Research Article

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On the N-spectrum of oriented graphs

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Abstract: Given any digraph D , its non-negative spectrum (or N-spectrum, shortly) consists of the eigenvalues of the matrix AA^T , where A is the adjacency matrix of D . In this study, we relate the classical spectrum of undirected graphs to the N-spectrum of their oriented counterparts, permitting us to derive spectral bounds. Moreover, we study the spectral effects caused by certain modifications of a given digraph.

Keywords: non-negative spectrum, digraph, singular values

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1 Introduction

The study of graph eigenvalues has a long history. Starting with classical applications in mathematical chemistry (see [1] and chapter 8 of [2]), further applications have been found in combinatorics (see [3–5]), combinatorial optimization (see [6,7]) and also in theoretical computer science (see [8]).

On the other hand, research on the spectrum of directed graphs is relatively rare compared to that of undirected graphs. The main reason is that the adjacency matrix of a digraph in general is difficult to work with. It is not symmetric (so we lose the property that eigenvalues are real) and not diagonalizable (hence the dimension of an eigenspace may differ from the algebraic multiplicity of the corresponding eigenvalue in the characteristic polynomial).

Many researchers have proposed definitions of Hermitian matrices to represent digraphs. For example, Guo and Mohar (see [9]) defined the so-called Hermitian adjacency matrix of a digraph D as follows: given a digraph D with vertex set V and arc set E , the Hermitian adjacency matrix of D is the matrix $H(D) = [h_{uv}]$ with

$$h_{uv} = \begin{cases} 1 & \text{if } uv \in E \text{ and } vu \in E, \\ i & \text{if } uv \in E \text{ and } vu \notin E, \\ -i & \text{if } uv \notin E \text{ and } vu \in E, \\ 0 & \text{otherwise.} \end{cases}$$

If D is an oriented graph, then the above definition can be tailored to the matrix $iA - iA^T$, where A is the well-known adjacency matrix of D .

Another way of defining an alternative version of the adjacency matrix – as it was done by Jovanović in [10] – is to consider the eigenvalues of the matrix $N_{\text{out}}(D) := AA^T$ for any given digraph D , where A is the classical adjacency matrix of D . $N_{\text{out}}(D)$ is symmetric, so it has many of the well-known benefits of the classical adjacency matrix of an undirected graph. Moreover, $N_{\text{out}}(D)$ is positive semi-definite, hence Jovanović called the spectrum of this matrix the non-negative spectrum of a digraph or, shortly, the

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When it comes to analyzing the N-spectrum of a digraph, it is possible to conveniently choose a vertex order such that the matrix AA^T assumes the block diagonal form. Following [11], we define a zig-zag trail (of length $2l$) between two vertices x and y to be a sequence of arcs

$$(x, u_1), (v_1, u_1), (v_1, u_2), (v_2, u_2), \dots, (u_l, v_l), (y, v_l)$$

such that it starts with a forward arc from x , then a reverse arc, then again a forward arc, and so on, until we reach y with a backward arc. A path of length zero is also considered a zig-zag trail. Next, two vertices shall be related to each other if they have a zig-zag trail between them. It is easy to see that this relation is an equivalence relation. Therefore, it partitions the vertex set V into subsets $\{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k\}$. This partition is called common out-neighbor partition. Renumbering the vertices of V starting with vertices in $\mathcal{B}_1$ then $\mathcal{B}_2$ and so on will result in the desired block diagonal matrix form. This follows directly from the interpretation of the entries of AA^T that we gave in the previous paragraph.

As an example, consider the digraph in Figure 1. The grey vertices form the largest block of the common out-neighbor partition. The black vertices form a block of two vertices, whereas each of the white vertices forms a singleton block. The chosen vertex order is consistent with the proposed way of numbering the vertices according to the common out-neighbor partition. Hence, we get the following matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the common out-neighbor partition, it is possible to relate the N-eigenvalues of certain directed bipartite graphs to the classical eigenvalues of their undirected counterparts. A bipartite graph is a graph whose vertex set V can be partitioned into two subsets V_1, V_2 such that neither of the two subgraphs induced by V_1 resp. V_2 contains any edges.

