|w|c}$, by $|wz| < (l+1) \frac{1-|w|c}{1+|w|c}$, we get

$$|w|v \frac{l+1+|wz|}{l+1-|wz|} \leq |w|c \frac{l+1+|wz|}{l+1-|wz|} < 1,$$

so (28) converges, hence the summation and the integral are interchangeable.

Remark. When $z = 0$, we have

$$Z_N(n, \lambda, w, 0) = \sum_{m=0}^{\infty} \frac{m!(1 - N\lambda|\lambda)_m w^m}{n!(m + N)!} \sum_{l=1}^n \binom{n}{l} \frac{(-1)^{l-1}}{l^m} P_m(cl/\lambda).$$

When $\lambda \rightarrow 0$, we have $P_m(c(l + 1 + wz)/\lambda) \rightarrow 1$ and hence

$$Z_N(n, \lambda, w, z) \rightarrow \sum_{j=0}^{\infty} \sum_{m=j}^{\infty} \frac{(1 - N\lambda|\lambda)_{m-j} m! (-1)^j}{n! j! (N + m - j)!} \zeta_n^{\star}(\{1\}_m) w^m z^j,$$

which reduces the result in [16]. Here,

$$\zeta_n^{\star}(\{1\}_r) := \begin{cases} \sum_{1 \leq k_1 \leq \dots \leq k_r \leq n} \frac{1}{k_1 \cdots k_r} & (r \geq 1), \\ 0 & (r = 0) \end{cases} = \sum_{l=1}^n \binom{n}{l} \frac{(-1)^{l-1}}{l^r} \quad (r \geq 0)$$

[21, (2)].

We give an expression in terms of the Bernoulli polynomials of higher order $B_n^{(k)}(z)$, defined by

$$\left(\frac{x}{e^x - 1} \right)^k e^{xz} = \sum_{n=0}^{\infty} B_n^{(k)}(z) \frac{x^n}{n!}.$$

When $z = 0$, $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of higher order.

Theorem 14. For $n \geq 0$,

$$Z_N(-n, \lambda, w, z) = C_{N,n}(\lambda, w; z) = (-1)^n \frac{n!}{N!} \sum_{i=0}^n \frac{B_i^{(n+1)}(n - wz)}{i!} \frac{(1 - N\lambda|\lambda)_{n-i} w^{n-i}}{(n - i + N)!}.$$

Proof. Assume that $\Re s > 0$. Write (23) with the Hankel contour H , where H starts at $+\infty$ and goes to $\delta > 0$, and goes around the origin with radius δ , and goes back to $+\infty$.

$$Z_N(s, \lambda, w, z) = \frac{1}{\Gamma(s)(e^{2\pi i s} - 1)} \int_H \frac{(1 - e^{-u})^{s-1}}{N!} {}_2F_1(1, N - 1/\lambda; N + 1; -w\lambda u) e^{-(1+wz)u} du.$$

This expression gives the meromorphic continuation on the whole space $\mathbb{C}$ in s . For a negative integer n ,

$$\int_{|u|=\varepsilon} \frac{(1 - e^{-u})^{-n-1}}{N!} {}_2F_1(1, N - 1/\lambda; N + 1; -w\lambda u) e^{-(1+wz)u} du = 2\pi i \operatorname{Res}_{u=0} \frac{1}{N!} \left(\frac{e^u}{e^u - 1} \right)^{n+1} \sum_{l=0}^{\infty} \frac{(1 - N\lambda|\lambda)_l (wu)^l}{(N + l)!} e^{-(1+wz)u}.$$

