Research Article

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The L -ordered L -semihypergroups
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Abstract: This study pursues an investigation on L -semihypergroups equipped with an L -order. First, the concept of L -ordered L -semihypergroups is introduced by L -posets and L -semihypergroups, and some related results are obtained. Then, prime, weakly prime, and semiprime L -hyperideals of L -ordered L -semihypergroups are studied. Moreover, the relationships among the three types of L -hyperideals are established. Finally, the intra-regular L -ordered L -semihypergroups are characterized in terms of these L -hyperideals. The results of the study show that some well-known results on ordered semihypergroups also hold in the case of L -ordered L -semihypergroups.

Keywords: L -ordered L -semihypergroup, L -hyperideal, prime L -hyperideal, semiprime L -hyperideal, weakly prime L -hyperideal

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1 Introduction

The theory of ordered semihypergroups generated by the fusion of ordered structures and algebraic hyperstructures is a newly developed research field of ordered algebra theory. In fact, it is a generalization of the theory of ordered semigroups. More precisely, an ordered semihypergroup [1] is a semihypergroup $(S, \circ)$ together with an order $\leq$ that is compatible with the hyperoperation, meaning that for all $x, y, z \in S$,

$$x \leq y \Rightarrow x \circ z \leq y \circ z \text{ and } z \circ x \leq z \circ y,$$

where $x \circ z \leq y \circ z$ means for all $u \in x \circ z$ there is $v \in y \circ z$ such that $u \leq v$ and $z \circ x \leq z \circ y$ is defined similarly. Later on, a lot of researchers focused on this topic (see [2–12]), for instance, Changphas and Davvaz [2] investigated the properties of hyperideals in an ordered semihypergroup; Davvaz et al. [3] studied the relationship between ordered semihypergroups and ordered semigroups by pseudoorders; Kehayopulu [4] introduced prime, weakly prime, and semiprime hyperideals of ordered semihypergroups and described ordered semihypergroups with these hyperideals. This theory can be penetrated into hyperalgebraic theories such as ordered hypergroups, ordered hyperrings, and ordered semihyperrings. It also plays a positive role in promoting the development of other disciplines.

Once the concept of fuzzy sets was put forward, it was quickly penetrated into other branches of mathematics and was widely used in many fields [13–15]. In computer science, an order usually expresses qualitative information between elements, so it cannot provide quantitative information needed for actual calculation. The introduction of fuzzy orders [16] makes up for this deficiency. It has been widely used in decision making, intelligent control, and so on. In particular, the combination of fuzzy orders and

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algebraic systems has appeared in recent years. For example, Šešlja et al. [17] induced fuzzy orders on fuzzy subsets through the classical order on a sequence group and proposed a concept of fuzzy ordered groups. The “fuzzy” is currently employed in the theory, which builds over the unit interval $[0, 1]$. If it builds over a complete lattice L , then the fuzzy order is usually called an L -order. In the last decade, many authors investigated L -order and applied it to various branches of mathematics and computer sciences (see [18–25]), such as its application in the semigroup theory. In 2012, Hao [21] and Wang [22] applied the theory of L -orders to ordered semigroups to construct the L -ordered semigroups, respectively. In 2015, Borzooei et al. [23] defined L -ordered groups and L -lattice ordered groups by directly combining the concept of L -orders with groups. All the aforementioned investigations focus mainly on L -order structures, so Huang et al. [24] studied the L -ordered semigroups from the perspective of more emphasis on algebras in 2018.

On the other hand, because the combination of fuzzy mathematics and hyperalgebra fully reflects the uncertainty of objective things themselves, fuzzy hyperalgebras have been concerned by many scholars since it was created. At present, it is one of the methods to study fuzzy hyperalgebras by defining and studying fuzzy hyperoperations. Similar to crisp hyperoperations, fuzzy hyperoperations map a pair of elements on a nonempty set S to a fuzzy subset of S . This method was first proposed by Corsini and Tofan [25]. Based on this method, Sen et al. [26] introduced the fuzzy semihypergroups; Leoreanu-Fotea and Davvaz [27] studied fuzzy hyperrings; Yin et al. [28] studied L -fuzzy hypergroups and L -fuzzy supermodules, and so on.

Motivated by the works of the L -ordered semigroups and fuzzy semihypergroups, we attempt in the present study to study the combination of L -orders and L -semihypergroups in detail.