Theorem 1. (Square Theorem [11]). *Let D be a bipartite digraph such that each vertex is either a source or a sink. For any vector $x \in \mathbb{R}^n$, we may construct its source part by setting all those entries of x to zero which correspond to the non-sources (i.e. sinks) of D . Likewise, we construct the sink part of x . Let k be the number of sources and l be the number of sinks in D . Furthermore, let G be the underlying undirected graph of D .*

- (i) *Given an eigenspace basis for eigenvalue $\lambda \neq 0$ of G , the source parts of these vectors form an N-eigenspace basis for N-eigenvalue λ^2 of D and their sink parts are all N-eigenvectors for N-eigenvalue 0 of D .*
- (ii) *Every eigenvector for eigenvalue 0 of G is also an N-eigenvector for N-eigenvalue 0 of D .*
- (iii) *If the source part of any N-eigenvector for N-eigenvalue 0 of D is not null, then this source part is an eigenvector for eigenvalue 0 of G .*
- (iv) *Given a basis of $\mathbb{R}^{k+l}$ of eigenvectors of G , an N-eigenspace basis for N-eigenvalue 0 of D can be constructed as follows. Collect the sink parts of all the vectors associated with positive eigenvalues of G , together with the vectors associated with eigenvalue 0. Alternatively, collect the source parts of all vectors of the given basis and determine a maximal linearly independent subset of resulting set, together with l unit vectors, one for each sink (such that it is non-zero exactly on the considered sink).*

Note that the digraph D mentioned in Theorem 1 is actually an oriented graph.

3 Upper bounds for the spectrum of a graph

Study of the spectral radius of graphs has been a prominent research topic in the spectral graph theory since its inception in the 1950s to this day. In 1990, Cvetković and Rowlinson (see [12]) surveyed numerous results on the spectral radius. This is complemented by a more recent survey by Stevanović (see [13]). In this section, we introduce new upper bounds for the spectral radius of graphs. We use the N-spectra of certain oriented graphs to establish these bounds.

Recall that a graph is bipartite if and only if its vertex set can be partitioned into two sets such that the end points of each edge belong to different sets. For what follows, we introduce the notion of zig-zag orientation of a bipartite graph. To this end, we simply choose one set of its vertex bipartition and orient all arcs of the graph toward this set. As a result, we obtain an oriented graph in which each vertex is either a source or a sink.

Theorem 2. *For any bipartite graph $G = (V, E)$, we have*

$$\lambda_1(G) \leq \max_{u \in V} \sqrt{\sum_{v \in N(u)} \deg(v)},$$

where $\lambda_1(G)$ is the largest eigenvalue of G .

Proof. Let D be a digraph obtained by zig-zag orientation of the given graph G such that the vertices are numbered according to the common out-neighbor partition. Let $N_{\text{out}}(D) = [a_{ij}]$. Using Theorem 1 and observing that the largest singular value of a matrix M is equal to its spectral matrix norm $\|M\|$ (see [14]), we obtain

$$\lambda_1^2(G) = \|N_{\text{out}}(D)\| \leq \|N_{\text{out}}(D)\|_{\infty} = \max_i \sum_j a_{ij},$$

where $\|N_{\text{out}}(D)\|_{\infty}$ is the maximum row sum norm of the matrix $N_{\text{out}}(D)$.

Note that the row in $N_{\text{out}}(D)$ corresponding to the vertex u has row sum equal to $\deg^+(u)$ plus the number of common out-neighbor vertices with vertices in the same partite class as u . However, the number of common out-neighbor vertices with vertices in the same partite class of u is exactly

$$\sum_{v \in N(u)} (\deg^-(v) - 1).$$

Since D has a zig-zag orientation, we have $\deg_D^-(v) = \deg_G(v)$ and $\deg_D^+(u) = \deg_G(u)$. Therefore,

$$\lambda_1^2(G) \leq \max_{u \in V} \left(\deg(u) + \sum_{v \in N(u)} (\deg(v) - 1) \right) = \max_{u \in V} \left(\deg(u) + \sum_{v \in N(u)} (v) - |N(u)| \right) = \max_{u \in V} \left(\sum_{v \in N(u)} \deg(v) \right).$$

□

Let us mention that the above upper bound is related to the Brualdi–Hoffman conjecture, which states $\lambda_1(G) \leq \sqrt{|E|}$ for any graph G (see [15]). This conjecture has already been settled by Bhattacharya et al. [16]. In the special case of bipartite graphs, the upper bound given in Theorem 2 is tighter than the upper bound given by the Brualdi–Hoffman conjecture.