Therefore,

$$\begin{aligned} Z_N(-n, \lambda, w, z) &= (-1)^n \frac{n!}{N!} \operatorname{Res}_{u=0} \left(\frac{e^u}{e^u - 1} \right)^{n+1} \sum_{l=0}^{\infty} \frac{(1 - N\lambda|\lambda)_l (wu)^l}{(N + l)!} e^{-(1+wz)u} \\ &= (-1)^n \frac{n!}{N!} \operatorname{Res}_{u=0} \left(\frac{u}{e^u - 1} \right)^{n+1} \frac{e^{(n-wz)u}}{u^{n+1}} \sum_{l=0}^{\infty} \frac{(1 - N\lambda|\lambda)_l (wu)^l}{(N + l)!} \\ &= (-1)^n \frac{n!}{N!} \operatorname{Res}_{u=0} \left(\sum_{m=0}^{\infty} \frac{u^m}{m!} B_m^{(n+1)}(n - wz) \right) \left(\sum_{l=0}^{\infty} \frac{(1 - N\lambda|\lambda)_l (wu)^l}{(N + l)!} \right) \frac{1}{u^{n+1}} \\ &= (-1)^n \frac{n!}{N!} \operatorname{Res}_{u=0} \sum_{m=0}^{\infty} \left(\sum_{i=0}^m \frac{B_i^{(n+1)}(n - wz)}{i!} \frac{(1 - N\lambda|\lambda)_{m-i} w^{m-i}}{(m - i + N)!} \right) u^{m-n-1} \\ &= (-1)^n \frac{n!}{N!} \sum_{i=0}^n \frac{B_i^{(n+1)}(n - wz)}{i!} \frac{(1 - N\lambda|\lambda)_{n-i} w^{n-i}}{(n - i + N)!}. \end{aligned}$$

□

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References

- [1] F. T. Howard, *Degenerate weighted Stirling numbers*, Discrete Math. **57** (1985), 45–58.
- [2] L. Carlitz, *Degenerate Stirling, Bernoulli, and Eulerian numbers*, Utilitas Math. **15** (1979), 51–88.
- [3] F. T. Howard, *Explicit formulas for degenerate Bernoulli numbers*, Discrete Math. **162** (1996), 175–185.
- [4] P. T. Young, *Degenerate Bernoulli polynomials, generalized factorial sums, and their applications*, J. Number Theory **128** (2008), 738–758.
- [5] T. Komatsu, *Hypergeometric degenerate Bernoulli polynomials and numbers*, Ars Math. Contemp. (to appear), DOI: 10.26493/1855-3974.1907.3c2.
- [6] M. Cenkci and F. T. Howard, *Notes on degenerate numbers*, Discrete Math. **307** (2007), 2359–2375.
- [7] M. Aoki and T. Komatsu, *Remarks on hypergeometric Cauchy numbers*, Math. Rep. (Bucur), (to appear), arXiv:1802.05455.
- [8] T. Komatsu, *Hypergeometric Cauchy numbers*, Int. J. Number Theory **9** (2013), 545–560.
- [9] F. T. Howard, *A sequence of numbers related to the exponential function*, Duke Math. J. **34** (1967), 599–615.
- [10] F. T. Howard, *Some sequences of rational numbers related to the exponential function*, Duke Math. J. **34** (1967), 701–716.
- [11] K. Kamano, *Sums of products of hypergeometric Bernoulli numbers*, J. Number Theory **130** (2010), 2259–2271.
- [12] T. Komatsu, *Complementary Euler numbers*, Period. Math. Hungar. **75** (2017), 302–314.
- [13] T. Komatsu and H. Zhu, *Hypergeometric Euler numbers*, AIMS Math. **5** (2020), 1284–1303.
- [14] J. W. L. Glaisher, *Expressions for Laplace's coefficients, Bernoullian and Eulerian numbers etc. as determinants*, Messenger **6** (1875), no. 2, 49–63.
- [15] M. Aoki, T. Komatsu, and G. K. Panda, *Several properties of hypergeometric Bernoulli numbers*, J. Inequal. Appl. **2019** (2019), 113, DOI: 10.1186/s13660-019-2066-y.
- [16] Y. Komori and A. Yoshihara, *Cauchy numbers and polynomials associated with hypergeometric Bernoulli numbers*, J. Comb. Number Theory **9** (2017), no. 2, 123–142.
- [17] S. Hu and T. Komatsu, *Explicit expressions for the related numbers of higher order Appell polynomials*, Queast. Math. (2019), DOI: 10.2989/16073606.2019.1596174.
- [18] M. Rahmani, *On p -Cauchy numbers*, Filomat **30** (2016), 2731–2742.
- [19] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.
- [20] D. Merlini, R. Sprugnoli, and M. C. Verri, *The Cauchy numbers*, Discrete Math. **306** (2006), 1906–1920.
- [21] M. Kuba, *On functions on Arakawa and Kaneko and multiple zeta values*, Appl. Anal. Discrete Math. **4** (2010), 45–53.

