Research Article

Shuangjian Guo, Xiaohui Zhang, and Shengxiang Wang*

On split involutive regular BiHom-Lie superalgebras

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Abstract: The goal of this paper is to examine the structure of split involutive regular BiHom-Lie superalgebras, which can be viewed as the natural generalization of split involutive regular Hom-Lie algebras and split regular BiHom-Lie superalgebras. By developing techniques of connections of roots for this kind of algebras, we show that such a split involutive regular BiHom-Lie superalgebra $\mathfrak{L}$ is of the form $\mathfrak{L} = U + \sum_{\alpha} I_{\alpha}$ with U a subspace of a maximal abelian subalgebra H and any I_{α} , a well-described ideal of $\mathfrak{L}$, satisfying $[I_{\alpha}, I_{\beta}] = 0$ if $[\alpha] \neq [\beta]$. In the case of $\mathfrak{L}$ being of maximal length, the simplicity of $\mathfrak{L}$ is also characterized in terms of connections of roots.

Keywords: involutive, BiHom-Lie superalgebra, root space, structure theory

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1 Introduction

The notion of Hom-Lie algebras was first introduced by Hartwig, Larsson and Silvestrov in [1], who developed an approach to deformations of the Witt and Virasoro algebras based on σ -deformations. In fact, Hom-Lie algebras include Lie algebras as a subclass, but the deformation of Lie algebras is twisted by a homomorphism.

A BiHom-algebra is an algebra in which the identities defining the structure are twisted by two homomorphisms ϕ and ψ . This class of algebras was introduced from a categorical approach in [2] which can be viewed as an extension of the class of Hom-algebras. If the two linear maps are the same automorphisms, BiHom-algebras will return to Hom-algebras. These algebraic structures include BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras. The representation theory of BiHom-Lie algebras was introduced by Cheng and Qi in [3], in which BiHom-cochain complexes, derivations, central extensions, derivation extensions, trivial representations and adjoint representations of BiHom-Lie algebras were studied. More applications of BiHom-algebras, BiHom-Lie superalgebras, BiHom-Lie color algebras and BiHom-Novikov algebras can be found in [4–7].

The class of the split algebras is especially related to addition quantum numbers, graded contractions and deformations. For instance, for a physical system which displays a symmetry, it is interesting to know the detailed structure of the split decomposition, since its roots can be seen as certain eigenvalues which are the additive quantum numbers characterizing the state of such system. Determining the structure of split algebras will become more and more meaningful in the area of research in mathematical physics.

* **Corresponding author: Shengxiang Wang**, School of Mathematics and Finance, Chuzhou University, Chuzhou 239000, People's Republic of China, e-mail: wangsx-math@163.com

Shuangjian Guo: School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, People's Republic of China

Xiaohui Zhang: School of Mathematical Sciences, Qufu Normal University, Qufu 273165, People's Republic of China

Recently, in [8–22], the structure of different classes of split algebras has been determined by the techniques of connections of roots. The purpose of this paper is to consider the structure of involutive regular BiHom-Lie superalgebras by the techniques of connections of roots based on some work in [11] and [12].

This paper is organized as follows. In Section 3, we prove that such an arbitrary involutive regular BiHom-Lie superalgebra $\mathcal{L}$ is of the form $\mathcal{L} = U + \sum_{\alpha} I_{\alpha}$ with U a subspace of a maximal abelian subalgebra H and any I_{α} , a well-described ideal of $\mathcal{L}$, satisfying $[I_{\alpha}, I_{\beta}] = 0$ if $[\alpha] \neq [\beta]$. In Section 4, under certain conditions, in the case of $\mathcal{L}$ being of maximal length, the simplicity of the algebra is characterized.

2 Preliminaries

Throughout this paper, we will denote by $\mathbb{N}$ the set of all nonnegative integers and by $\mathbb{Z}$ the set of all integers. Split involutive regular BiHom-Lie superalgebras are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. Now we recall some basic definitions and results related to our paper from [11] and [12].

Definition 2.1. A BiHom-Lie superalgebra $\mathcal{L}$ is a $\mathbb{Z}_2$ -graded algebra $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$, endowed with an even bilinear mapping $[\cdot, \cdot]: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ and two homomorphisms $\phi, \psi: \mathcal{L} \rightarrow \mathcal{L}$ satisfying the following conditions, for all $x \in \mathcal{L}_{\bar{i}}, y \in \mathcal{L}_{\bar{j}}, z \in \mathcal{L}_{\bar{k}}$ and $\bar{i}, \bar{j}, \bar{k} \in \mathbb{Z}_2$:

$$\begin{aligned} [x, y] &\in \mathcal{L}_{\bar{i}+\bar{j}}, \\ \phi \circ \psi &= \psi \circ \phi, \\ [\psi(x), \phi(y)] &= -(-1)^{\bar{i}\bar{j}} [\psi(y), \phi(x)], \\ (-1)^{\bar{k}\bar{i}} [\psi^2(x), [\psi(y), \phi(z)]] &+ (-1)^{\bar{i}\bar{j}} [\psi^2(y), [\psi(z), \phi(x)]] + (-1)^{\bar{j}\bar{k}} [\psi^2(z), [\psi(x), \phi(y)]] = 0. \end{aligned}$$

When ϕ and ψ are algebra automorphisms, it is said that $\mathcal{L}$ is a regular BiHom-Lie superalgebra.

Let $\mathcal{L}$ be a BiHom-Lie superalgebra and $-: \mathbb{K} \rightarrow \mathbb{K}$ an involutive automorphism (we say that $-$ is a conjugation on $\mathbb{K}$). An involution on $\mathcal{L}$ is a conjugate-linear map $*$: $\mathcal{L} \rightarrow \mathcal{L}$, ($x \rightarrow x^*$), such that $(x^*)^* = x$, $[\psi(x), \phi(y)]^* = [\psi(y^*), \phi(x^*)]$ and $\phi(x)^* = \phi(x^*)$, $\psi(x)^* = \psi(x^*)$ for any $x, y \in \mathcal{L}$. A regular BiHom-Lie superalgebra endowed with an involution is called an involutive regular BiHom-Lie superalgebra. An involutive subset of an involutive algebra is a subset globally invariant by the involution.

Note that $\mathcal{L}_0$ is a BiHom-Lie algebra. The usual regularity concepts will be understood in the graded sense. That is, an involutive super-subalgebra $A = A_0 \oplus A_1$ of $\mathcal{L}$ is a graded subspace $A = A_0 \oplus A_1$ satisfying

$$[A, A] \subset \mathcal{L}, \quad \phi(A) = A, \quad \psi(A) = A.$$

An involutive ideal I of $\mathcal{L}$ is a graded subspace $I = I_0 \oplus I_1$ of $\mathcal{L}$ such that

$$[I, \mathcal{L}] \subset I, \quad \phi(I) = I, \quad \psi(I) = I.$$

We say that $\mathcal{L}$ is simple if the product is nonzero and its only ideals are $\{0\}$ and $\mathcal{L}$. From now on, $(\mathcal{L}, *)$ always denotes an involutive regular BiHom-Lie superalgebra.