The following theorem gives an upper bound for the spectral radius of any graph G in terms of the N-spectral radius of some orientation of this graph into a digraph D .

Theorem 3. *Let G be a graph and D an oriented graph whose unoriented counterpart is G . Then,*

$$\lambda_1(G) \leq 2\sqrt{\mu_1(D)},$$

where $\lambda_1(G)$ is the largest eigenvalue of G and $\mu_1(D)$ is the largest N-eigenvalue of D .

Figure 2: A grid $P_5 \times P_6$ oriented according to Lemma 2.

Proof. Suppose that the adjacency matrix of D is A . Then, the adjacency matrix of G is $A + A^T$. Now observe that

$$2\|A^2\| \leq \|A^T A + A A^T\|.$$

It follows that

$$\begin{aligned} \|(A + A^T)^2\| &= \|A^2 + A A^T + A^T A + (A^T)^2\| \leq \|A^2\| + \|(A^T)^2\| + \|A A^T\| + \|A^T A\| \\ &\leq \frac{1}{2}\|A^T A + A A^T\| + \frac{1}{2}\|A^T A + A A^T\| + 2\|A A^T\| \leq 4\|A A^T\|. \end{aligned}$$

Consequently, $\lambda_1^2(G) \leq 4\mu_1(D)$ and finally $\lambda_1(G) \leq 2\sqrt{\mu_1(D)}$. $\square$

An immediate question arises when thinking about applying Theorem 3: which orientation yields the lowest upper bound? In the following we turn our attention to grid graphs and show that the inequality mentioned in Theorem 3 is sharp.

Let P_i denote the path with i vertices. A grid is a graph that is isomorphic to the Cartesian product $P_n \times P_m$. For the sake of simplicity, we shall assume that its vertex set is the set of all pairs (i, j) such that $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Thus, two vertices $(i, j), (k, l)$ of a grid are adjacent if and only if either $|i - k| = 1$ and $j = l$ or if $i = k$ and $|j - l| = 1$. Figure 2 depicts a grid $P_5 \times P_6$ that has been oriented. Note that the chosen layout gives us a notion of the vertical/horizontal direction. We find that vertex (i, j) lies in row number i and column number k of the grid. With this image before our eyes, we define the rows $\mathcal{R}_k = \{(k, j)\}_{j=1, \dots, m}$ and columns $C_k = \{(i, k)\}_{i=1, \dots, n}$ of a grid.

Lemma 1. Let D be an oriented graph obtained from zig-zag orientation of a given path P_n such that at least one of its leaves becomes a source. Then, the characteristic polynomial of the principal submatrix of $N_{\text{out}}(D)$ corresponding to the sources of D is

$$\prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left(x - 4 \cos^2 \left(\frac{\pi j}{n+1} \right) \right).$$

Proof. The claim is implicit in the proof of Corollary 4 in [11]. To prove it, we recall the well-known fact (see e.g. [4]) that the eigenvalues of P_n are

$$2 \cos \left(\frac{\pi j}{n+1} \right), \quad \text{for } j = 1, \dots, n. \quad (1)$$

For $j = 1, \dots, \lfloor \frac{n}{2} \rfloor$ these eigenvalues are distinct positive numbers. Using Theorem 1, they can be directly attributed to the sources of D . If n is odd, one needs to add a zero eigenvalue, namely the one obtained for $j = \lfloor \frac{n}{2} \rfloor$. Thus, we get the formula

$$x^{\lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor} \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left(x - 4 \cos^2 \left(\frac{\pi j}{n+1} \right) \right),$$

which then can be simplified to get the desired result. $\square$

Lemma 2. Let D be the oriented graph obtained from orienting a given grid $P_n \times P_m$ as follows:

- Any path induced by an odd column C_{2l+1} has zig-zag orientation, with starting arc $(2, 2l+1)(1, 2l+1)$.
- Any path induced by an even column C_{2l} has zig-zag orientation, with starting arc $(1, 2l)(2, 2l)$.
- Any path induced by an odd row R_{2k+1} has zig-zag orientation, with starting arc $(2k+1, 1)(2k+1, 2)$.
- Any path induced by an even row R_{2k} has zig-zag orientation, with starting arc $(2k, 2)(2k, 1)$.