The contents are arranged as follows: in Section 2, some basic notions and conclusions that will be used throughout this study are listed. In Section 3, the concept of an L -ordered L -semihypergroup is proposed. In addition, the relationship between ordered semihypergroups and L -ordered L -semihypergroups is discussed. In Section 4, the notions of L -left (resp. L -right, L -) hyperideals of an L -ordered L -semihypergroup are introduced and investigated. In Section 5, prime, weakly prime, and semiprime L -hyperideals of L -ordered L -semihypergroups are introduced and studied. It is proved that: (1) the L -hyperideals of an L -ordered L -semihypergroup are weakly prime if and only if they are idempotent and form a weak chain; (2) the L -hyperideals of an L -ordered L -semihypergroup are prime if and only if they form a weak chain and the L -ordered L -semihypergroup is intra-regular. Finally, some conclusions are presented in Section 6.

2 Preliminaries

For the convenience of the reader, in this section, some basic concepts are reviewed. Because a complete residuated lattice is significant in fuzzy logic, it is used as the structure of truth values throughout this study. If no other conditions are imposed, in the sequel, L always denotes a complete residuated lattice.

Definition 2.1. [29,30] A complete residuated lattice is an algebraic structure $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ such that

- (1) $(L, \wedge, \vee, 0, 1)$ is a complete lattice with the least element 0 and the greatest element 1 ;
- (2) $(L, *, 1)$ is a commutative monoid, i.e., $*$ is commutative, associative, and $a * 1 = a$ holds for all $a \in L$;
- (3) $*$ and $\rightarrow$ form an adjoint pair, i.e., $\alpha * \beta \leq \gamma \Leftrightarrow \alpha \leq \beta \rightarrow \gamma$ for all $\alpha, \beta, \gamma \in L$.

Proposition 2.2. [29,30] Let L be a complete residuated lattice, $\alpha, \beta, \gamma \in L$, and $\{\alpha_k\}_{k \in K}, \{\beta_k\}_{k \in K} \subseteq L$, we have

- (1) $0 * \alpha = 0, 1 \rightarrow \alpha = \alpha, 0 \rightarrow \alpha = 1$;
- (2) $1 \leq \alpha \rightarrow \beta \Leftrightarrow \alpha \leq \beta$;
- (3) $\alpha \rightarrow \beta = \vee \{ \gamma : \alpha * \gamma \leq \beta \}$;
- (4) $(\alpha \rightarrow \beta) * (\beta \rightarrow \gamma) \leq (\alpha \rightarrow \gamma)$;
- (5) $\alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha * \beta) \rightarrow \gamma = \beta \rightarrow (\alpha \rightarrow \gamma)$;

- (6) $\alpha * \bigvee_{k \in K} \beta_k = \bigvee_{k \in K} \alpha * \beta_k$;
- (7) $\alpha * \bigwedge_{k \in K} \beta_k \leq \bigwedge_{k \in K} \alpha * \beta_k$;
- (8) $\left(\bigvee_{k \in K} \alpha_k \right) \rightarrow \beta = \bigwedge_{k \in K} (\alpha_k \rightarrow \beta)$;
- (9) $\alpha \rightarrow \left(\bigwedge_{k \in K} \beta_k \right) = \bigwedge_{k \in K} (\alpha \rightarrow \beta_k)$;
- (10) $(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma) \geq \beta \rightarrow \gamma$;
- (11) $(\alpha \rightarrow \beta) \rightarrow (\gamma \rightarrow \beta) \geq \gamma \rightarrow \alpha$;
- (12) $\alpha * (\alpha \rightarrow \beta) \leq \beta$;
- (13) $\alpha \leq (\alpha \rightarrow \beta) \rightarrow \beta$;
- (14) $\alpha \rightarrow (\beta * \gamma) \geq (\alpha \rightarrow \beta) * \gamma$;
- (15) $(\alpha * \beta) \rightarrow (\alpha * \gamma) \geq \beta \rightarrow \gamma$.

Obviously, every complete Heyting algebra, i.e., frame, is a complete residuated lattice. Moreover, in this case, $*$ = $\wedge$.

Let S be a nonempty set. An arbitrary mapping $\mu: S \rightarrow L$ is called an L -subset of S and the symbol L^S denotes the set of all L -subsets of S . Moreover, $\mu \in L^S$ with $\bigvee_{x \in S} \mu(x) = 1$ is called a nonempty L -subset of S and the symbol L^{S*} denotes the set of all nonempty L -subsets of S . For $\alpha \in L$ and $A \subseteq L$, the L -subset α_A is defined by $\alpha_A(x) = \alpha$ if $x \in A$ and $\alpha_A(x) = 0$ otherwise. In particular, when $\alpha = 1$, α_A is said to be the characteristic function of A , denoted by χ_A and when $A = \{x\}$, α_A is said to be an L -point with support x and value α and denoted by x_α . For $\mu, \nu \in L^S$, we write fuzzy subsets $\mu \cap \nu$, $\mu \cup \nu$, and $\mu \subseteq \nu$ by $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$, $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$, and $\mu(x) \leq \nu(x)$ for all $x \in S$.