Appendix

Bernoulli numbers B_n in (4)

$$\begin{aligned} B_0 &= 1, & B_1 &= -\frac{1}{2}, & B_2 &= \frac{1}{6}, & B_3 &= 0, & B_4 &= -\frac{1}{30}, & B_5 &= 0, & B_6 &= \frac{1}{42}, \\ B_7 &= 0, & B_8 &= -\frac{1}{30}, & B_9 &= 0, & B_{10} &= \frac{5}{66} \end{aligned}$$

Cauchy numbers c_n in (5) with (6)

$$\begin{aligned} c_0 &= 1, & c_1 &= \frac{1}{2}, & c_2 &= -\frac{1}{6}, & c_3 &= \frac{1}{4}, & c_4 &= -\frac{19}{30}, & c_5 &= \frac{9}{4}, & c_6 &= -\frac{863}{84}, \\ c_7 &= \frac{1,375}{24}, & c_8 &= -\frac{33,953}{90}, & c_9 &= \frac{57,281}{20}, & c_{10} &= -\frac{32,50,433}{132} \end{aligned}$$

Degenerate Bernoulli numbers $\beta_n := \beta_n(\lambda)$ in (3)

$$\begin{aligned} \beta_0 &= 1, & \beta_1 &= \frac{\lambda}{2} - \frac{1}{2}, & \beta_2 &= -\frac{\lambda^2}{6} + \frac{1}{6}, & \beta_3 &= \frac{\lambda^3}{4} - \frac{\lambda}{4}, & \beta_4 &= -\frac{19\lambda^4}{720} + \frac{2\lambda^2}{3} - \frac{1}{30}, \\ \beta_5 &= \frac{9\lambda^5}{4} - \frac{5\lambda^3}{2} + \frac{\lambda}{4}, & \beta_6 &= -\frac{863\lambda^6}{84} + 12\lambda^4 - \frac{7\lambda^2}{4} + \frac{1}{42}, \\ \beta_7 &= \frac{1,375\lambda^7}{24} - 70\lambda^5 + \frac{105\lambda^3}{8} - \frac{5\lambda}{12}, & \beta_8 &= -\frac{33,953\lambda^8}{90} + 480\lambda^6 - \frac{1,624\lambda^4}{15} + \frac{50\lambda^2}{9} - \frac{1}{30}, \\ \beta_9 &= \frac{57,281\lambda^9}{20} - 3,780\lambda^7 + \frac{9,849\lambda^5}{10} - 70\lambda^3 + \frac{21\lambda}{20}, \\ \beta_{10} &= -\frac{32,50,433\lambda^{10}}{132} + 33,600\lambda^8 - \frac{29,531\lambda^6}{3} + \frac{5,345\lambda^4}{6} - \frac{91\lambda^2}{4} + \frac{5}{66} \end{aligned}$$

Degenerate Cauchy numbers $\gamma_n := \gamma_n(\lambda)$ in (9)

$$\begin{aligned} \gamma_0 &= 1, & \gamma_1 &= -\frac{\lambda}{2} + \frac{1}{2}, & \gamma_2 &= \frac{\lambda^2}{6} - \frac{1}{6}, & \gamma_3 &= -\frac{\lambda^2}{4} + \frac{1}{4}, & \gamma_4 &= -\frac{\lambda^4}{30} + \frac{2\lambda^2}{3} - \frac{19}{30}, \\ \gamma_5 &= \frac{\lambda^4}{4} - \frac{5\lambda^2}{2} + \frac{9}{4}, & \gamma_6 &= \frac{\lambda^6}{42} - \frac{7\lambda^4}{4} + 12\lambda^2 - \frac{863}{84}, \\ \gamma_7 &= -\frac{5\lambda^6}{12} + \frac{105\lambda^4}{8} - 70\lambda^2 + \frac{1,375}{24}, & \gamma_8 &= -\frac{\lambda^8}{30} + \frac{50\lambda^6}{9} - \frac{1,624\lambda^4}{15} + 480\lambda^2 - \frac{33,953}{90}, \\ \gamma_9 &= \frac{21\lambda^8}{20} - 70\lambda^6 + \frac{9,849\lambda^4}{10} - 3780\lambda^2 + \frac{57,281}{20}, \\ \gamma_{10} &= \frac{5\lambda^{10}}{66} - \frac{91\lambda^8}{4} + \frac{5,345\lambda^6}{6} - \frac{29,531\lambda^4}{3} + 33,600\lambda^2 - \frac{32,50,433}{132} \end{aligned}$$