Let us introduce the class of split algebras in the framework of involutive regular BiHom-Lie superalgebras. Denote by $H = H_0 \oplus H_1$ a maximal abelian subalgebra H of $\mathcal{L}$. For a linear functional commuting with the involution

$$\alpha: (H_0, *) \rightarrow (\mathbb{K}, -),$$

that is, $\alpha(h^*) = \overline{\alpha(h)}$ for any $h \in H_0$, we define the root space of $\mathcal{L}$ associated with α as the subspace

$$\mathcal{L}_{\alpha} := \{v_{\alpha} \in \mathcal{L}: [h_0, \phi(v_{\alpha})] = \alpha(h)\phi\psi(v_{\alpha}), \text{ for any } h_0 \in H_0\}.$$

The elements $\alpha: H_0 \rightarrow \mathbb{K}$ satisfying $\mathfrak{L}_\alpha \neq 0$ are called roots of $\mathfrak{L}$ with respect to H , and we denote $\Lambda := \{\alpha: (H_0, *) \rightarrow (\mathbb{K}, -): \mathfrak{L}_\alpha \neq 0\}$. We call that $\mathfrak{L}$ is a split involutive regular BiHom-Lie superalgebra with respect to H if

$$\mathfrak{L} = H \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{L}_\alpha.$$

We also say that Λ is the root system of $\mathfrak{L}$.

It is easy to see that $H^* = H$. To simplify notations, the mappings $\phi|_H, \psi|_H, \phi^{-1}|_H, \psi^{-1}|_H: H \rightarrow H$ will be denoted by ϕ, ψ and ϕ^{-1}, ψ^{-1} , respectively.

The following two lemmas are analogous to the results of [11] and [12].

Lemma 2.2. *Let $(\mathfrak{L}, *)$ be a split involutive regular BiHom-Lie superalgebra. Then, for any $\alpha, \beta \in \Lambda \cup \{0\}$,*

- (1) $\phi(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\phi^{-1}}$ and $\phi^{-1}(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\phi}$;
- (2) $\psi(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\psi^{-1}}$ and $\psi^{-1}(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\psi}$;
- (3) $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \subset \mathfrak{L}_{\alpha\phi^{-1}+\beta\psi^{-1}}$;
- (4) $(\mathfrak{L}_\alpha)^* = \mathfrak{L}_{-\alpha}$.

Lemma 2.3. *The following assertions hold:*

- (1) *If $\alpha \in \Lambda$, then $\alpha\phi^{-z_1}\psi^{-z_2} \in \Lambda$ for any $z_1, z_2 \in \mathbb{Z}$;*
- (2) $\mathfrak{L}_0 = H$.

Definition 2.4. A root system Λ of a split involutive regular BiHom-Lie superalgebra is called symmetric if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$.

3 Decompositions

In what follows, $\mathfrak{L}$ denotes a split involutive regular BiHom-Lie superalgebra and

$$\mathfrak{L} = H \oplus \left(\bigoplus_{\alpha \in \Lambda} \mathfrak{L}_\alpha \right)$$

is the corresponding root-space decomposition. We begin by developing the techniques of connections of roots in this section.

Definition 3.1. Let α and β be two nonzero roots. We will say that α is connected to β if either

$$\beta = \varepsilon \alpha \phi^{z_1} \psi^{z_2} \quad \text{for some } z_1, z_2 \in \mathbb{Z} \text{ and } \varepsilon \in \{-1, 1\}$$

or there exists $\{\alpha_1, \dots, \alpha_k\} \subset \Lambda$ with $k \geq 2$, such that

1. $\alpha_1 \in \{\alpha \phi^{-n} \psi^{-r} : n, r \in \mathbb{N}\}$;

2.

$$\begin{aligned} & \alpha_1 \phi^{-1} + \alpha_2 \psi^{-1} \in \Lambda, \\ & \alpha_1 \phi^{-2} + \alpha_2 \phi^{-1} \psi^{-1} + \alpha_3 \psi^{-1} \in \Lambda, \\ & \alpha_1 \phi^{-3} + \alpha_2 \phi^{-2} \psi^{-1} + \alpha_3 \phi^{-1} \psi^{-1} + \alpha_4 \psi^{-1} \in \Lambda, \\ & \dots \\ & \alpha_1 \phi^{-i} + \alpha_2 \phi^{-i+1} \psi^{-1} + \alpha_3 \phi^{-i+2} \psi^{-1} + \dots + \alpha_i \phi^{-1} \psi^{-1} + \alpha_{i+1} \psi^{-1} \in \Lambda, \\ & \alpha_1 \phi^{-k+2} + \alpha_2 \phi^{-k+3} \psi^{-1} + \alpha_3 \phi^{-k+4} \psi^{-1} + \dots + \alpha_{k-2} \phi^{-1} \psi^{-1} + \alpha_{k-1} \psi^{-1} \in \Lambda; \end{aligned}$$

3. $\alpha_1 \phi^{-k+1} + \alpha_2 \phi^{-k+2} \psi^{-1} + \alpha_3 \phi^{-k+3} \psi^{-1} + \dots + \alpha_i \phi^{-k+i} \psi^{-1} + \dots + \alpha_{k-1} \phi^{-1} \psi^{-1} + \alpha_k \psi^{-1} \in \{\pm \beta \phi^{-m} \psi^{-s} : m, s \in \mathbb{N}\}$.

We will also say that $\{\alpha_1, \dots, \alpha_k\}$ is a connection from α to β .

The proof of the next result is analogous to the one in [12].

Proposition 3.2. *The relation $\sim$ in Λ , defined by $\alpha \sim \beta$ if and only if α is connected to β , is an equivalence relation.*

By Proposition 3.2, we can consider the quotient set

$$\Lambda/\sim = \{[\alpha] : \alpha \in \Lambda\},$$

with $[\alpha]$ being the set of nonzero roots which are connected to α . Our next goal is to associate an ideal $I_{[\alpha]}$ with $[\alpha]$. Fixing $[\alpha] \in \Lambda/\sim$, we start by defining

$$I_{H,[\alpha]} = \text{span}_{\mathbb{K}} \{ [\mathfrak{L}_{\beta\psi^{-1}}, (\mathfrak{L}_{\beta\phi^{-1}})^*] : \beta \in [\alpha] \}.$$

Now we define

$$V_{[\alpha]} := \bigoplus_{\beta \in [\alpha]} \mathfrak{L}_{\beta}.$$

Finally, we denote by $I_{[\alpha]}$ the direct sum of the two subspaces above:

$$I_{[\alpha]} := I_{H,[\alpha]} \oplus V_{[\alpha]}.$$

Proposition 3.3. *For any $[\alpha] \in \Lambda/\sim$, the following assertions hold:*

- (1) $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$;
- (2) $\phi(I_{[\alpha]}) = I_{[\alpha]}$ and $\psi(I_{[\alpha]}) = I_{[\alpha]}$;
- (3) $I_{[\alpha]}^* = I_{[\alpha]}$;
- (4) for any $[\beta] \neq [\alpha]$, we have $[I_{[\alpha]}, I_{[\beta]}] = 0$.