Definition 2.3. [18,19] An L -poset is a pair (S, R) such that S is a nonempty set and $R: S \times S \rightarrow L$ is a mapping, called an L -order, that satisfies for any $x, y, z \in S$,

- (1) $R(x, x) = 1$ (reflexivity);
- (2) $R(x, y) * R(y, z) \leq R(x, z)$ (transitivity);
- (3) $R(x, y) = R(y, x) = 1$ implies $x = y$ (antisymmetry).

For a given L -poset (S, R) , define a binary mapping $\text{sub}(-, -): L^S \times L^S \rightarrow L$ as $\text{sub}(\mu, \nu) = \bigwedge_{x \in S} \mu(x) \rightarrow \nu(x)$ for any pair $(\mu, \nu) \in L^S \times L^S$. Then, (L^S, sub) is an L -poset.

Let (S, R) be an L -poset. $\mu \in L^S$ is called a lower L -subset of S if $\mu(x) * R(y, x) \leq \mu(y)$ for any $x, y \in S$. $[\mu] \in L^S$ is defined as follows:

$$\forall y \in S, [\mu](y) = \bigvee_{x \in S} \mu(x) * R(y, x).$$

It is simple to check that

- (1) $\mu \subseteq [\mu]$;
- (2) μ is a lower L -subset if and only if $[\mu] = \mu$;
- (3) $\mu \subseteq \nu$ implies $[\mu] \subseteq [\nu]$ for all $\mu, \nu \in L^S$;
- (4) $\text{sub}(\mu, \nu) \leq \text{sub}([\mu], [\nu])$ for all $\mu, \nu \in L^S$.

Definition 2.4. [26] An L -semihypergroup is a pair $(S, \circ)$ such that S is a nonempty set and $\circ: S \times S \rightarrow L^{S*}$ is a mapping, called an L -hyperoperation and written as $(x, y) \mapsto x \circ y$, that satisfies for any $x, y, z \in S$, $(x \circ y) \circ z = x \circ (y \circ z)$, where $(\mu \circ \nu)(x) = \bigvee_{y, z \in S} \mu(y) * \nu(z) * (y \circ z)(x)$ for all $\mu, \nu \in L^{S*}$ and all $x \in S$.

Remark 2.5. (1) When $L = [0, 1]$ and L^{S*} is replaced by L^S , the L -semihypergroup is the fuzzy semihypergroup defined in [26].

(2) For convenience, we usually use χ_x in this article to represent the characteristic function $\chi_{\{x\}}$ and let $x \circ \mu = \chi_x \circ \mu$, $\mu \circ x = \mu \circ \chi_x$, and $x \circ x = \chi_x \circ \chi_x$.

For an L -semihypergroup $(S, \circ)$, it is obvious that $\mu \subseteq \nu$ implies $\mu \circ \delta \subseteq \nu \circ \delta$ and $\delta \circ \mu \subseteq \delta \circ \nu$ for all $\mu, \nu, \delta \in L^{S^*}$. Moreover, $(L^{S^*}, \circ)$ is a semihypergroup. In fact, for all $\mu, \nu, \delta \in L^{S^*}$ and all $x \in S$,

$$\begin{aligned} ((\mu \circ \nu) \circ \delta)(x) &= \bigvee_{y, z \in S} (\mu \circ \nu)(y) * \delta(z) * (y \circ z)(x) \\ &= \bigvee_{y, z, s, t \in S} \mu(s) * \nu(t) * (s \circ t)(y) * \delta(z) * (y \circ z)(x) \\ &= \bigvee_{z, s, t \in S} \mu(s) * \nu(t) * \delta(z) * \left(\bigvee_{y \in S} (s \circ t)(y) * (y \circ z)(x) \right) \\ &= \bigvee_{z, s, t \in S} \mu(s) * \nu(t) * \delta(z) * ((s \circ t) \circ z)(x) \\ &= \bigvee_{z, s, t \in S} \mu(s) * \nu(t) * \delta(z) * (s \circ (t \circ z))(x) \\ &= (\mu \circ (\nu \circ \delta))(x). \end{aligned}$$