Hypergeometric degenerate Cauchy numbers $\gamma_{N,n} := \gamma_{N,n}(\lambda)$ in (15)

$$\begin{aligned} \gamma_{N,0} &= 1, \\ \gamma_{N,1} &= -\frac{\lambda - N}{N + 1}, \\ \gamma_{N,2} &= \frac{2(\lambda^2 - (N - 1)\lambda - N)}{(N + 1)^2(N + 2)}, \\ \gamma_{N,3} &= \frac{3!((N - 1)\lambda^3 - (2N^2 - N + 3)\lambda^2 + (N^2 + 2)(N - 1)\lambda + N(N^2 + N + 2))}{(N + 1)^2(N + 2)(N + 3)}, \end{aligned}$$

$$\begin{aligned}
\gamma_{N,4} &= 4!((N^3 - N^2 - 6N + 2)\lambda^4 - (N - 1)(3N^3 + 5N^2 + 2N + 12)\lambda^3 \\
&\quad + (3N^5 + 6N^4 + 13N^3 + 30N^2 + 6M + 22)\lambda^2 \\
&\quad - (N - 1)(N^5 + 3N^4 + 8N^3 + 16N^2 + 10N + 12)\lambda \\
&\quad - N(N^5 + 5N^4 + 14N^3 + 24N^2 + 20N + 12)) \\
&\quad \times ((N + 1)^4(N + 2)^2(N + 3)(N + 4))^{-1}, \\
\gamma_{N,5} &= ((N - 1)(N^3 - 3N^2 - 14N + 2)\lambda^5 \\
&\quad - (4N^5 - 5N^4 - 28N^3 + 9N^2 - 72N + 20)\lambda^4 \\
&\quad + 2(N - 1)(3N^5 + 7N^4 + 11N^3 + 37N^2 + 5N + 35)\lambda^3 \\
&\quad - 2(2N^7 + 7N^6 + 24N^5 + 65N^4 + 64N^3 + 118N^2 + 30N + 50)\lambda^2 \\
&\quad + (N - 1)(N^7 + 5N^6 + 20N^5 + 56N^4 + 82N^3 + 118N^2 + 64N + 48)\lambda \\
&\quad + N(N^7 + 8N^6 + 35N^5 + 96N^4 + 160N^3 + 184N^2 + 116N + 48)) \\
&\quad \times ((N + 1)^5(N + 2)^2(N + 3)(N + 4)(N + 5))^{-1}
\end{aligned}$$

H-degenerate Cauchy numbers $C_{N,n} := C_{N,n}(\lambda)$ in (19)

$$\begin{aligned}
C_{N,0} &= 1, \\
C_{N,1} &= -\frac{N\lambda - 1}{N + 1}, \\
C_{N,2} &= \frac{(N\lambda - 1)(2(N + 1)\lambda + N)}{(N + 1)(N + 2)}, \\
C_{N,3} &= \frac{2(N\lambda - 1)(3(N + 1)\lambda^2 + 3N\lambda + N)}{(N + 1)(N + 3)}, \\
C_{N,4} &= 2(N\lambda - 1)(12(N + 1)(N + 2)(N + 3)\lambda^3 + 6(3N^3 + 15N^2 + 18N + 2)\lambda^2 \\
&\quad + (11N^3 + 52N^2 + 47N - 12)\lambda + N(3N^2 + 16N + 19)) \\
&\quad \times ((N + 1)(N + 2)(N + 3)(N + 4))^{-1}, \\
C_{N,5} &= 2(N\lambda - 1)(60(N + 1)(N + 2)(N + 3)(N + 4)\lambda^4 \\
&\quad + 120(N^4 + 9N^3 + 26N^2 + 26N + 5)\lambda^3 \\
&\quad + 15(7N^4 + 60N^3 + 151N^2 + 98N - 20)\lambda^2 \\
&\quad + 5(10N^4 + 88N^3 + 221N^2 + 119N - 60)\lambda \\
&\quad + N(12N^3 + 118N^2 + 357N + 323)) \\
&\quad \times ((N + 1)(N + 2)(N + 3)(N + 4)(N + 5))^{-1}
\end{aligned}$$