Proof.

- (1) First, we check that $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$, and we can write

$$[I_{[\alpha]}, I_{[\alpha]}] = [I_{H,[\alpha]} \oplus V_{[\alpha]}, I_{H,[\alpha]} \oplus V_{[\alpha]}] \subset [I_{H,[\alpha]}, V_{[\alpha]}] + [V_{[\alpha]}, I_{H,[\alpha]}] + [V_{[\alpha]}, V_{[\alpha]}]. \quad (3.1)$$

Given $\beta \in [\alpha]$, we have $[I_{H,[\alpha]}] \subset \mathfrak{L}_{\beta\psi^{-1}}$. Since $\beta\psi^{-1} \in [\alpha]$, it follows that $[I_{H,[\alpha]}, \mathfrak{L}_{\beta}] \subset V_{[\alpha]}$.

By a similar argument, we get $[\mathfrak{L}_{\beta}, I_{H,[\alpha]}] \subset V_{[\alpha]}$.

Next we consider $[V_{[\alpha]}, V_{[\alpha]}]$. If we take $\beta, \gamma \in [\alpha]$ and $\bar{i}, \bar{j} \in \mathbb{Z}_2$ such that $[\mathfrak{L}_{\beta,\bar{i}}, \mathfrak{L}_{\gamma,\bar{j}}] \neq 0$, then $[\mathfrak{L}_{\beta}, \mathfrak{L}_{\gamma}] \subset \mathfrak{L}_{\beta\phi^{-1} + \gamma\psi^{-1}, \bar{i} + \bar{j}}$. If $\beta\phi^{-1} + \gamma\psi^{-1} = 0$, we get $[\mathfrak{L}_{\beta}, \mathfrak{L}_{\gamma}] = [\mathfrak{L}_{\beta}, \mathfrak{L}_{-\beta}] = [\mathfrak{L}_{\beta}, (\mathfrak{L}_{\beta})^*] \subset H$ and so $[\mathfrak{L}_{\beta}, \mathfrak{L}_{\gamma}] \subset I_{H,[\alpha]}$. Suppose that $\beta\phi^{-1} + \gamma\psi^{-1} \in \Lambda$. We infer that $\{\beta, \gamma\}$ is a connection from β to $\beta\phi^{-1} + \gamma\psi^{-1}$. The transitivity of $\sim$ shows that $\beta\phi^{-1} + \gamma\psi^{-1} \in [\alpha]$ and so $[\mathfrak{L}_{\beta}, \mathfrak{L}_{\gamma}] \subset V_{[\alpha]}$. Hence,

$$[V_{[\alpha]}, V_{[\alpha]}] \in I_{[\alpha]}. \quad (3.2)$$

From (3.1) and (3.2), we get $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$.

- (2) It is easy to check that $\phi(I_{[\alpha]}) = I_{[\alpha]}$ and $\psi(I_{[\alpha]}) = I_{[\alpha]}$.
- (3) It is easy to check that $I_{[\alpha]}^* = I_{[\alpha]}$.
- (4) For the expression $[I_{[\alpha]}, I_{[\beta]}]$, we first note that

$$[I_{[\alpha]}, I_{[\beta]}] = [I_{H,[\alpha]} \oplus V_{[\alpha]}, I_{H,[\beta]} \oplus V_{[\beta]}] \subset [I_{H,[\alpha]}, V_{[\beta]}] + [V_{[\alpha]}, I_{H,[\beta]}] + [V_{[\alpha]}, V_{[\beta]}]. \quad (3.3)$$

First, we consider the summand $[V_{[\alpha]}, V_{[\beta]}]$ and suppose that there exist $\alpha_1 \in [\alpha]$, $\beta_1 \in [\beta]$ and $\bar{j}, \bar{k} \in \mathbb{Z}_2$ such that $[\mathfrak{L}_{\alpha_1,\bar{j}}, \mathfrak{L}_{\beta_1,\bar{k}}] \neq 0$. By the hypothesis, $\alpha_1\phi^{-1} \neq -\beta_1\psi^{-1}$, and thus $\alpha_1\phi^{-1} + \beta_1\psi^{-1} \in \Lambda$. So $\{\alpha_1, \beta_1, -\alpha_1\phi^{-2}\psi\}$ is a connection between α_1 and β_1 . By the transitivity of the connection relation we see $\alpha \in [\beta]$, a contradiction. Hence, $[\mathfrak{L}_{\alpha_1,\bar{j}}, \mathfrak{L}_{\beta_1,\bar{k}}] = 0$, and so

$$[V_{[\alpha]}, V_{[\beta]}] = 0. \quad (3.4)$$

Next we consider the first summand $[I_{H,[\alpha]}, V_{[\beta]}]$ on the right-hand side of (3.3). Suppose that there exist $\beta \in [\alpha]$ and $\eta \in [\beta]$ such that

$$[[\mathcal{L}_{\beta\psi^{-1}}, \mathcal{L}_{-\beta\phi^{-1}}], \phi^2(\mathcal{L}_{\eta})] = 0.$$

If

$$[[\mathcal{L}_{\beta\psi^{-1}}, \mathcal{L}_{-\beta\phi^{-1}}], \phi^2(\mathcal{L}_{\eta})] \neq 0,$$

then BiHom-Jacobi superidentity gives

$$[\psi^2(\mathcal{L}_{\eta}), [\mathcal{L}_{-\beta\psi^{-1}}, \mathcal{L}_{\beta\phi^{-1}}]] \neq 0.$$

Hence, there exists $\bar{i}, \bar{j}, \bar{k} \in \mathbb{Z}_2$ such that $[\psi^2(\mathcal{L}_{\eta, \bar{i}}), [\psi(\mathcal{L}_{-\beta, \bar{j}}), \phi(\mathcal{L}_{\beta, \bar{k}})]] \neq 0$. We get either $[\psi(\mathcal{L}_{\beta, \bar{k}}), \phi(\mathcal{L}_{\eta, \bar{i}})] \neq 0$ or $[\psi(\mathcal{L}_{\eta, \bar{i}}), \phi(\mathcal{L}_{-\beta, \bar{j}})] \neq 0$, which contradicts (3.4). Therefore, $[[\mathcal{L}_{\beta\psi^{-1}}, \mathcal{L}_{-\beta\phi^{-1}}], \phi^2(\mathcal{L}_{\eta})] = 0$. Consequently, $[I_{H,[\alpha]}, V_{[\beta]}] = 0$. In a similar way, one can prove $[V_{[\alpha]}, I_{H,[\beta]}] = 0$ and the proof is completed. $\square$