Definition 2.6. [1] Let S be a nonempty set and $P^*(S)$ the family of nonempty subsets of S . An ordered semihypergroup is a triple $(S, \bullet, \leq)$ consisting of S together with a relation $\leq$ and a hyperoperation $\bullet: S \times S \rightarrow P^*(S)$ on S such that

- (1) $(S, \leq)$ is a poset;
- (2) $(S, \bullet)$ is a semihypergroup;
- (3) $x \leq y \Rightarrow z \bullet x \leq z \bullet y$ and $x \bullet z \leq y \bullet z$ for all $x, y, z \in S$, where for all $A, B \in P^*(S)$, $A \leq B$ means for each $x \in A$, there is a $y \in B$ such that $x \leq y$.

3 L -ordered L -semihypergroups

In this section, we introduce the concept of the L -ordered L -semihypergroups and give some examples to explain it. Moreover, we also discuss the relationship between L -ordered L -semihypergroups and ordered semihypergroups.

Definition 3.1. An L -ordered L -semihypergroup is a triple $(S, \circ, R)$ consisting of a nonempty set S together with an L -relation R and an L -hyperoperation $\circ$ on S such that

- (1) (S, R) is an L -poset;
- (2) $(S, \circ)$ is an L -semihypergroup;
- (3) $R(x, y) \leq R(z \circ x, z \circ y)$ and $R(x, y) \leq R(x \circ z, y \circ z)$ for all $x, y, z \in S$, where $R(\mu, \nu) = \bigwedge_{x \in S} \mu(x) \rightarrow \bigvee_{y \in S} \nu(y) * R(x, y)$ for all $\mu, \nu \in L^{S^*}$.

An L -ordered L -semihypergroup $(S, \circ, R)$ is commutative if $x \circ y = y \circ x$ for all $x, y \in S$.

Obviously, $R(\mu, \nu) = \text{sub}(\mu, \nu)$ when $\nu = \{\nu\}$. And for every L -ordered L -semihypergroup $(S, \circ, R)$, $(L^{S^*}, \circ, \text{sub})$ is an L -ordered semigroup, which is introduced in [24]. Conversely, for every L -ordered semigroup $(S, \bullet, R)$, we can define an L -hyperoperation on S as follows: for all $x, y, z \in S$,

$$(x \circ y)(z) = \begin{cases} 1, & z = x \bullet y, \\ 0, & z \neq x \bullet y. \end{cases}$$

Then, for all $x, y, z \in S$,

$$R(x \circ z, y \circ z) = \bigwedge_{s \in S} (x \circ z)(s) \rightarrow \bigvee_{t \in S} (y \circ z)(t) * R(s, t) = 1 \rightarrow R(x \bullet z, y \bullet z) \geq R(x, y)$$

and $R(z \circ x, z \circ y) \geq R(x, y)$ similarly. Therefore, $(S, \circ, R)$ is an L -ordered L -semihypergroup.

Example 3.2. Let S be a nonempty set and (S, R) an L -poset. Define an L -hyperoperation $\circ$ on S by $x \circ y = \chi_{\{x, y\}}$ for all $x, y \in S$. Then, $(S, \circ)$ is a L -semihypergroup by [5,6]. Moreover, for all $x, y, z \in S$, we have

$$\begin{aligned} R(x \circ z, y \circ z) &= \bigwedge_{a \in S} (x \circ z)(a) \rightarrow \bigvee_{b \in S} (y \circ z)(b) * R(a, b) \\ &= \bigwedge_{a \in S} \chi_{\{x, z\}}(a) \rightarrow \bigvee_{b \in S} \chi_{\{y, z\}}(b) * R(a, b) \\ &= (R(x, y) \vee R(x, z)) * (R(z, y) \vee R(z, z)) \\ &\geq R(x, y) \end{aligned}$$

and $R(x, y) \leq R(z \circ x, z \circ y)$ similarly. Therefore, $(S, \circ, R)$ is an L -ordered L -semihypergroup.

Example 3.3. Let $S = \{x, y\}$ and $L = [0, 1]$ define an L -hyperoperation $\circ$ on set S by $(x \circ x)(x) = 0.1$, $(x \circ x)(y) = 1$, $(x \circ y)(x) = 0.4$, $(x \circ y)(y) = 1$, $(y \circ x)(x) = 0.4$, $(y \circ x)(y) = 1$, $(y \circ y)(x) = 0.4$, $(y \circ y)(y) = 1$, and an L -order on S by $R(x, y) = 1$, $R(y, x) = 0$, $R(x, x) = 1$, $R(y, y) = 1$. By simple calculation, we can obtain that $(S, \circ, R)$ is an L -ordered L -semihypergroup.