Then,

$$\sigma_N(D) = \{0^{(d)}\} \cup \left\{ 4 \cos^2 \left(\frac{\pi i}{n+1} \right)^{\left(\lfloor \frac{m}{2} \rfloor \right)} \right\}_{i=1}^n \cup \left\{ 4 \cos^2 \left(\frac{\pi j}{m+1} \right)^{\left(\lfloor \frac{n}{2} \rfloor \right)} \right\}_{j=1}^m,$$

where $d = \left(\left\lceil \frac{m}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor \right) \left\lfloor \frac{n}{2} \right\rfloor + \left(\left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor \right) \left\lceil \frac{m}{2} \right\rceil$.

Proof. First observe that, by the definition of the particular orientation, there exist neither sources nor sinks in D . Now suppose that (i, j) and $(i+1, j+1)$ have a common out-neighbor, say $(i, j+1)$. Since both the path induced by the vertices $(k, j+1)$ with $k \in \{1, \dots, n\}$ and the path induced by the vertices (i, k) with $k \in \{1, \dots, m\}$ have zig-zag orientation, it actually follows that $(i, j+1)$ must be a sink – a contradiction. As a result, common neighbors can only occur along those zig-zag paths that were explicitly mentioned in the definition of the chosen orientation. No other zig-zag paths exist in D .

Let us divide the vertices according to whether they belong to an even or odd diagonal of the grid. To this end, let $\mathcal{D}_1 = \{(i, j) : i - j \text{ is even}\}$ and $\mathcal{D}_2 = \{(i, j) : i - j \text{ is odd}\}$.

Observe that $\mathcal{B}_{1,k} := \mathcal{D}_1 \cap R_k$ contains exactly the sources of the subgraph of D induced by the vertex set R_k . Moreover, $\mathcal{B}_{2,l} := \mathcal{D}_2 \cap C_l$ contains exactly the sources of the subgraph of D induced by the vertex set C_l . Since no other zig-zag paths exist in D apart from those induced by the separate rows and columns of the grid it follows that the sets

$$\mathcal{B}_{1,1}, \dots, \mathcal{B}_{1,n}, \mathcal{B}_{2,1}, \dots, \mathcal{B}_{2,m}$$

form the common out-neighbor partition of D . Hence, we may use Lemma 1 to determine the characteristic polynomial of $N_{\text{out}}(D)$ on a per-block basis, where each block can also be obtained “locally” by considering a suitable zig-zag-oriented path and determining the block of its N_{out} matrix corresponding to the sources of that path. Moreover, keep in mind that the zig-zag paths of two neighboring columns or rows can be transformed into each other by reversing the orientation. Hence, their N-spectrum is the same.

Let us define

$$S_k(x) := \prod_{j=1}^{\lfloor \frac{k}{2} \rfloor} \left(x - 4 \cos^2 \left(\frac{\pi j}{k+1} \right) \right).$$

If both n and m are even, then it is a direct consequence of Lemma 1 that $\chi(D; x) = S_n(x)^m S_m(x)^n$. In the case of odd n we need to consider that the zig-zag paths contained in the grid columns, alternatingly, contain $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ sinks (as can easily be seen from Figure 2). For each column that contains $\lceil \frac{n}{2} \rceil$ local sinks we need to add a zero N-eigenvalue. This happens $\lceil \frac{m}{2} \rceil$ times. For the case when m is odd we need to consider that the zig-zag paths contained in the grid rows, alternatingly, contain $\lfloor \frac{m}{2} \rfloor$ and $\lceil \frac{m}{2} \rceil$ sinks. A zero N-eigenvalue needs to be added whenever there are $\lceil \frac{m}{2} \rceil$ local sinks per row. Observing the slightly different pattern, we see that this happens only $\lfloor \frac{n}{2} \rfloor$ times. All in all, we obtain $\chi(D; x) = S_n(x)^m S_m(x)^n x^d$. $\square$

Theorem 4. *The minimum N -spectral radius of any oriented grid $P_n \times P_n$ can be achieved by using the orientation described in Lemma 2.*

Proof. Let D be the digraph with underlying graph $P_n \times P_n$ and orientation as described in Lemma 2. It follows from Theorem 3 that

$$\lambda_1(P_n \times P_n) \leq 2\sqrt{\mu_1(D)},$$

but by Eq. (1) and Lemma 2 we have

$$2\sqrt{\mu_1(D)} = 2\sqrt{4\cos^2\left(\frac{\pi}{n+1}\right)} = 4\cos\left(\frac{\pi}{n+1}\right) = 2\lambda_1(P_n) = \lambda_1(P_n \times P_n). \quad \square$$

Example 1. Consider the grid $P_5 \times P_5$. It has a spectral radius of approximately 3.46. As predicted by Theorem 4, this matches the bound we deduce from the N -spectral radius 3 achieved by zig-zag orientation of the given grid. Alternatively, orient all “horizontal” arcs from left to right and all “vertical” arcs from top to bottom. Then, we obtain an N -spectral radius of approximately 3.62, which only yields an upper bound of 3.81 for the spectral radius of the grid $P_5 \times P_5$.

4 Effects of modification

It is interesting to study how modifying a given digraph reflects in changes of its N -spectrum. Several modifications have already been considered in [10] and [11]. In this section, we want to focus on N -eigenvalue interlacing. Probably, the most classical theorem about eigenvalue interlacing is the following.

Theorem 5. (cf. [4]). *Let G be a graph (resp. a symmetric matrix) with eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$ and H an induced subgraph (resp. a principal submatrix) of G with eigenvalues $\mu_1 \geq \dots \geq \mu_m$. Then,*

$$\lambda_k \geq \mu_{k-n+m} \geq \lambda_{k-n+m} \quad (\text{for } k = 1, \dots, 2n - m).$$

The special case $m = n - 1$ gives us the following inequality chain:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \dots \geq \mu_{n-2} \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

When it comes to the N -spectrum, deriving any helpful interlacing theorems at all turns out to be quite a challenge. At least, Jovanović has been able to derive the following “edge version” of the interlacing theorem.

Theorem 6. (cf. [10]). *Let D be a weakly connected digraph of order n , with N -eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$. Assume that D has at least one vertex v with in-degree one and obtain D' from D by removing the incoming arc at v . Now let $\mu_1 \geq \dots \geq \mu_n$ be the N -eigenvalues of D' . Then,*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \lambda_n \geq \mu_n.$$

Let us derive another N -eigenvalue interlacing theorem. To this end, let $D = (V, E)$ be a digraph and $I \subset V$. By D^{-I} (resp. D^{+I}) we denote the digraph obtained by adding a sink (resp. source) to the digraph D such that its set of in-neighbors (resp. out-neighbors) is I . Furthermore, let A^{-I} (resp. A^{+I}) denote the adjacency matrix of D^{-I} (resp. D^{+I}).

Theorem 7. *Let $D = (V, E)$ be a digraph and $I = \{i_1, i_2, \dots, i_r\} \subset V$. Suppose that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are the N -eigenvalues of the digraph D and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n+1}$ are the N -eigenvalues of D^{-I} (resp. D^{+I}). Then,*

$$\lambda_1 + r \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \mu_{n+1} = 0.$$

Proof. In this proof and also later on we shall use subscripts of the form $r \times s$ to document the dimensions of some annotated matrix, in order to make the overall block structure more transparent. Moreover, if a mere number is affixed with such a dimensional annotation, then this denotes a matrix of the mentioned dimensions where all entries are equal to that number.

The adjacency matrix of D^{-I} is given by

$$A^{-I} = \begin{pmatrix} (A)_{n \times n} & \left(\sum_{j=1}^r e_{ij} \right)_{n \times 1} \\ (0)_{1 \times n} & (0)_{1 \times 1} \end{pmatrix},$$

where, as usual, e_k denotes the k th unit vector (of the dimension implied by the context). Therefore,

$$(A^{-I})^T (A^{-I}) = \begin{pmatrix} (A^T)_{n \times n} & (0)_{n \times 1} \\ \left(\sum_{j=1}^r e_{ij}^T \right)_{1 \times n} & (0)_{1 \times 1} \end{pmatrix} \begin{pmatrix} (A)_{n \times n} & \left(\sum_{j=1}^r e_{ij} \right)_{n \times 1} \\ (0)_{1 \times n} & (0)_{1 \times 1} \end{pmatrix} = \begin{pmatrix} (A^T A)_{n \times n} & \left(A^T \sum_{j=1}^r e_{ij} \right)_{n \times 1} \\ \left(\sum_{j=1}^r e_{ij}^T A \right)_{1 \times n} & (r)_{1 \times 1} \end{pmatrix}.$$

Note that $A^T A$ is the principal submatrix of $(A^{-I})^T (A^{-I})$. Applying the interlacing Theorem 5 on $(A^{-I})^T (A^{-I})$ and using the fact that the matrices $(A^{-I})^T (A^{-I})$ and $(A^{-I}) (A^{-I})^T$ have the same spectra (and likewise AA^T and $A^T A$), we get

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \mu_{n+1} = 0.$$

It remains to prove $\mu_1 \leq \lambda_1 + r$. In order to do that consider

$$(A^{-I}) (A^{-I})^T = \begin{pmatrix} \left(AA^T + \sum_{j=1}^r e_{ij} \sum_{k=1}^r e_{ik}^T \right)_{n \times n} & (0)_{n \times 1} \\ (0)_{1 \times n} & (0)_{1 \times 1} \end{pmatrix}.$$

Now observe $\sum_{j=1}^r e_{ij} \sum_{k=1}^r e_{ik}^T = [b_{st}]$, where

$$b_{st} = \begin{cases} 1 & s = i_j \text{ and } t = i_k \text{ for some } 1 \leq j, k \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $\sum_{j=1}^r e_{ij} \sum_{k=1}^r e_{ik}^T$ has a maximum row sum of r . Now, since $(A^{-I}) (A^{-I})^T$ is a positive semi-definite matrix, we have

$$\mu_1 = \|AA^T + \sum_{j=1}^r e_{ij} \sum_{k=1}^r e_{ik}^T\| \leq \|AA^T\| + \left\| \sum_{j=1}^r e_{ij} \sum_{k=1}^r e_{ik}^T \right\| = \lambda_1 + \left\| \sum_{j=1}^r e_{ij} \sum_{k=1}^r e_{ik}^T \right\| \leq \lambda_1 + \left\| \sum_{j=1}^r e_{ij} \sum_{k=1}^r e_{ik}^T \right\|_{\infty} = \lambda_1 + r.$$

The proof for the case D^{+I} works analogously. $\square$

The effect of adding a vertex with both incoming and outgoing arcs on the entire N-spectrum can be hardly predicted. However, the following two theorems describe the effect of such an operation on the N-spectral radius of oriented digraphs.

Theorem 8. Let $D = (V, E)$ be a digraph, $I = \{v_{i_1}, v_{i_2}, \dots, v_{i_r}\} \subset V$ any set of source vertices in D and $D^{\pm I}$ the digraph obtained from D by adding a new vertex η and connecting it to each vertex of I (with an arc of arbitrary orientation). Then,

$$\lambda_1 - \deg^-(\eta) \leq \mu_1 \leq \lambda_1 + \deg^-(\eta),$$

where λ_1 is the N-spectral radius of D and μ_1 is the N-spectral radius of $D^{\pm I}$.

Proof. Observe that η only gets connected to vertices that originally were sources, but some of the old source vertices contained in I may lose their source property in the new digraph. This needs to be analyzed. The adjacency matrix of $D^{\pm I}$ is given by

$$A^{\pm I} = \begin{pmatrix} (A)_{n \times n} & \left(\sum_{v_{ij} \in N^-(\eta)} e_{ij} \right)_{n \times 1} \\ \left(\sum_{v_{ij} \in N^+(\eta)} e_{ij}^T \right)_{1 \times n} & (0)_{1 \times 1} \end{pmatrix},$$

so it follows that

$$A^{\pm I} (A^{\pm I})^T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

where

$$P = \left(AA^T + \sum_{v_{ij} \in N^-(\eta)} e_{ij} \sum_{v_{ij} \in N^-(\eta)} e_{ij}^T \right)_{n \times n} \quad Q = \left(A \sum_{v_{ij} \in N^+(\eta)} e_{ij} \right)_{n \times 1} \quad R = Q^T \quad S = \left(\sum_{v_{ij} \in N^+(\eta)} e_{ij}^T \sum_{v_{ij} \in N^+(\eta)} e_{ij} \right)_{1 \times 1}.$$

Note that Q describes the new common out-neighbors of the vertices in V with the vertex η . Since all vertices in I are source vertices in D , there is no new common out-neighbor between V and η in $D^{\pm I}$. Therefore, $Q = (0)_{n \times 1}$, and by symmetry $R = (0)_{1 \times n}$. Furthermore, the single entry in S is equal to $\deg^-(\eta)$, so

$$\sigma_N(D^{\pm I}) = \sigma(P) \cup \{\deg^-(\eta)\}.$$

Using the interlacing Theorem 5, we therefore obtain

$$\begin{aligned} \|A^{\pm I} (A^{\pm I})^T\| = \mu_1 &\geq \|AA^T + \sum_{v_{ij} \in N^-(\eta)} e_{ij} \sum_{v_{ij} \in N^-(\eta)} e_{ij}^T\| \geq \|AA^T\| - \left\| \sum_{v_{ij} \in N^-(\eta)} e_{ij} \sum_{v_{ij} \in N^-(\eta)} e_{ij}^T \right\| \\ &\geq \lambda_1 - \left\| \sum_{v_{ij} \in N^-(\eta)} e_{ij} \sum_{v_{ij} \in N^-(\eta)} e_{ij}^T \right\|_{\infty}, \end{aligned}$$

but $\sum_{v_{ij} \in N^-(\eta)} e_{ij} \sum_{v_{ij} \in N^-(\eta)} e_{ij}^T$ is the matrix that describes the new common out-neighbors of the vertices of I . In fact, any two vertices in $N^-(\eta)$ have common out-neighbors, therefore, the matrix $\sum_{v_{ij} \in N^-(\eta)} e_{ij} \sum_{v_{ij} \in N^-(\eta)} e_{ij}^T$ has a maximum row sum of $\deg^-(\eta)$. This implies $\lambda_1 - \deg^-(\eta) \leq \mu_1$. The other inequality can be proved in the same way. $\square$

Corollary 1. Let D be a digraph, $I = \{v_{i_1}, v_{i_2}, \dots, v_{i_r}\} \subset V(D)$ any set of sink vertices in D and $D^{\pm I}$ the digraph obtained from D by adding a new vertex η with incoming and outgoing arcs attached to the vertices in I . Then,

$$\lambda_1 - \deg^+(\eta) \leq \mu_1 \leq \lambda_1 + \deg^+(\eta),$$

where λ_1 is the N -spectral radius of D and μ_1 is the N -spectral radius of $D^{\pm I}$.

Corollary 2. Adding a new source (resp. sink) vertex to a digraph D attached to some source (resp. sink) vertices in D does not change the N -spectral radius.

Theorem 9. Let $G = (V, E)$ be a graph and $u \in V$ such that no two vertices in $N(u)$ are adjacent. Furthermore, let D be an arbitrary orientation of G , as long as all vertices in $N(u)$ become sinks (resp. sources) in $D - u$ and u becomes a sink (resp. source) in D . Then,

$$\lambda_1(G) \leq 2\sqrt{\mu_1(D - u)},$$

where $\lambda_1(G)$ denotes the largest eigenvalue of G and $\mu_1(D - u)$ denotes the largest N -eigenvalue of $D - u$.

Proof. Using Theorem 3, we get $\lambda_1(G) \leq 2\sqrt{\mu_1(D)}$. But Corollary 2 implies $\mu_1(D) = \mu_1(D - u)$, hence the proof is complete. $\square$

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