Theorem 3.4. *The following assertions hold:*

- (1) *For any $[\alpha] \in \Lambda/\sim$, the involutive subalgebra $I_{[\alpha]} = I_{H,[\alpha]} + V_{[\alpha]}$ of $\mathcal{L}$ associated with $[\alpha]$ is an involutive ideal of $\mathcal{L}$.*
- (2) *If $\mathcal{L}$ is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$ and $H = \sum_{\alpha \in \Lambda} ([\mathcal{L}_{\alpha\psi^{-1}}, (\mathcal{L}_{\alpha\phi^{-1}})^*])$.*

Proof. (1) Since $[I_{[\alpha]}, H] \subset I_{[\alpha]}$, by Lemmas 2.2 and 2.3, we have

$$[I_{[\alpha]}, \mathcal{L}] = \left[I_{[\alpha]}, H \oplus \left(\bigoplus_{\beta \in [\alpha]} \mathcal{L}_{\beta} \right) \oplus \left(\bigoplus_{\gamma \notin [\alpha]} \mathcal{L}_{\gamma} \right) \right] \subset I_{[\alpha]}.$$

According to Propositions 3.2 and 3.3, we have

$$[I_{[\alpha]}, \mathcal{L}] = \left[I_{[\alpha]}, \left(H \oplus \left(\bigoplus_{\beta \in [\alpha]} \mathcal{L}_{\beta} \right) \oplus \left(\bigoplus_{\gamma \notin [\alpha]} \mathcal{L}_{\gamma} \right) \right) \right] \subset I_{[\alpha]}.$$

As we also have $\phi(I_{[\alpha]}) = I_{[\alpha]}$ and $\psi(I_{[\alpha]}) = I_{[\alpha]}$, we conclude that $I_{[\alpha]}$ is an ideal of $\mathcal{L}$.

- (2) The simplicity of $\mathcal{L}$ implies $I_{[\alpha]} = \mathcal{L}$, then $[\alpha] = \Lambda$ and $H = \sum_{\alpha \in \Lambda} ([\mathcal{L}_{\alpha\psi^{-1}}, (\mathcal{L}_{\alpha\phi^{-1}})^*])$. $\square$

Theorem 3.5. *We have*

$$\mathcal{L} = U + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]},$$

where U is a linear complement in H of $\text{span}_{\mathbb{K}} \{[\mathcal{L}_{\alpha\psi^{-1}}, (\mathcal{L}_{\alpha\phi^{-1}})^*]: \alpha \in \Lambda\}$ and any $I_{[\alpha]}$ is one of the involutive ideals of $\mathcal{L}$ described in Theorem 3.4, satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.

Proof. $I_{[\alpha]}$ is well defined and is an ideal of $\mathcal{L}$, being clear that

$$\mathcal{L} = H \oplus \sum_{[\alpha] \in \Lambda} \mathcal{L}_{[\alpha]} = U + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}.$$

Finally, Proposition 3.3 gives us $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. $\square$

Let us denote by $Z(\mathcal{L})$ the center of $\mathcal{L}$, that is,

$$Z(\mathcal{L}) := \{v \in \mathcal{L}: [v, \mathcal{L}] = 0\}.$$

Corollary 3.6. *If $Z(\mathcal{L}) = 0$ and $H = \sum_{\alpha \in \Lambda} ([\mathcal{L}_{\alpha\psi^{-1}}, (\mathcal{L}_{\alpha\phi^{-1}})^*])$, then $\mathcal{L}$ is the direct sum of the involutive ideal given in Theorem 3.4:*

$$\mathcal{L} = \bigoplus_{[\alpha] \in \Lambda / \sim} I_{[\alpha]}.$$

Furthermore, $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.

Proof. Since $H = \sum_{\alpha \in \Lambda} ([\mathcal{L}_{\alpha\psi^{-1}}, (\mathcal{L}_{\alpha\phi^{-1}})^*])$, it follows that $\mathcal{L} = \sum_{[\alpha] \in \Lambda / \sim} I_{[\alpha]}$. To verify the direct character of the sum, take some $v \in I_{[\alpha]} \cap (\sum_{[\beta] \in \Lambda / \sim, [\beta] \neq [\alpha]} I_{[\beta]})$. Since $v \in I_{[\alpha]}$, the fact $[I_{[\alpha]}, I_{[\beta]}] = 0$ when $[\alpha] \neq [\beta]$ gives us

$$\left[v, \sum_{[\beta] \in \Lambda / \sim, [\beta] \neq [\alpha]} I_{[\beta]} \right] = 0.$$

In a similar way, $v \in \sum_{[\beta] \in \Lambda / \sim, [\beta] \neq [\alpha]} I_{[\beta]}$ implies $[v, I_{[\alpha]}] = 0$. That is, $v \in Z(\mathcal{L})$ and so $v = 0$. $\square$

4 The simple components

In this section, we focus on the simplicity of the split involutive regular BiHom-Lie superalgebra $\mathcal{L}$ by centering our attention in those of maximal length. From now on, $\text{char}(\mathbb{K}) = 0$.

For an ideal I , we have

$$I = (I \cap H) \oplus \left(\bigoplus_{\alpha \in \Lambda} (I \cap \mathcal{L}_{\alpha}) \right).$$

Lemma 4.1. Let $\mathcal{L}$ be a split regular BiHom-Lie superalgebra with $Z(\mathcal{L}) = 0$ and I an ideal of $\mathcal{L}$. If $I \subset H$, then $I = \{0\}$.

Proof. It is analogous to [12]. $\square$

Definition 4.2. We say that a split involutive regular BiHom-Lie superalgebra $\mathcal{L}$ is root-multiplicative if given $\alpha \in \Lambda_{\bar{i}}, \beta \in \Lambda_{\bar{j}}$, for $\bar{i}, \bar{j} \in \mathbb{Z}_2$, such that $\alpha\psi^{-1} + \beta\phi^{-1} \in \Lambda_{\bar{i}+\bar{j}}$, then $[\mathcal{L}_{\alpha, \bar{i}}, \mathcal{L}_{\beta, \bar{j}}] \neq 0$.

Definition 4.3. A split involutive regular BiHom-Lie superalgebra $\mathcal{L}$ is of maximal length if $\dim \mathcal{L}_{\alpha, \bar{i}} = 1$ for any $\alpha \in \Lambda$ and $\bar{i} \in \mathbb{Z}_2$.

Observe that if $\mathcal{L}$ is of maximal length, then we have

$$I = \left((I_0 \cap H_0) \oplus \left(\bigoplus_{\alpha \in \Lambda_0^I} \mathcal{L}_{\alpha, 0} \right) \right) \oplus \left((I_1 \cap H_1) \oplus \left(\bigoplus_{\alpha \in \Lambda_1^I} \mathcal{L}_{\alpha, 1} \right) \right), \quad (4.1)$$

where $\Lambda_{\bar{i}}^I = \{\alpha \in \Lambda: I_{\bar{i}} \cap \mathcal{L}_{\alpha, \bar{i}} \neq 0\}$, $\bar{i} \in \mathbb{Z}_2$.

Definition 4.4. A BiHom-Lie superalgebra $\mathcal{L}$ is called perfect if $Z(\mathcal{L}) = 0$ and $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$.

Proposition 4.5. Let $\mathcal{L}$ be a perfect split involutive regular BiHom-Lie superalgebra of maximal length and root-multiplicative. If $\mathcal{L}$ has all of its roots connected, then any ideal I of $\mathcal{L}$ satisfies $I^* = I$.

Proof. Let I be a nonzero ideal of $\mathcal{L}$, and we can write

$$I = \left((I_0 \cap H_0) \oplus \left(\bigoplus_{\alpha \in \Lambda_0^I} \mathcal{L}_{\alpha, 0} \right) \right) \oplus \left((I_1 \cap H_1) \oplus \left(\bigoplus_{\alpha \in \Lambda_1^I} \mathcal{L}_{\alpha, 1} \right) \right), \quad (4.2)$$

with $\Lambda_i^I \subset \Lambda_{\bar{i}}$, $\bar{i} \in \mathbb{Z}_2$, and some $\Lambda_i^I \neq \emptyset$. Hence, let us fix some $\alpha_0 \in \Lambda_i^I$ such that

$$0 \neq \mathcal{L}_{\alpha_0, \bar{i}_0} \subset I, \quad \text{for some } \bar{i}_0 \in \mathbb{Z}_2. \quad (4.3)$$

By Lemma 2.2, $\phi(\mathcal{L}_{\alpha_0}) = \mathcal{L}_{\alpha_0 \phi^{-1}}$ and $\psi(\mathcal{L}_{\alpha_0}) = \mathcal{L}_{\alpha_0 \psi^{-1}}$. By (4.3), we have $\phi(\mathcal{L}_{\alpha_0, \bar{i}_0}) \subset \phi(I) = I$ and $\psi(\mathcal{L}_{\alpha_0, \bar{i}_0}) \subset \psi(I) = I$. So $\mathcal{L}_{\alpha_0 \phi^{-1} \psi^{-1}, \bar{i}_0} \subset I$. Similarly, we get

$$\mathcal{L}_{\alpha_0 \phi^{-n} \psi^{-r}, \bar{i}_0} \subset I, \quad \text{for } n, r \in \mathbb{N}. \quad (4.4)$$

For any $\beta \in \Lambda$, $\beta \notin \pm \alpha_0 \phi^{-n+1} \psi^{-1}$, for $n \in \mathbb{N}$, the fact that α_0 and β are connected gives us a connection $\{\gamma_1, \gamma_2, \dots, \gamma_n\} \subset \Lambda$ from α_0 to β such that

$$\begin{aligned} \gamma_1 &= \alpha_0 \in \Lambda_{\bar{i}_0}, \gamma_k \in \Lambda_{\bar{i}_k}, \quad \text{for } k = 2, \dots, n, \\ \gamma_1 \phi^{-1} + \gamma_2 \psi^{-1} &\in \Lambda_{\bar{i}_1 + \bar{i}_2}, \gamma_1 \phi^{-n+1} + \gamma_2 \phi^{-n} \psi^{-1} + \gamma_3 \phi^{-n+1} \psi^{-1} + \dots + \gamma_{n-1} \phi^{-1} \psi^{-1} + \gamma_n \psi^{-1} \in \Lambda_{\bar{i}_1 + \bar{i}_2 + \dots + \bar{i}_n}, \\ \gamma_1 \phi^{-n+1} + \gamma_2 \phi^{-n} \psi^{-1} + \gamma_3 \phi^{-n+1} \psi^{-1} + \dots + \gamma_i \phi^{-n+i} \psi^{-1} + \dots + \gamma_{n-1} \phi^{-1} \psi^{-1} + \gamma_n \psi^{-1} &\in \{\pm \beta \phi^{-m} \psi^{-s} : m, s \in \mathbb{N}\} \\ \text{and } \bar{i}_1 + \bar{i}_2 + \dots + \bar{i}_n &= \bar{j}, \end{aligned}$$

Taking into account that γ_1, γ_2 and $\gamma_1 \phi^{-1} + \gamma_2 \psi^{-1}$. Since $\gamma_1 = \alpha_0 \in \Lambda_{\bar{i}_0}$ and $\gamma_2 \in \Lambda_{\bar{i}_2}$, the root-multiplicativity and maximal length of $\mathcal{L}$ show $\gamma_1 \phi^{-1} + \gamma_2 \psi^{-1} \in \Lambda_{\bar{i}_1 + \bar{j}}$.

$$0 \neq [\mathcal{L}_{\gamma_1, \bar{i}_0}, \mathcal{L}_{\gamma_2, \bar{i}_2}] = \mathcal{L}_{\gamma_1 \phi^{-1} + \gamma_2 \psi^{-1}, \bar{i}_0 + \bar{i}_2},$$

and by (4.4), we have

$$0 \neq \mathcal{L}_{\gamma_1 \phi^{-1} + \gamma_2 \psi^{-1}, \bar{i}_0 + \bar{i}_2} \subset I.$$

We can argue in a similar way from $\gamma_1 \phi^{-1} + \gamma_2 \psi^{-1}$, γ_3 and $\gamma_1 \phi^{-2} + \gamma_2 \phi^{-1} \psi^{-1}$, $\gamma_3 \psi^{-1}$. Hence,

$$0 \neq [\mathcal{L}_{\gamma_1 \phi^{-1} + \gamma_2 \psi^{-1}, \bar{i}_0 + \bar{i}_2}, \mathcal{L}_{\gamma_3, \bar{i}_3}] = \mathcal{L}_{\gamma_1 \phi^{-2} + \gamma_2 \phi^{-1} \psi^{-1} + \gamma_3 \psi^{-1}, \bar{i}_0 + \bar{i}_2 + \bar{i}_3},$$

and by above, we have

$$0 \neq \mathcal{L}_{\gamma_1 \phi^{-2} + \gamma_2 \phi^{-1} \psi^{-1} + \gamma_3 \psi^{-1}, \bar{i}_0 + \bar{i}_2 + \bar{i}_3} \subset I.$$