Example 3.4. Let (S, R) be an L -poset. Define an L -hyperoperation $\circ$ on set S by $(x \circ y)(t) = R(t, x) \vee R(t, y)$ for all $x, y, t \in S$. By careful calculation, we can obtain that $(S, \circ, R)$ is an L -ordered L -semihypergroup.

Starting now with an L -ordered L -semihypergroup $(S, \circ, R)$ and defining the following hyperoperation and the order on S : for all $x, y \in S$, $x \bullet y = \{z \in S \mid (x \circ y)(z) > 0\}$ and $x \leq_R y$ when $R(x, y) = 1$, then we can obtain an ordered semihypergroup, as follows.

Theorem 3.5. If $(S, \circ, R)$ is an L -ordered L -semihypergroup, then $(S, \bullet, \leq_R)$ is an ordered semihypergroup, which is called the associated ordered semihypergroup.

Proof. Clearly, identity (1) of Definition 3.1 holds.

(2) We can easily verify that

$$(x \bullet y) \bullet z = \bigcup_{a \in \{b \mid ((x \circ y) \circ z)(b) > 0\}} a = \bigcup_{a \in \{b \mid (x \circ (y \circ z))(b) > 0\}} a = x \bullet (y \bullet z)$$

for all $x, y, z \in S$. Hence, identity (2) of Definition 3.1 holds.

(3) For all $x, y, z \in S$, let $x \leq_R y$, then

$$1 = R(x, y) \leq R(x \circ z, y \circ z) = \bigwedge_{a \in S} (x \circ z)(a) \rightarrow \bigvee_{b \in S} (y \circ z)(b) * R(a, b).$$

This implies that for all $a \in S$, $(x \circ z)(a) \leq \bigvee_{b \in S} (y \circ z)(b) * R(a, b)$. Now, let $a \in x \bullet z$, then

$$\begin{aligned} 0 < (x \circ z)(a) &\leq \bigvee_{b \in S} (y \circ z)(b) * R(a, b) \Rightarrow \text{there is } b \in S \text{ such that } (y \circ z)(b) * R(a, b) > 0 \\ &\Rightarrow \text{there is } b \in S \text{ such that } (y \circ z)(b) > 0, R(a, b) > 0, \end{aligned}$$

which implies there is $b \in (y \bullet z)$ such that $a \leq_R b$. Hence, $x \bullet z \leq_R y \bullet z$. Similarly, we can check that $z \bullet x \leq_R z \bullet y$.

Combining the aforementioned arguments, we have that $(S, \bullet, \leq_R)$ is an ordered semihypergroup. $\square$

Starting now with an ordered semihypergroup $(S, \bullet, \leq)$ and defining the following L -hyperoperation and the L -order on S : for all $x, y \in S$, $x \circ y = \chi_{x \bullet y}$ and

$$R_{\leq}(x, y) = \begin{cases} 1, & x \leq y, \\ 0, & x \not\leq y, \end{cases}$$

then we can obtain an L -ordered L -semihypergroup, as follows.

Theorem 3.6. *If $(S, \bullet, \leq)$ is an ordered semihypergroup, then $(S, \circ, R_\leq)$ is an L -ordered L -semihypergroup, which is called the associated L -ordered L -semihypergroup.*

Proof. Clearly, identity (1) of Definition 3.1 holds.

(2) We can easily check that

$$((x \circ y) \circ z)(a) = (\chi_{x \bullet y} \circ z)(a) = \bigvee_{b \in (x \bullet y)} (b \circ z)(a) = \bigvee_{b \in (x \bullet y)} \chi_{b \bullet z}(a) = \chi_{(x \bullet y) \bullet z}(a) = \chi_{x \bullet (y \bullet z)}(a) = (x \circ (y \circ z))(a)$$

for all $x, y, z, a \in S$. Hence, identity (2) of Definition 3.1 holds.

(3) For all $x, y, z \in S$, let $R_\leq(x, y) = 1$, then for each $a \in x \bullet z$, there is $b \in y \bullet z$ such that $R_\leq(a, b) = 1$, and so

$$\begin{aligned} R_\leq(x \circ z, y \circ z) &= \bigwedge_{a \in S} (x \circ z)(a) \rightarrow \bigvee_{b \in S} (y \circ z)(b) * R_\leq(a, b) = \bigwedge_{a \in S} \chi_{x \bullet z}(a) \rightarrow \bigvee_{b \in S} \chi_{y \bullet z}(b) * R_\leq(a, b) \\ &= \bigwedge_{a \in x \bullet z} \bigvee_{b \in y \bullet z} R_\leq(a, b) = 1 \geq R_\leq(x, y). \end{aligned}$$

Similarly, we can check that $R_\leq(z \circ x, z \circ y) \geq R_\leq(x, y)$.

This completes the proof. $\square$

4 L -hyperideals of L -ordered L -semihypergroups

In this section, we introduce the concept of L -left (resp. L -right, L -) hyperideals of an L -ordered L -semihypergroup and develop some characterizations for them, some of which are useful in sequel.

Definition 4.1. Let $(S, \circ, R)$ be an L -ordered L -semihypergroup and $\mu \in L^{S^*}$. μ is called an L -left (resp. L -right) hyperideal if it satisfies $\mu = (\mu]$ and $\chi_S \circ \mu \subseteq \mu$ (resp. $\mu \circ \chi_S \subseteq \mu$).

Clearly, Definition 4.1 is a generalization for the concept of left (resp. right) hyperideals on crisp ordered semihypergroups. It should be noticed that, whenever a statement is made about L -left hyperideals, it is to be understood that the analogous statement is also made about L -right hyperideals. By an L -hyperideal, we mean the one which is both an L -left and L -right hyperideal.

By [26], we have that an L -lower subset μ is an L -left (resp. L -right) hyperideal if and only if $x \circ \mu \subseteq \mu$ (resp. $\mu \circ x \subseteq \mu$) for all $x \in S$. Moreover, it is easily checked that χ_S is an L -hyperideal.

By [26] and the properties of L -lower subsets, we have the following results.

Theorem 4.2. *Let $(S, \circ, R)$ be an L -ordered L -semihypergroup and $\mu \in L^{S^*}$. Then, $(\mu \cup \chi_S \circ \mu]$ is the smallest L -left hyperideal of $(S, \circ, R)$ containing μ , in this case, $(\mu \cup \chi_S \circ \mu]$ is called the L -left hyperideal generated by μ , denoted by $L(\mu)$.*

Proof. Let $L(\mu) = (\mu \cup \chi_S \circ \mu]$, then $L(\mu) = (L(\mu)]$ and $L(\mu)$ is a nonempty L -subset containing μ . Moreover,

$$\chi_S \circ L(\mu) = \chi_S \circ (\mu \cup \chi_S \circ \mu] \subseteq (\chi_S \circ \mu) \cup (\chi_S \circ \mu) = ((\chi_S \circ \mu) \cup (\chi_S \circ \chi_S \circ \mu)) = (\chi_S \circ \mu) \subseteq L(\mu).$$

Let ν be an L -left ideal such that $\mu \subseteq \nu$. Then,

$$L(\mu) = (\mu \cup \chi_S \circ \mu] \subseteq (\nu \cup \chi_S \circ \nu) \subseteq (\nu] = \nu.$$

Therefore, $L(\mu) = (\mu \cup \chi_S \circ \mu]$ is the smallest L -left hyperideal containing μ . $\square$

Similarly, for all $\mu \in L^{S^*}$, the L -right hyperideal generated by μ is denoted by $R(\mu)$ and $R(\mu) = (\mu \cup \mu \circ \chi_S]$.

Lemma 4.3. Let $(S, \circ, R)$ be an L -ordered L -semihypergroup. Then,

- (1) for all $\mu, \nu \in L^{S^*}$, $(\mu] \circ (\nu] \subseteq (\mu \circ \nu]$, $((\mu] \circ (\nu]) = (\mu \circ \nu]$;
- (2) for all L -hyperideals μ and ν , $(\mu \circ \nu]$ is also an L -hyperideal;
- (3) for all $\mu \in L^{S^*}$, $(\chi_S \circ \mu \circ \chi_S]$ is an L -hyperideal;
- (4) for all L -hyperideals δ and all $\mu, \nu \in L^{S^*}$, $R(\mu, \nu) * \text{sub}(\nu, \delta) \leq \text{sub}(\mu, \delta)$;
- (5) for all $\mu, \nu, \delta \in L^{S^*}$, $\text{sub}(\mu, \nu) \leq \text{sub}(\delta \circ \mu, \delta \circ \nu)$, $\text{sub}(\mu, \nu) \leq \text{sub}(\mu \circ \delta, \nu \circ \delta)$;
- (6) for all $\mu, \nu \in L^{S^*}$, $\text{sub}(\mu, \nu) \leq \text{sub}(\mu \circ \chi_S, \nu \circ \chi_S)$, $\text{sub}(\mu, \nu) \leq \text{sub}(\chi_S \circ \mu, \chi_S \circ \nu)$,
 $\text{sub}(\mu, \nu) \leq \text{sub}(\mu \cup \chi_S \circ \mu, \nu \cup \chi_S \circ \nu)$, $\text{sub}(\mu, \nu) \leq \text{sub}(\mu \cup \mu \circ \chi_S, \nu \cup \nu \circ \chi_S)$.

Proof.

(1) Let $x \in S$. Then, we have

$$\begin{aligned}
 (\mu \circ \nu](x) &= \bigvee_{y \in S} (\mu \circ \nu)(y) * R(x, y) = \bigvee_{y \in S} \left[\left(\bigvee_{a, b \in S} \mu(a) * \nu(b) * (a \circ b)(y) \right) * R(x, y) \right] \\
 &= \bigvee_{a, b \in S} \left[\mu(a) * \nu(b) * \left(\bigvee_{y \in S} (a \circ b)(y) * R(x, y) \right) \right], \\
 (\mu] \circ (\nu](x) &= \bigvee_{y, z \in S} (\mu](y) * (\nu](z) * (y \circ z)(x) \\
 &= \bigvee_{y, z \in S} \left[\left(\bigvee_{a \in S} \mu(a) * R(y, a) \right) * \left(\bigvee_{b \in S} \nu(b) * R(z, b) \right) * (y \circ z)(x) \right] \\
 &= \bigvee_{a, b \in S} \left[\mu(a) * \nu(b) * \left(\bigvee_{z, y \in S} R(y, a) * R(z, b) \right) * (y \circ z)(x) \right] \\
 &\leq \bigvee_{a, b \in S} \left[\mu(a) * \nu(b) * \left(\bigvee_{z, y \in S} R(y \circ b, a \circ b) * R(y \circ z, y \circ b) \right) * (y \circ z)(x) \right] \\
 &\leq \bigvee_{a, b \in S} \left[\mu(a) * \nu(b) * \left(\bigvee_{y \in S} R(y \circ b, a \circ b) * \left(\bigvee_{c \in S} (y \circ b)(c) * R(x, c) \right) \right) \right] \\
 &\leq \bigvee_{a, b \in S} \left[\mu(a) * \nu(b) * \left(\bigvee_{c, d \in S} (a \circ b)(d) * R(c, d) * R(x, c) \right) \right] \\
 &\leq \bigvee_{a, b \in S} \left[\mu(a) * \nu(b) * \left(\bigvee_{d \in S} (a \circ b)(d) * R(x, d) \right) \right],
 \end{aligned}$$

and

$$\begin{aligned}
 ((\mu] \circ (\nu)](x) &= \bigvee_{y, z, s \in S} (\mu](z) * (\nu](s) * (z \circ s)(y) * R(x, y) \\
 &= \bigvee_{y, z, s \in S} \left[\left(\bigvee_{t \in S} \mu(t) * R(z, t) \right) * \left(\bigvee_{u \in S} \nu(u) * R(s, u) \right) * (z \circ s)(y) * R(x, y) \right] \\
 &= \bigvee_{y, z, s \in S} \left[\left(\bigvee_{t \in S} \mu(t) * R(z, t) \right) * \left(\bigvee_{u \in S} \nu(u) * R(s, u) \right) * (z \circ s)(y) * R(x, y) \right] \\
 &\geq \bigvee_{t, u \in S} \left[\mu(t) * \nu(u) * \left(\bigvee_{y \in S} (t \circ u)(y) * R(x, y) * R(t, t) \right) * R(u, u) \right] \\
 &= \bigvee_{t, u \in S} \left[\mu(t) * \nu(u) * \left(\bigvee_{y \in S} (t \circ u)(y) * R(x, y) \right) \right],
 \end{aligned}$$

so $(\mu] \circ (v] \subseteq (\mu \circ v]$ and $((\mu] \circ (v]) \supseteq (\mu \circ v]$. Moreover, it is clear $((\mu] \circ (v]) \subseteq (\mu \circ v]$ by $(\mu] \circ (v] \subseteq (\mu \circ v]$. Therefore, $((\mu] \circ (v]) = (\mu \circ v]$.

- (2) Let μ and v be L -hyperideals. Then, we have $\chi_S \circ (\mu \circ v] = (\chi_S] \circ (\mu \circ v] \subseteq (\chi_S \circ \mu \circ v] \subseteq (\mu \circ v]$ and $(\mu \circ v] \circ \chi_S = (\mu \circ v] \circ (\chi_S] \subseteq (\mu \circ v \circ \chi_S] \subseteq (\mu \circ v]$. Therefore, $(\mu \circ v]$ is an L -hyperideal.
- (3) Let $\mu \in L^{S*}$, then $\chi_S \circ (\chi_S \circ \mu \circ \chi_S] = (\chi_S] \circ (\chi_S \circ \mu \circ \chi_S] \subseteq (\chi_S \circ \chi_S \circ \mu \circ \chi_S] \subseteq (\chi_S \circ \mu \circ \chi_S]$ and $(\chi_S \circ \mu \circ \chi_S] \circ \chi_S = (\chi_S \circ \mu \circ \chi_S] \circ (\chi_S] \subseteq (\chi_S \circ \mu \circ \chi_S \circ \chi_S] \subseteq (\chi_S \circ \mu \circ \chi_S]$.
- (4) Let δ be an L -hyperideal and $\mu, v \in L^{S*}$. Then, $R(\mu, v) * \text{sub}(v, \delta) \leq \text{sub}(\mu, (v]) * \text{sub}((v], (\delta]) \leq \text{sub}(\mu, (\delta]) = \text{sub}(\mu, \delta)$.
- (5) Let $\mu, v, \delta \in L^{S*}$. Then,

$$\begin{aligned} \text{sub}(\delta \circ \mu, \delta \circ v) &= \bigwedge_{x \in S} (\delta \circ \mu)(x) \rightarrow (\delta \circ v)(x) \\ &= \bigwedge_{x, z, s \in S} (\delta(z) * \mu(s) * (z \circ s)(x)) \rightarrow \left(\bigvee_{t, u \in S} \delta(t) * v(u) * (t \circ u)(x) \right) \\ &\geq \bigwedge_{x, z, s \in S} (\delta(z) * \mu(s) * (z \circ s)(x)) \rightarrow (\delta(z) * v(s) * (z \circ s)(x)) \\ &\geq \bigwedge_{s \in S} \mu(s) \rightarrow v(s) = \text{sub}(\mu, v). \end{aligned}$$

Similarly, we have $\text{sub}(\mu \circ \delta, v \circ \delta) \geq \text{sub}(\mu, v)$.

- (6) Let $\mu, v, \delta \in L^{S*}$. Then,

$$\begin{aligned} \text{sub}(\mu \circ \chi_S, v \circ \chi_S) &= \bigwedge_{x \in S} (\mu \circ \chi_S)(x) \rightarrow (v \circ \chi_S)(x) \\ &\geq \bigwedge_{x, y, z \in S} (\mu(y) * (y \circ z)(x)) \rightarrow (v(y) * (y \circ z)(x)) \\ &\geq \bigwedge_{y \in S} \mu(y) \rightarrow v(y) = \text{sub}(\mu, v). \end{aligned}$$

Similarly, we have $\text{sub}(\chi_S \circ \mu, \chi_S \circ v) \geq \text{sub}(\mu, v)$. Moreover,

$$\begin{aligned} \text{sub}(\mu \cup \chi_S \circ \mu, v \cup \chi_S \circ v) &= \bigwedge_{x \in S} (\mu \cup \chi_S \circ \mu)(x) \rightarrow (v \cup \chi_S \circ v)(x) \\ &\geq \bigwedge_{x \in S} (\mu(x) \vee (\chi_S \circ \mu)(x)) \rightarrow (v(x) \vee (\chi_S \circ v)(x)) \\ &\geq \bigwedge_{x \in S} ((\mu(x) \rightarrow v(x)) \vee (\mu(x) \rightarrow (\chi_S \circ v)(x))) \\ &\quad \wedge ((\chi_S \circ \mu)(x) \rightarrow v(x)) \vee ((\chi_S \circ \mu)(x) \rightarrow (\chi_S \circ v)(x))) \\ &\geq \bigwedge_{x \in S} (\mu(x) \rightarrow v(x)) \wedge ((\chi_S \circ \mu)(x) \rightarrow (\chi_S \circ v)(x)) \\ &\geq \bigwedge_{x \in S} \mu(x) \rightarrow v(x) = \text{sub}(\mu, v). \end{aligned}$$

Similarly, we have $\text{sub}(\mu, v) \leq \text{sub}(\mu \cup \mu \circ \chi_S, v \cup v \circ \chi_S)$. □

Proposition 4.4. Let $(S, \circ, R)$ be an L -ordered L -semihypergroup and μ a nonempty L -subset of S . