Research Article

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On the finite approximate controllability for Hilfer fractional evolution systems with nonlocal conditions

<https://doi.org/10.1515/math-2020-0158>

received August 21, 2019; accepted April 6, 2020

Abstract: The aim of this study is to investigate the finite approximate controllability of certain Hilfer fractional evolution systems with nonlocal conditions. To achieve this, we first transform the mild solution of the Hilfer fractional evolution system into a fixed point problem for a condensing map. Then, by using the topological degree approach, we present sufficient conditions for the existence and uniqueness of the solution of the Hilfer fractional evolution systems. Using the variational approach, we obtain sufficient conditions for the finite approximate controllability of semilinear controlled systems. Finally, an example is provided to illustrate main results.

Keywords: finite approximate controllability, Hilfer fractional derivative, inequalities, variational approach, topological degree approach

MSC 2010: 34A08, 34A37, 34C25

1 Introduction

In this study, we investigate the following Hilfer fractional evolution system:

$$\begin{cases} D_{0^+}^{\nu, \mu} x(t) = Ax(t) + f(t, x(t)) + Bu(t), & t \in J' = (0, b] \\ I_{0^+}^{(1-\nu)(1-\mu)} x(0) = x_0 + \psi(x), \end{cases} \quad (1.1)$$

where $D_{0^+}^{\nu, \mu}$ represents the Hilfer fractional derivative, $0 \leq \nu \leq 1$, $\frac{1}{2} < \mu < 1$; $x(\cdot)$ is assumed to be in a Hilbert space H , $I_{0^+}^{(1-\nu)(1-\mu)}$ is called the Riemann-Liouville fractional integral of order $(1-\nu)(1-\mu)$; $A: D(A) \subseteq H \rightarrow H$ is the infinitesimal generator of a compact, uniformly bounded and C_0 -semigroup $\{T(t), t \geq 0\}$ on a separable Hilbert space H . And $f: J' \times H \rightarrow H$ is a given function that will be specified later. The control function u is taken in $L^2(J', U)$ and the admissible control set U is a Hilbert space, B is a bounded linear operator from U into H . Finally, x_0 is an element of H [1–10].

Hilfer [11] presented the generalized Riemann-Liouville fractional derivative, which is called the Hilfer fractional derivative. In [12], by using the fixed point theory, semigroup theory, the authors investigated the approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions. More results on this topic can be found in [13, 14].

It is known that finite approximate controllability is a consequence of approximate controllability. For a finite dimensional subspace of bounded domains, there exists an orthogonal projection over it of the solution at the end of control time, such that the final state satisfies simultaneously a finite number of

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exact constraints. Due to the strong practicality and applicability, in recent years, some researchers have studied the finite approximate controllability of some differential systems. In [15], semilinear variational inequalities with distributed controls have been studied. Zuazua [16] presented finite dimensional version of null controllability for the semilinear heat equation in bounded domains with Dirichlet boundary conditions. By applying the fixed point result of Leray-Schauder, Menezes [17] investigated the finite approximate controllability for a nonlocal parabolic problem. Mahmudov [18] replaced the uniform boundedness of the nonlinear function by some weaker natural conditions and examined the simultaneous approximate controllability and finite approximate controllability of some semilinear abstract equations.

The topological degree method is a powerful tool to certify the existence of solutions to fractional systems with the weaker conditions. In [19], by using the coincidence degree theory approach, Mawhin studied the existence of solutions to nonlinear boundary value problems. Dinca et al. [20] used the topological degree method to prove the existence of solutions of the Dirichlet problems with p -Laplacian. Isaia [21] applied the topological degree method along with condensing maps and proved the existence of solutions of a nonlinear integral equation. Wang et al. [22] used the topological degree method to solve a class of fractional equations. In [23], Iqbal et al. studied the solutions of coupled systems of multipoint boundary value problems of fractional order hybrid differential equations.

Motivated by the aforementioned works, specially [18] and [22], we offer to study the finite approximate controllability of certain Hilfer fractional evolution systems with nonlocal conditions. Our aim is to obtain some suitable growth conditions for the existence of solutions of system (1.1) by using the topological degree approach under the assumption that the operators are bounded and to apply the variational approach to prove the finite approximate controllability.

2 Preliminaries

In this section, we recall some definitions, notations, and preliminary results concerning fractional calculus and finite approximate controllability, which will be used later, and furthermore, we consider the mild solution of (1.1) using two classes of operators which will be specified later.

Let $J = [0, b]$ and E be a Banach space with norm $\|\cdot\|_E$. Now E^* denotes its dual and $\langle \cdot, \cdot \rangle_E$ denotes the duality pairing between E^* and E . We use $L_b(E, E)$ to denote the space of bounded linear operators with the norm $\|\cdot\|_{L_b(E, E)}$. Let $C(J, E)$ be the Banach space of all continuous functions from J to E . Set $\gamma = \nu + \mu - \nu\mu$, $0 < \gamma < 1$, and then $1 - \gamma = (1 - \nu)(1 - \mu)$. Define

$$Y := C_{1-\gamma}(J', H) = \left\{ x \in C(J', H) : \lim_{t \rightarrow 0^+} t^{1-\gamma} x(t) \text{ exists and is finite} \right\}$$

endowed with the norm $\|x\|_Y = \sup_{t \in J'} \|t^{1-\gamma} x(t)\|_H$. Clearly, $(Y, \|\cdot\|_Y)$ is a Banach space.

For brevity, we denote $L_H^p = L^p(J, H)$, $L_{R^+}^p = L^p(J, R^+)$, and $L_U^p = L^p(J, U)$ for $1 \leq p < \infty$.

We collect some definitions and remarks of fractional calculus of Hilfer type, finite approximate controllability, non-compactness of Kuratowski type, and the topological degree theories. For more details, we refer to [13–16, 21, 24–33].

Definition 2.1. System (1.1) is finite approximately controllable on J' , if there exist $x_b \in Y$ and $\varepsilon > 0$, for the control $u_\varepsilon \in L_U^2$, the solution $x_\varepsilon(b)$ of system (1.1) satisfies the following conditions:

$$\|x_\varepsilon(b) - x_b\| < \varepsilon \quad (2.1)$$

$$\Pi_{\mathcal{E}} x_\varepsilon(b) = \Pi_{\mathcal{E}} x_b, \quad (2.2)$$

where $\mathcal{E}$ is a finite dimensional subspace of H and $\Pi_{\mathcal{E}}$ is the orthogonal projection from H to $\mathcal{E}$.

Definition 2.1. It states that the approximate control u_ε can be chosen such that condition 2.1 holds and simultaneously a finite number of exact constraints that condition 2.2 also holds.

In this study, $\mathfrak{B}$ denotes the family of all bounded sets for the Banach space X , $Q \in \mathfrak{B}$.

Definition 2.2. The Kuratowski measure ν of non-compactness of Q , denoted by $\nu(Q)$, is the infimum of the set of all numbers $k > 0$ such that Q admits a finite cover by sets with diameters $k > 0$, that is

$$\nu(Q) = \inf \left\{ k > 0: Q \subset \bigcup_{i=0}^n X_i, X_i \in X, \text{diam}(X_i) < k, \quad i = 1, 2, 3, \dots, n \right\}.$$

Proposition 2.3. The Kuratowski measure ν of non-compactness has the following properties:

- (i) *Monotone:* if the bounded subsets Q_1, Q_2 of $\mathcal{X}$, $Q_1 \subseteq Q_2$ implies $\nu(Q_1) \leq \nu(Q_2)$.
- (ii) *Nonsingular:* if $\nu(\{a\} \cup \Omega) = \nu(\Omega)$ for every $a \in \mathcal{X}$ and every nonempty subset $\Omega \subseteq \mathcal{X}$.
- (iii) *Regular:* if $\nu(\Omega) = 0$ is equivalent to the relative compactness of Ω .
- (iv) *Algebraic semi-additivity:* $\nu(Q_1 + Q_2) \leq \nu(Q_1) + \nu(Q_2)$, where $Q_1 + Q_2 = \{x + y: x \in Q_1, y \in Q_2\}$.
- (v) *Semi-additivity:* $\nu(Q_1 \cup Q_2) = \max \{\nu(Q_1), \nu(Q_2)\}$.
- (vi) $\nu(\lambda Q) \leq \|\lambda\| \nu(Q)$.
- (vii) $\nu(Q) = \nu(\bar{Q})$.

Definition 2.4. Assume that the function $\mathcal{F}: \Omega \rightarrow X$ is continuous and bounded mapping for $\Omega \in X$. Then, $\mathcal{F}$ is ν -Lipschitz, if there exists a constant $L \geq 0$, all $Q \in \Omega$ such that

$$\nu(\mathcal{F}(Q)) \leq L\nu(Q).$$

Furthermore, if $L < 1$, $\mathcal{F}$ is called a strict ν -contraction.

Definition 2.5. For every $Q \in \Omega$, the function $\mathcal{F}: \Omega \rightarrow X$ is ν -condensing if

$$\nu(\mathcal{F}(Q)) < \nu(Q).$$

In other words, $\nu(\mathcal{F}(Q)) \geq \nu(Q)$ represents $\nu(Q) = 0$.

Obviously, ν -condensing mapping is ν -Lipschitz as $L = 1$.

Denote the collection of all strict ν -contractions $\mathcal{F}: \Omega \rightarrow X$ by $SC_F^\nu(\Omega)$, and the collection of all ν -condensing mappings $\mathcal{F}: \Omega \rightarrow X$ by $C_F^\nu(\Omega)$, it is clear that $SC_F^\nu(\Omega) \subset C_F^\nu(\Omega)$. We recall that $\mathcal{F}: \Omega \rightarrow X$ is Lipschitz, if for $x, y \in \Omega$, there exists $L > 0$, such that

$$\|\mathcal{F}_x - \mathcal{F}_y\|_H \leq L\|x - y\|_H.$$

If $L < 1$, $\mathcal{F}$ is called the strict contraction.

Proposition 2.6. If $\mathcal{F}: \Omega \rightarrow X$ is compact, then $\mathcal{F}$ is ν -Lipschitz with zero constant.

Proposition 2.7. If $\mathcal{F}: \Omega \rightarrow X$ is Lipschitz with constant L_1 , then $\mathcal{F}$ is ν -Lipschitz with the same constant L_1 .

Proposition 2.8. If $\mathcal{F}_1: \Omega \rightarrow X$, $\mathcal{F}_2: \Omega \rightarrow X$ are Lipschitz with constants L_1 and L_2 , respectively, then $\mathcal{F}_1 + \mathcal{F}_2: \Omega \rightarrow X$ is ν -Lipschitz with the constant $L_1 + L_2$.

Theorem 2.9. Assume that $\mathcal{F}: X \rightarrow X$ is ν -condensing and

$$\mathcal{N} = \{x \in X: (\exists) \lambda \in [0, 1] \text{ such that } x = \lambda \mathcal{F}x\}.$$

If $\mathcal{N}$ is a bounded set in X , and there exists a real number $r > 0$ with $\mathcal{N} \subset Q_r(0)$. Then,

$$\deg(I - \lambda \mathcal{F}, Q_r(0), 0) = 1, \quad \text{for all } \lambda \in [0, 1].$$

Therefore, the operator $\mathcal{F}$ has at least one fixed point and the set of fixed point of $\mathcal{F}$ lies in $Q_r(0)$.

The following definition is based on Definition 2.3 in [14] and Definition 5 in [34].

Definition 2.10. For each $u \in L^2_U$, a function $x \in Y$ is a mild solution of (1.1), if $I^{(1-\nu)(1-\mu)}_{0^+} x(0) = x_0 + \psi(x)$ and there exists $f \in L^2_H$ a.e. on $t \in J'$ as follows:

$$x(t) = \mathcal{L}_{v,\mu}(t)(x_0 + \psi(x)) + \int_0^t \mathcal{T}_\mu(t-s)[f(s, x(s)) + Bu(s)]ds, \quad t \in J', \quad (2.3)$$

where

$$\mathcal{P}_\mu(t) := \int_0^\infty \mu \theta M_\mu(\theta) T(t^\mu \theta) d\theta, \quad \mathcal{T}_\mu(t) := t^{\mu-1} \mathcal{P}_\mu(t), \quad \mathcal{L}_{v,\mu}(t) := I^{v(1-\mu)}_{0^+} \mathcal{T}_\mu(t),$$

where $M_\mu(\theta)$ is the M -Wright function defined by

$$M_\mu(\theta) = \sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)! \Gamma(1-\mu n)}, \quad 0 < \mu < 1, \theta \in \mathbb{C}$$

and it satisfies $M_\mu(\theta) > 0$, $\int_0^\infty M_\mu(\theta) d\theta = 1$, and $\int_0^\infty \theta^\delta M_\mu(\theta) d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\mu\delta)}$, $\delta \in (-1, \infty)$.

In Section 1, we assumed that $T(t)(t \geq 0)$ is uniformly bounded, so there exists an $M > 1$ with $\sup_{t \in [0, \infty)} \|T(t)\| \leq M$.

Lemma 2.11 [14, Proposition 2.16] For any fixed $t > 0$, $\mathcal{T}_\mu(t)$ and $\mathcal{L}_{v,\mu}(t)$ are linear and bounded operators, i.e., for any $x \in Y$,

$$\|\mathcal{T}_\mu(t)x\|_H \leq \frac{Mt^{\mu-1}}{\Gamma(\mu)} \|x\|_H \quad \text{and} \quad \|\mathcal{L}_{v,\mu}(t)x\|_H \leq \frac{Mt^{\nu-1}}{\Gamma(\nu)} \|x\|_H, \quad \nu = v + \mu - \nu\mu.$$

Remark 2.12. By Lemma 2.11, we see that $\mathcal{T}_\mu(\cdot)$ and $\mathcal{L}_{v,\mu}(\cdot)$ are continuous in the uniform operator topology for $t > 0$, i.e., $\|\mathcal{T}_\mu(t_2) - \mathcal{T}_\mu(t_1)\|_{L_b(H,H)} \rightarrow 0$, $\|\mathcal{L}_{v,\mu}(t_2) - \mathcal{L}_{v,\mu}(t_1)\|_{L_b(H,H)} \rightarrow 0$ as $t_2 \rightarrow t_1$.

3 Existence of mild solutions

Before proceeding to the proof of the existence of mild solutions for (1.1), we propose the following assumptions:

$H(f): f: J' \times H \rightarrow R$ is a function such that:

- (i) the function $t \mapsto f(t, x)$ is measurable for all $x \in H$;
- (ii) the function $x \mapsto f(t, x)$ is continuous for $t \in J'$;
- (iii) for arbitrary $(t, x) \in J' \times H$, there exist $a_1, m_1 > 0$, $q_1 \in [0, 1)$ such that

$$\|f(t, x)\|_H \leq a_1 \|x\|_H^{q_1} + m_1.$$

- (iv) for arbitrary $x \in Y$, there exist $a_2, m_2 > 0$, $q_2 \in [0, 1)$ such that

$$\|\psi(x)\|_H \leq a_2 \|x\|_H^{q_2} + m_2.$$

And for any $x_1, x_2 \in Y$, there exists a constant $L_g \in [0, 1)$ such that

$$\|\psi(x_1) - \psi(x_2)\|_H \leq L_g \|x_1 - x_2\|_H.$$

$H(B)$: the following linear fractional control system

$$\begin{cases} D_{0^+}^{\nu, \mu} x(t) = Ax(t) + Bu(t), \\ I_{0^+}^{(1-\nu)(1-\mu)} x(0) = x_0 + \psi(x) \end{cases}$$

is approximately controllable on J' .

Under the aforementioned assumptions of $H(f), H(\psi)$, we will prove that the Hilfer fractional evolution system (1.1) has at least one solution $x \in Y$. Define operators

$$\begin{aligned} \mathcal{G}: Y &\rightarrow Y, & (\mathcal{G}x)(t) &= \mathcal{L}_{\nu, \mu}(t)(x_0 + \psi(x)), \\ \mathcal{F}: Y &\rightarrow Y, & (\mathcal{F}x)(t) &= \int_0^t \mathcal{T}_\mu(t-s)f(s, x(s))ds, \\ \mathcal{T}: Y &\rightarrow Y, & \mathcal{T}x &= \mathcal{G}x + \mathcal{F}x. \end{aligned}$$

In view of (2.3), it can be written as $x = \mathcal{T}x$, so the existence of the solution to Hilfer fractional Eq. (1.1) is equivalent to the existence of the fixed point for the operator $\mathcal{T}$.

Lemma 3.1. Let $t \in \left(\left(\frac{M}{\Gamma(\nu)} \right)^{\frac{1}{1-\nu}}, 1 \right)$, the operator $\mathcal{G}: Y \rightarrow Y$ is Lipschitz with constant L_g . Consequently, $\mathcal{G}$ is ν -Lipschitz with the same constant L_g . Furthermore, $\mathcal{G}$ satisfies the following growth condition:

$$\|\mathcal{G}\|_H \leq L_g (\|x_0\|_H + a_2 \|x\|_H^q + m_2).$$

Proof. From $H(\psi)$, for every $x_1, x_2 \in Y$, $\|\mathcal{G}x_1 - \mathcal{G}x_2\|_H \leq \frac{Mt^{\nu-1}}{\Gamma(\nu)} \|\psi(x_1) - \psi(x_2)\|_H \leq L_g \|x_1 - x_2\|_H$. In view of Proposition 2.8, $\mathcal{G}$ is ν -Lipschitz with the same constant L_g . $\square$

Lemma 3.2. The operator $\mathcal{F}$ is continuous and satisfies the following growth condition:

$$\|\mathcal{F}x\|_H \leq \frac{Mb^\mu (a_1 \|x\|_H^q + m_1)}{\Gamma(\mu + 1)}.$$

Proof. For a bounded set $B_k = \{x \in Y, \|x\|_H \leq k\}$, let $\{x_n\}$ be a sequence of B_k and $x_n \rightarrow x$ in B_k . In view of the continuity of f , one can assume $f(s, x_n(s)) \rightarrow f(s, x(s))$ as $n \rightarrow +\infty$. By means of the Lebesgue dominated convergence theorem, $\int_0^t \mathcal{T}_\mu(t-s)[f(s, x(s)) - f(s, x_n(s))]ds \rightarrow 0$ as $n \rightarrow +\infty$. Then, for all $t \in J'$,

$$|(\mathcal{F}x)(t) - (\mathcal{F}x_n)(t)| \leq \int_0^t \mathcal{T}_\mu(t-s)|f(s, x(s)) - f(s, x_n(s))|ds \rightarrow 0.$$

Thus, the operator $\mathcal{F}$ is continuous and the growth condition is a simple consequence of $H(f)$. $\square$

Lemma 3.3. The operator $\mathcal{F}$ is compact, and $\mathcal{F}$ is ν -Lipschitz with zero constant.

Proof. To prove the compactness of $\mathcal{F}$, consider the bounded set $\mathcal{D} \subset B_k$, and we need to show that $\mathcal{F}(\mathcal{D})$ is relatively compact in Y .

In fact, from Lemma 3.2, we know $\mathcal{F}(\mathcal{D})$ is bounded. For $x_n \in \mathcal{D}$ and $0 < t_1 < t_2 \leq b$,

$$|(\mathcal{F}(x_n))(t_1) - (\mathcal{F}(x_n))(t_2)| \leq \int_0^{t_1} |\mathcal{T}_\mu(t_1-s) - \mathcal{T}_\mu(t_2-s)| |f(s, x(s))| ds + \int_{t_2}^{t_1} |\mathcal{T}_\mu(t_2-s)f(s, x(s))| ds.$$

By following Remark 2.12, $|\mathcal{T}_\mu(t_1 - s) - \mathcal{T}_\mu(t_2 - s)| \rightarrow 0$ as $t_1 \rightarrow t_2$, thus $|(\mathcal{F}(x_n))(t_1) - (\mathcal{F}(x_n))(t_2)| \rightarrow 0$, and hence $\{\mathcal{F}(x_n)\}$ is equicontinuous. In view of the Arzela–Ascoli theorem, $\mathcal{F}(\mathfrak{D})$ is relatively compact. With the aid of Proposition 2.6, $\mathcal{F}$ is ν -Lipschitz with zero constant. $\square$

Theorem 3.4. *If the assumptions $H(f)$, $H(\psi)$ hold, then the Hilfer fractional equation (1.1) has at least one solution $x \in Y$ and the set of the solution is bounded in Y .*

Proof. In this section, we have defined three operators $\mathcal{G}$, $\mathcal{F}$, $\mathcal{T}$ and they are continuous and bounded. Furthermore, the operator $\mathcal{G}: Y \rightarrow Y$ is Lipschitz with constant L_g , $\mathcal{F}$ is ν -Lipschitz with zero constant. Define

$$D = \{x \in Y, (\exists) \lambda \in [0, 1] \text{ such that } x = \lambda \mathcal{T}x\}$$

for each $x \in D$,

$$\|x\|_H = \lambda \|\mathcal{T}x\|_H \leq \lambda (\|\mathcal{G}x\|_H + \|\mathcal{F}x\|_H) \leq \lambda \left[L_g (\|x_0\|_H + a_2 \|x\|_H^2 + m_2) + \frac{Mb^\mu (a_1 \|x\|_H^q + m_1)}{\Gamma(\mu + 1)} \right].$$

By Theorem 2.9, $\mathcal{T}$ has at least one fixed point and the set of the solution is bounded in Y , then the Hilfer fractional Eq. (1.1) has at least one solution $x \in Y$ and the set of the solution is bounded in Y . $\square$

4 Finite approximate controllability for the semilinear case

In this section, we study the finite approximate controllability of (1.1).

Take into account two relevant operators:

$$I_0^b = \int_0^b \mathcal{T}_\mu(b-s) B B^* \mathcal{T}_\mu^*(b-s) ds,$$

and

$$R_\varepsilon^b = (\varepsilon \mathbb{I} + I_0^b)^{-1}, \quad \varepsilon > 0,$$

where $\mathbb{I}$ denotes the identity operator, B^* denotes the adjoint of B , and $\mathcal{T}_\mu^*(\cdot)$ is the adjoint of $\mathcal{T}_\mu(\cdot)$. We can choose the functional defined by

$$\mathcal{J}_\varepsilon(\Psi; x) = \varepsilon \|(I - \Pi_\varepsilon) R_\varepsilon^b \Psi\|_H + \frac{1}{2} \int_0^b \|B^* \mathcal{T}_\mu^*(b-t) R_\varepsilon^b \Psi\|_H^2 dt - \langle H(x), R_\varepsilon^b \Psi \rangle, \quad (4.1)$$

where $\varepsilon > 0$, $x, x_b \in Y$ and

$$\begin{aligned} H(x) &= x_b - \mathcal{L}_{\nu, \mu}(b)(x_0 + \varphi(x)) - \int_0^b \mathcal{T}_\mu(b-s) f(s, x) ds, \\ u(s, x) &= B^* \mathcal{T}_\mu^*(b-s) R_\varepsilon^b \Psi(x), \\ x_\varepsilon(t) &= \mathcal{L}_{\nu, \mu}(t)(x_0 + \varphi(x)) - \int_0^t \mathcal{T}_\mu(t-s) [f(s, x(s)) + Bu_\varepsilon(s, x)] ds. \end{aligned}$$

For each $x \in Y$, the functional $J_\varepsilon(\Psi; x)$ is continuous and strictly convex, and we can define a function $\Psi: Y \rightarrow Y$ such that there exists a unique minimum Ψ_ε according to the functional $J_\varepsilon(\Psi; x)$.

Next, we define $B_r^{(1-\gamma)}(J') = \{x \in Y: \|x\|_Y \leq r\}$ and $B_r(J) = \{x \in C(J, H): \|x\|_C \leq r\}$. Obviously, the set $\mathcal{H} = \{H(x): x \in B_r^{(1-\gamma)}(J')\}$ is relatively compact in H , $H(x): B_r^{(1-\gamma)}(J') \rightarrow H$ is continuous.

Lemma 4.1. *For any $B_r^{(1-\gamma)}(J')$, there exists $\varepsilon_1 = R_\varepsilon^b \varepsilon$*

$$\liminf_{\|\Psi\|_H \rightarrow +\infty} \inf_{x \in B_r^{(1-\gamma)}(J')} \frac{\mathcal{J}_\varepsilon(\Psi; x)}{\|\Psi\|_H} \geq \varepsilon_1.$$

Proof. The idea comes from [16,18]. If it is the contrary case, for given sequences $\{\Psi_n\} \in H$, $\{x_n\} \subset Y$ with $\|\Psi_n\| \rightarrow +\infty$, such that

$$\liminf_{n \rightarrow +\infty} \frac{\mathcal{J}_\varepsilon(\Psi_n; x_n)}{\|\Psi_n\|_H} < \varepsilon_1,$$

we may normalize that $\hat{\Psi}_n = \frac{\Psi_n}{\|\Psi_n\|}$, then $\|\hat{\Psi}_n\| = 1$. In this case, we choose a subsequence that is denoted by $\hat{\Psi}_n$, and $\hat{\Psi}_n \xrightarrow{w} \hat{\Psi}$. From the compactness of $\mathcal{T}_\mu(t)$, one can get

$$B^* \mathcal{T}_\mu^*(b-t) R_\varepsilon^b \hat{\Psi}_n \xrightarrow{s} B^* \mathcal{T}_\mu^*(b-t) R_\varepsilon^b \hat{\Psi}.$$

From 4.1, we have

$$\frac{\mathcal{J}_\varepsilon(\Psi_n; x_n)}{\|\Psi_n\|_H^2} = \frac{\varepsilon}{\|\Psi_n\|_H} \|(I - \Pi_\varepsilon) R_\varepsilon^b \hat{\Psi}_n\|_H + \frac{1}{2} \int_0^b \|B^* \mathcal{T}_\mu^*(b-t) R_\varepsilon^b \hat{\Psi}_n\|_H^2 dt - \frac{1}{\|\Psi_n\|_H} \langle H(x), R_\varepsilon^b \hat{\Psi}_n \rangle, \quad (4.2)$$

as $\|\Psi_n\| \rightarrow +\infty$, by the Fatou lemma,

$$\int_0^b \|B^* \mathcal{T}_\mu^*(b-t) R_\varepsilon^b \hat{\Psi}\|_H^2 dt \leq \liminf_{n \rightarrow +\infty} \int_0^b \|B^* \mathcal{T}_\mu^*(b-t) R_\varepsilon^b \hat{\Psi}_n\|_H^2 dt = 0,$$

which implies $\hat{\Psi}_n \xrightarrow{w} 0$. In view of Definition 2.1, we have $\Pi_\varepsilon \hat{\Psi}_n \rightarrow 0$ in H and

$$\|(I - \Pi_\varepsilon) R_\varepsilon^b \hat{\Psi}_n\|_H \rightarrow R_\varepsilon^b.$$

Therefore,

$$\frac{\mathcal{J}_\varepsilon(\Psi_n; x_n)}{\|\Psi_n\|_H} = \varepsilon \|(I - \Pi_\varepsilon) R_\varepsilon^b \hat{\Psi}_n\|_H + \frac{\|\Psi_n\|_H}{2} \int_0^b \|B^* \mathcal{T}_\mu^*(b-t) R_\varepsilon^b \hat{\Psi}_n\|_H^2 dt - \langle H(x), R_\varepsilon^b \hat{\Psi}_n \rangle, \quad (4.3)$$

$$\liminf_{n \rightarrow +\infty} \frac{\mathcal{J}_\varepsilon(\Psi_n; x_n)}{\|\Psi_n\|_H} \geq \liminf_{n \rightarrow +\infty} (\varepsilon \|(I - \Pi_\varepsilon) R_\varepsilon^b \hat{\Psi}_n\|_H - \langle H(x), R_\varepsilon^b \hat{\Psi}_n \rangle) = R_\varepsilon^b \varepsilon = \varepsilon_1,$$

which contradicts our assumption. Thus,

$$\liminf_{n \rightarrow +\infty} \frac{\mathcal{J}_\varepsilon(\Psi_n; x_n)}{\|\Psi_n\|_H} \geq \varepsilon_1. \quad \square$$

Without loss of generality, similar to the normalized $\hat{\Psi}_n$, let $\Psi_{\varepsilon,n}(x)$ be a minimum sequence of $\mathcal{J}_\varepsilon(\Psi(x); x)$ for $x \in B_r^{(1-\gamma)}(J')$, and suppose $\Psi_{\varepsilon,n}(x) \xrightarrow{w} \widetilde{\Psi_\varepsilon(x)}$. Obviously, $\mathcal{J}_\varepsilon(\widetilde{\Psi_\varepsilon(x)}; x) \leq \liminf_{n \rightarrow +\infty} \mathcal{J}_\varepsilon(\Psi_{\varepsilon,n}(x); x)$. Because of the convexity of $\mathcal{J}_\varepsilon(\cdot; x)$, the minimum can be achieved at the unique point $\widetilde{\Psi_\varepsilon(x)}$.

Theorem 4.2. *Suppose that the assumptions on $H(f)$ and $H(B)$ hold, then (1.1) is finite approximately controllable on J' .*

Proof. From 4.1, one can see that the function $\mathcal{J}_\varepsilon(\Psi; x)$ is strictly convex, so we can assume that $\widetilde{\Psi}_\varepsilon$ is the unique critical point which minimizes $\mathcal{J}_\varepsilon(\Psi; x)$. That is,

$$\mathcal{J}_\varepsilon(\widetilde{\Psi}_\varepsilon; x) = \min_{\Psi \in H} \mathcal{J}_\varepsilon(\Psi; x).$$

Since $\mathcal{J}_\varepsilon(\Psi; x)$ is the Gateaux differentiable at $\widetilde{\Psi}_\varepsilon$, for any $\Psi_0 \in H$ and $\theta > 0$, we have

$$\begin{aligned} \mathcal{J}_\varepsilon(\widetilde{\Psi}_\varepsilon + \theta\Psi_0; x) - \mathcal{J}_\varepsilon(\widetilde{\Psi}_\varepsilon; x) &= \varepsilon\|(I - \Pi_\varepsilon)R_\varepsilon^b(\widetilde{\Psi}_\varepsilon + \theta\Psi_0)\|_H + \frac{1}{2} \int_0^b \|B^*\mathcal{T}_\mu^*(b-t) \times R_\varepsilon^b(\widetilde{\Psi}_\varepsilon + \theta\Psi_0)\|_H^2 dt \\ &\quad - \langle H(x), R_\varepsilon^b(\widetilde{\Psi}_\varepsilon + \theta\Psi_0) \rangle - \varepsilon\|(I - \Pi_\varepsilon)R_\varepsilon^b\widetilde{\Psi}_\varepsilon\|_H \\ &\quad - \frac{1}{2} \int_0^b \|B^*\mathcal{T}_\mu^*(b-t) \times R_\varepsilon^b\widetilde{\Psi}_\varepsilon\|_H^2 dt + \langle H(x), R_\varepsilon^b\widetilde{\Psi}_\varepsilon \rangle \\ &= \varepsilon\theta\|(I - \Pi_\varepsilon)R_\varepsilon^b\Psi_0\|_H + \theta \int_0^b \langle B^*\mathcal{T}_\mu^*(b-t) \times R_\varepsilon^b\widetilde{\Psi}_\varepsilon, B^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\Psi_0 \rangle dt \\ &\quad + \frac{\theta^2}{2} \int_0^b \|B^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\Psi_0\|_H^2 dt - \theta \langle H(x), \Psi_0 \rangle \end{aligned}$$

such that

$$\begin{aligned} 0 &= \lim_{\theta \rightarrow 0^+} \frac{\mathcal{J}_\varepsilon(\widetilde{\Psi}_\varepsilon + \theta\Psi_0; x) - \mathcal{J}_\varepsilon(\widetilde{\Psi}_\varepsilon; x)}{\theta} \\ &= \lim_{\theta \rightarrow 0^+} \left(\varepsilon\|(I - \Pi_\varepsilon)R_\varepsilon^b\Psi_0\|_H + \int_0^b \langle B^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\widetilde{\Psi}_\varepsilon, B^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\Psi_0 \rangle dt \right. \\ &\quad \left. + \frac{\theta}{2} \int_0^b \|B^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\Psi_0\|_H^2 dt - \langle H(x), R_\varepsilon^b\Psi_0 \rangle \right) \\ &= \varepsilon\|(I - \Pi_\varepsilon)R_\varepsilon^b\Psi_0\|_H + \int_0^b \langle B^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\widetilde{\Psi}_\varepsilon, B^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\Psi_0 \rangle dt - \langle H(x), R_\varepsilon^b\Psi_0 \rangle dt. \end{aligned}$$

Since,

$$\begin{aligned} \int_0^b \langle B^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\widetilde{\Psi}_\varepsilon, B^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\Psi_0 \rangle dt &= \int_0^b \langle \mathcal{T}_\mu(b-t)BB^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\widetilde{\Psi}_\varepsilon, R_\varepsilon^b\Psi_0 \rangle dt \\ &= \int_0^b \langle \mathcal{T}_\mu(b-t)Bu_\varepsilon(s, x), R_\varepsilon^b\Psi_0 \rangle dt, \end{aligned}$$

we have,

$$\begin{aligned} \langle H(x), R_\varepsilon^b\Psi_0 \rangle &= \varepsilon\|(I - \Pi_\varepsilon)R_\varepsilon^b\Psi_0\|_H + \int_0^b \langle \mathcal{T}_\mu(b-t)BB^*\mathcal{T}_\mu^*(b-t)R_\varepsilon^b\widetilde{\Psi}_\varepsilon, R_\varepsilon^b\Psi_0 \rangle dt \\ &= \varepsilon\|(I - \Pi_\varepsilon)R_\varepsilon^b\Psi_0\|_H + \int_0^b \langle \mathcal{T}_\mu(b-t)Bu_\varepsilon(s, x), R_\varepsilon^b\Psi_0 \rangle dt. \end{aligned}$$

From the definition of $H(x)$, $x_\varepsilon(b)$, one can get

$$H(x) = x_b - x_\varepsilon(b) + \int_0^t \mathcal{J}_\mu(t-s)Bu_\varepsilon(s, x)ds.$$

Then,

$$|\langle x_b - x_\varepsilon(b), \Psi_0 \rangle| = \varepsilon \|(I - \Pi_\varepsilon)\Psi_0\|_H \leq \varepsilon \|\Psi_0\|_H,$$

which is equivalent to

$$\|x_b - x_\varepsilon(b)\|_H \leq \varepsilon.$$

On the other hand, as $\theta < 0$, similar argument can be obtained, thus are omitted.

From the aforementioned results, we conclude that system 1 is approximately controllable on J' given $\Psi_0 \in H$, and if $\Psi_0 \in \mathcal{E}$, system 1 is finite approximately controllable on J' , that is, $\Pi_\varepsilon x_\varepsilon(b) = \Pi_\varepsilon x_b$. $\square$

5 An example

As an application of our result, we consider the following Hilfer fractional partial equation:

$$\begin{cases} D_{0^+}^{\nu, \mu} x(t, y) = x_{yy}(t, y) + Bu(t, y) + \int_0^1 e^{-s} \frac{|x(s, y)|}{1 + |x(s, y)|} ds, & 0 < t \leq 1, 0 \leq y \leq \pi, \\ x(t, 0) = x(t, \pi) = 0, & 0 < t \leq 1, \\ I_{0^+}^{(1-\nu)(1-\mu)} x(0, y) = x_0(y) + y, & 0 \leq y \leq \pi, \end{cases} \quad (5.1)$$

where $\nu = 1/2$, $\mu = 3/4$, $x(t, y)$ represents the temperature function at point $y \in [0, \pi]$ and time $t \in (0, 1]$. Now, set $H = L^2[0, \pi]$ and $e_n(y) = \sqrt{2/\pi} \sin(ny)$, $n = 1, 2, \dots$. Then, $\{e_n(y)\}$ is an orthonormal base on H . Define $A: D(A) \subset H \rightarrow H$ by $Ax = x_{yy}$ with domain

$$\{x \in H: x, x' \text{ are absolutely continuous, } x'' \in H, x(0) = x(\pi) = 0\}.$$

Then,

$$Ax = \sum_{n=1}^{\infty} (-n^2) \langle x, e_n \rangle e_n, \quad x \in D(A)$$

one can see that A generates a compact semigroup $T(t)$ ($t > 0$) on H and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n, \quad x \in H.$$

Hence, $T(t)$ is compact and $\|T(t)\| \leq 1$.

The infinite dimensional Hilbert space U can be defined as follows:

$$U := \left\{ u: u = \sum_{n=2}^{\infty} u_n e_n, \sum_{n=2}^{\infty} u_n^2 < \infty \right\},$$

and under the norm $\|u\|_U = (\sum_{n=2}^{\infty} u_n^2)^{1/2}$. We define a map $B \in \mathcal{L}(U, H)$ by

$$Bu = 5u_2 e_1 + 4u_2 e_2 + \sum_{n=3}^{\infty} u_n e_n \quad \text{for } u = \sum_{n=2}^{\infty} u_n e_n \in U,$$

and for $v = \sum_{n=1}^{\infty} v_n e_n \in H$, the inner product $\langle Bu, v \rangle = \langle u, B^*v \rangle$, and thus

$$B^*v = (5v_1 + 4v_2)e_2 + \sum_{n=3}^{\infty} v_n e_n$$

and

$$B^*T^*(t)x = (5x_1 e^{-t} + 4x_2 e^{-4t})e_2 + \sum_{n=3}^{\infty} e^{-n^2 t} x_n e_n.$$

Assume $\|B^*T^*(t)x\| = 0$ for some $t \in J'$, it follows that

$$\|5x_1 e^{-t} + 4x_2 e^{-4t}\|^2 + \sum_{n=3}^{\infty} \|e^{-n^2 t} x_n\|^2 = 0,$$

thus $x = 0$, and the linear part of (8) is approximately controllable on J' . Furthermore,

$$f(t, x(t, y)) = \int_0^1 e^{-s} \frac{|x(s, y)|}{1 + |x(s, y)|} ds \leq \int_0^1 e^{-s} ds,$$

obviously, all the conditions of $H(f)$ hold, then (8) is finite approximately controllable on J' .

Funding: The work was supported by Guizhou Province Innovative talents fund [2016]046, Qian Ke He Ping Tai Ren Cai [2018]5784-08.

Author contributions: The authors contributed equally to this work, and all the authors read and approved the final manuscript.

Conflicts of interest: The authors declare that they have no competing interests.

Data availability: The data used to support the findings of this study are included within the article.

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