Following this process with the connection $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$, we obtain that

$$0 \neq \mathcal{L}_{\gamma_1 \phi^{-n+1} + \gamma_2 \phi^{-n} \psi^{-1} + \gamma_3 \phi^{-n+1} \psi^{-1} + \dots + \gamma_n \psi^{-1}, \bar{i}_0 + \bar{i}_2 + \bar{i}_3 + \dots + \bar{i}_n} \subset I.$$

And so that we get either

$$\mathcal{L}_{\beta \phi^{-m} \psi^{-s}, \bar{j}} \subset I \quad \text{or} \quad \mathcal{L}_{-\beta \phi^{-m} \psi^{-s}, \bar{j}} \subset I, \quad (4.5)$$

for any $\beta \in \Lambda$, $m \in \mathbb{N}$. From Lemma 2.2, we get

$$\mathcal{L}_{\beta, \bar{j}} \subset I \quad \text{or} \quad \mathcal{L}_{-\beta, \bar{j}} = (\mathcal{L}_{\beta, \bar{j}})^* \subset I. \quad (4.6)$$

In both cases

$$[\mathcal{L}_{\beta}, (\mathcal{L}_{\beta})^*] \subset I. \quad (4.7)$$

And so $[e_{\beta}, e_{-\beta}] \in I$. As given any $e_{-\alpha_0} \in \mathcal{L}_{-\alpha_0}$, we have

$$e_{-\alpha_0} = -\alpha_0 ([e_{\beta}, e_{-\beta}])^{-1} [e_{\beta}, e_{-\beta}], e_{-\alpha_0} \in I,$$

we conclude $\mathcal{L}_{-\alpha_0} \subset I$, and then we get $(\mathcal{L}_{\alpha_0})^* \subset I$. Hence, $(\oplus_{\alpha \in \Lambda'} \mathcal{L}_{\alpha})^* = \oplus_{\alpha \in \Lambda'} \mathcal{L}_{\alpha}$.

Finally, the fact $H = \sum_{\alpha \in \Lambda} ([\mathcal{L}_{\alpha \psi^{-1}}, \mathcal{L}_{-\alpha \phi^{-1}}])$ and (4.7) give us

$$H \subset I. \quad (4.8)$$

As $H^* = H$, we get $(I \cap H)^* = (I \cap H)$, taking into account $(\oplus_{\alpha \in \Lambda'} \mathcal{L}_{\alpha})^* = \oplus_{\alpha \in \Lambda'} \mathcal{L}_{\alpha}$. The decomposition of I in (4.1) finally gives us $I^* = I$. $\square$

Theorem 4.6. *Let $\mathcal{L}$ be a perfect split involutive regular BiHom-Lie superalgebra of maximal length and root-multiplicative. Then $\mathcal{L}$ is simple if and only if it has all its nonzero roots connected.*

Proof. The first implication is Theorem 3.4. To prove the converse, write $\mathcal{L} = H + \sum_{[\alpha] \in \Lambda / \sim} L_{[\alpha]}$ and consider I a nonzero ideal of $\mathcal{L}$. By (4.8), we have $H \subset I$. Given any $\alpha \in \Lambda$ and taking into account $\alpha \neq 0$ and the maximal length of $\mathcal{L}$, we have $\mathcal{L}_\alpha \subset I$. We conclude $I = \mathcal{L}$, and therefore $\mathcal{L}$ is simple. $\square$

Theorem 4.7. *Let $\mathcal{L}$ be a perfect split involutive regular BiHom-Lie superalgebra of maximal length and root-multiplicative. Then $\mathcal{L}$ is the direct sum of the family of its minimal involutive ideals, each one being a simple split involutive regular BiHom-Lie superalgebra and having all its nonzero roots connected.*

Proof. By Corollary 3.6, $\mathcal{L} = \oplus_{[\alpha] \in \Lambda / \sim} I_{[\alpha]}$ is the direct sum of the family of ideals

$$I_{[\alpha]} = I_{H, [\alpha]} \oplus V_{[\alpha]} = \text{span}_{\mathbb{K}} \left\{ \left[\mathcal{L}_{\beta\psi^{-1}}, \left(\mathcal{L}_{\beta\phi^{-1}} \right)^* \right] : \beta \in [\alpha] \right\} \oplus_{\beta \in [\alpha]} \mathcal{L}_\beta,$$

where each $I_{[\alpha]}$ is a split involutive regular BiHom-Lie superalgebra with root system $\mathcal{L}_{I_{[\alpha]}} = [\alpha]$. To make use of Theorem 4.6 in each $I_{[\alpha]}$, we observe that the root-multiplicativity of $\mathcal{L}$ and Proposition 3.3 show that $\mathcal{L}_{I_{[\alpha]}}$ has all of its elements $\mathcal{L}_{I_{[\alpha]}}$ connected, that is, connected through connections contained in $\mathcal{L}_{I_{[\alpha]}}$. Moreover, each $I_{[\alpha]}$ is root-multiplicative by the root-multiplicativity of $\mathcal{L}$. So we infer that $I_{[\alpha]}$ is of maximal length and finally its center $Z(I_{[\alpha]}) = 0$. As a consequence, $[I_\alpha, I_\beta] = 0$ if $[\alpha] \neq [\beta]$. Applying Theorem 4.6, we conclude that $I_{[\alpha]}$ is simple and $\mathcal{L} = \oplus_{[\alpha] \in \Lambda / \sim} I_{[\alpha]}$. $\square$

5 Example

In this section, we provide an example to clarify the results in Section 3, generalizing the example of [23].

In [24], Rittenberg and Wyler introduced the definition of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebras. Let $\mathcal{L}$ be a linear space and $a = (a_1, a_2)$, $b = (b_1, b_2)$ be elements of $\Pi = \mathbb{Z}_2 \times \mathbb{Z}_2$ such that

$$a + b = (a_1 + b_1, a_2 + b_2), \quad a \cdot b = (a_1 b_1 + a_2 b_2).$$

Suppose that $\mathcal{L}$ is a direct sum of graded components,

$$\mathcal{L} = \bigoplus_{\lambda \in \Pi} \mathcal{L}^\lambda = \mathcal{L}^{(0,0)} \oplus \mathcal{L}^{(0,1)} \oplus \mathcal{L}^{(1,0)} \oplus \mathcal{L}^{(1,1)}.$$

If $\mathcal{L}$ satisfies skew-symmetry and Jacobi identity, then $\mathcal{L}$ is referred to as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebra.

In [25], denote by $e_{i,j}$ the matrix with zero everywhere except a 1 on position (i, j) , where the row and the column indices run from 1 to $2m + 2n + 1$. We introduce the following elements:

$$c_j^+ = \sqrt{2} (e_{j, 2m+1} - e_{2m+1, m+j}), \quad c_j^- = \sqrt{2} (e_{2m+1, j} - e_{m+j, 2m+1}) \quad \text{for } j = 1, 2, 3, \dots, m$$

and

$$c_{m+j}^+ = \sqrt{2} (e_{2m+1, 2m+1+n+j} + e_{2m+1+j, 2m+1}), \quad c_{m+j}^- = \sqrt{2} (e_{2m+1, 2m+1+j} - e_{2m+1+n+j, 2m+1}) \quad \text{for } j = 1, 2, 3, \dots, n.$$

Tolstoy proved that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebra $\mathcal{L}$ is defined by $2m + 2n$ generators $c_j^\pm$ ($j = 1, 2, 3, \dots, m$) and $c_{m+k}^\pm$ ($k = 1, 2, 3, \dots, n$), subject to the relations

$$\begin{aligned} [[c_j^\mu, c_k^\eta], c_l^\nu] &= |\nu - \eta| \delta_{kl} c_j^\mu - |\nu - \mu| \delta_{jl} c_k^\eta, \\ [\{c_{m+j}^\mu, c_{m+k}^\eta\}, c_{m+l}^\nu] &= (\nu - \mu) \delta_{jl} c_{m+k}^\eta + (\nu - \eta) \delta_{kl} c_{m+j}^\mu, \\ [[c_j^\mu, c_{m+k}^\eta], c_l^\nu] &= -|\nu - \mu| \delta_{jl} c_{m+k}^\eta, [\{c_j^\mu, c_{m+k}^\eta\}, c_{m+l}^\nu] = (\nu - \eta) \delta_{kl} c_j^\mu, \end{aligned}$$

where $j, k \in \{1, 2, 3, \dots, m\}$ and $\mu, \nu, \eta \in \{+, -\}$ is isomorphic to $\text{ps}(2m + 1, 2n)$.

Using Theorem 2.7 in [5], we can construct an involutive BiHom-Lie superalgebra from the Lie superalgebra $\text{psl}(2m+1, 2n)$.

We consider even automorphisms $\phi, \psi: \text{psl}(2m+1, 2n) \rightarrow \text{psl}(2m+1, 2n)$ such that $\phi^2 = \psi^2 = \text{id}$. More precisely, ϕ and ψ are involutions given by

$$\phi(e_{2m+1+j,k}) = -e_{2m+1+j,k}, \quad \psi(e_{2m+1+j,k}) = e_{2m+1+j,k}, \quad j = 1, 2, 3, \dots, 2m, \phi(x) = x, \psi(x) = -x, \text{ otherwise.}$$

One can easily check that the tuple $(\mathfrak{L} = \text{psl}(2m+1, 2n), [x, y]_{\phi, \psi} = [\phi(x), \psi(y)], \phi, \psi)$ is an involutive BiHom-Lie superalgebra.

As a basis in the maximal involutive abelian subalgebra H of $(\mathfrak{L}, \phi, \psi)$, consider

$$h_i = e_{i,i} - e_{m+i,m+i} = \frac{1}{2}[c_i^-, c_i^+], \quad i = 1, 2, 3, \dots, m,$$

$$h_{m+j,m+j} = e_{2m+1+j,2m+1+j} - e_{2m+1+n+j,2m+1+n+j} = \frac{1}{2}[c_{m+j,m+j}^-, c_{m+j,m+j}^+], \quad j = 1, 2, 3, \dots, n,$$

which belong to $\mathfrak{L}_0^{(0,0)}$. They span the space H of diagonal matrices in $\mathfrak{L}$. That is,

$$H = \text{span}_{\mathbb{K}} \{h_i, h_{m+j}: i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n\}.$$

Let $\{\varepsilon_i, \delta_j: i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n\}$ be the dual basis of H^* given by

$$\varepsilon_i(h_k) = \delta_{ik}, \varepsilon_i(h_{m+j}) = 0; \delta_j(\varepsilon_i) = 0, \delta_j(h_i) = \delta_{ji}, \delta_{2m+2n+1} = 0.$$

Then, the root vectors and the corresponding root spaces are given by

$$\begin{aligned} e_{j,k} - e_{m+k,m+j} &\in \mathfrak{L}_{\varepsilon_k - \varepsilon_j}^{(0,0)}, \quad j \neq k = 1, 2, 3, \dots, m, \\ e_{j,m+k} - e_{k,m+j} &\in \mathfrak{L}_{-(\varepsilon_k + \varepsilon_j)}^{(0,0)}, \quad j < k = 1, 2, 3, \dots, m, \\ e_{m+j,k} - e_{m+k,j} &\in \mathfrak{L}_{\varepsilon_k + \varepsilon_j}^{(0,0)}, \quad j < k = 1, 2, 3, \dots, m, \\ e_{j,2m+1} - e_{2m+1,m+j} &\in \mathfrak{L}_{-\varepsilon_j}^{(1,1)}, \quad j = 1, 2, 3, \dots, m, \\ e_{m+j,2m+1} - e_{2m+1,j} &\in \mathfrak{L}_{\varepsilon_j}^{(1,1)}, \quad j = 1, 2, 3, \dots, m, \\ e_{2m+1+j,2m+1+k} - e_{2m+1+n+k,2m+1+n+j} &\in \mathfrak{L}_{\delta_k - \delta_j}^{(0,1)}, \quad j \neq k = 1, 2, 3, \dots, n, \\ e_{2m+1+j,2m+1+n+k} + e_{2m+1+k,2m+1+n+j} &\in \mathfrak{L}_{-(\delta_k + \delta_j)}^{(0,0)}, \quad j \neq k = 1, 2, 3, \dots, n, \\ e_{2m+1+n+j,2m+1+k} + e_{2m+1+n+k,2m+1+j} &\in \mathfrak{L}_{\delta_k + \delta_j}^{(0,0)}, \quad j \neq k = 1, 2, 3, \dots, n, \\ e_{j,2m+1+k} - e_{2m+1+n+k,m+j} &\in \mathfrak{L}_{\delta_k - \varepsilon_j}^{(0,1)}, \quad j = 1, 2, 3, \dots, m; \quad k = 1, 2, 3, \dots, n, \\ e_{m+j,2m+1+k} - e_{2m+1+n+k,j} &\in \mathfrak{L}_{\delta_k + \varepsilon_j}^{(0,1)}, \quad j = 1, 2, 3, \dots, m; \quad k = 1, 2, 3, \dots, n, \\ e_{2m+1,2m+1+k} - e_{2m+1+n+k,2m+1} &\in \mathfrak{L}_{\delta_k}^{(1,0)}, \quad k = 1, 2, 3, \dots, n, \\ e_{j,2m+1+n+k} + e_{2m+1+k,m+j} &\in \mathfrak{L}_{-(\varepsilon_j + \delta_k)}^{(0,1)}, \quad j = 1, 2, 3, \dots, m; \quad k = 1, 2, 3, \dots, n, \\ e_{m+j,2m+1+n+k} - e_{2m+1+k,j} &\in \mathfrak{L}_{\varepsilon_j - \delta_k}^{(0,1)}, \quad j = 1, 2, 3, \dots, m; \quad k = 1, 2, 3, \dots, n, \\ e_{2m+1,2m+1+n+k} - e_{2m+1+k,2m+1} &\in \mathfrak{L}_{-\delta_k}^{(1,0)}, \quad k = 1, 2, 3, \dots, n. \end{aligned}$$

The set of roots is given by

$$\Lambda = \{\pm \varepsilon_j, \pm(\varepsilon_j \pm \varepsilon_k), \pm \delta_k, \pm(\delta_j \pm \delta_k), \pm(\varepsilon_j \pm \delta_k): j = 1, 2, 3, \dots, m; k = 1, 2, 3, \dots, n\}.$$

So, we have a split involutive BiHom-Lie superalgebra with respect to H , given by

$$\mathfrak{L} = H \oplus \left(\bigoplus_{\lambda \in \Pi} \bigoplus_{\alpha \in \Lambda} \mathfrak{L}_{\alpha}^{\lambda} \right).$$

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References

- [1] Jonas T. Hartwig, Daniel Larsson, and Sergei D. Silvestrov, *Deformations of Lie algebras using σ -derivations*, J. Algebra **295** (2006), no. 2, 314–361, DOI: 10.1016/j.jalgebra.2005.07.036.
- [2] Giacomo Graziani, Abdenacer Makhlouf, Claudia Menini, and Florin Panaite, *BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras*, SIGMA **11** (2015), no. 86, 1–34, DOI: 10.3842/SIGMA.2015.086.
- [3] Yongsheng Cheng and Huang Qi, *Representations of BiHom-Lie algebras*, arXiv: 1610.04302v1, 2016.
- [4] Ling Liu, Abdenacer Makhlouf, Claudia Menini, and Florin Panaite, *BiHom-pre-Lie algebras, BiHom-Leibniz algebras and Rota-Baxter operators on BiHom-Lie algebras*, arxiv: 1706.00474v1, 2017.
- [5] Shengxiang Wang and Shuangjian Guo, *BiHom-Lie superalgebra structures*, arXiv: 1610.02290v1, 2016.
- [6] Kadri Abdaoui, Abdelkader Ben Hassine, and Abdenacer Makhlouf, *BiHom-Lie colour algebras structures*, arxiv: 1706.02188v1, 2017.
- [7] Shuangjian Guo, Xiaohui Zhang, and Shengxiang Wang, *The construction and deformation of BiHom-Novikov algebras*, J. Geom. Phys. **132** (2018), no. 10, 460–472, DOI: 10.1016/j.geomphys.2018.06.011.
- [8] Helena Albuquerque, Elisabete Barreiro, A. J. Calderón, and José M. Sánchez, *On split regular Hom-Lie superalgebras*, J. Geom. Phys. **128** (2018), no. 6, 1–11, DOI: 10.1016/j.geomphys.2018.01.025.
- [9] Yan Cao, Liangyun Chen, and Bing Sun, *On split regular Hom-Leibniz algebras*, J. Algebra Appl. **17** (2018), no. 10, 1850185, DOI: 10.1142/S0219498818501852.
- [10] Shuangjian Guo and Yuanyuan Ke, *On split regular BiHom-Poisson superalgebras*, Colloq. Math. **160** (2020), 269–282, DOI: 10.4064/cm7809-3-2019.
- [11] Jian Zhang, Chiping Zhang, and Yan Cao, *On the structure of split involutive regular Hom-Lie algebras*, Oper. Matrices **11** (2017), no. 3, 783–792, DOI: 10.7153/oam-11-55.
- [12] Jian Zhang, Liangyun Chen, and Chiping Zhang, *On split regular BiHom-Lie superalgebras*, J. Geom. Phys. **128** (2018), no. 6, 38–47, DOI: 10.1016/j.geomphys.2018.02.005.
- [13] María J. Aragón Perinán and Antonio J. Calderón, *Split regular Hom-Lie algebras*, J. Lie Theory **25** (2015), no. 3, 875–888.
- [14] María J. Aragón Perinán and Antonio J. Calderón, *The structure of split regular Hom-Poisson algebras*, Colloq. Math. **145** (2016), no. 1, 1–13, DOI: 10.4064/cm6568-9-2015.
- [15] Antonio J. Calderón, *On split Lie algebras with symmetric root systems*, Proc. Indian Acad. Sci. (Math. Sci.) **118** (2018), no. 3, 351–356, DOI: 10.1007/s12044-008-0027-3.
- [16] Antonio J. Calderón, *On split Lie triple systems*, Proc. Indian Acad. Sci. (Math. Sci.) **119** (2019), no. 2, 165–177, DOI: 10.1007/s12044-009-0017-0.
- [17] Antonio J. Calderón and José M. Sánchez, *On split Leibniz algebras*, Linear Algebra Appl. **436** (2012), no. 6, 1648–1660, DOI: 10.1016/j.laa.2011.02.046.
- [18] Antonio J. Calderón and José M. Sánchez, *On split Leibniz superalgebras*, Linear Algebra Appl. **438** (2013), no. 12, 4709–4725, DOI: 10.1016/j.laa.2013.01.017.
- [19] Antonio J. Calderón and José M. Sánchez, *On the structure of split Lie color algebras*, Linear Algebra Appl. **436** (2012), no. 2, 307–315, DOI: 10.1016/j.laa.2011.02.003.
- [20] Antonio J. Calderón and José M. Sánchez, *The structure of split regular BiHom-Lie algebras*, J. Geom. Phys. **110** (2016), no. 12, 296–305, DOI: 10.1016/j.geomphys.2016.08.004.
- [21] Yan Cao and Liangyun Chen, *On the structure of split Leibniz triple systems*, Acta Math. Sin. (Engl. Ser.) **31** (2015), no. 10, 1629–1644, DOI: 10.1007/s10114-015-4800-9.
- [22] Yan Cao and Liangyun Chen, *On split Leibniz triple systems*, J. Korean Math. Soc. **54** (2017), no. 4, 1265–1279, DOI: 10.4134/JKMS.j160465.
- [23] Valiollah Khalili, *On the structure of split involutive Hom-Lie color algebras*, Rev. Unión, Mat. Argent. **60** (2019), no. 1, 61–77, DOI: 10.33044/revuma.v60n1a05.
- [24] V. Rittenberg and D. Wyler, *Sequences of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded Lie algebras and superalgebras*, J. Math. Phys. **19** (1978), no. 10, 2193–2200, DOI: 10.1063/1.523552.
- [25] V. N. Tolstoy, *Once more on parastatistics*, Phys. Part. Nucl. Lett. **11** (2014), no. 7, 933–937, DOI: 10.1134/S1547477114070449.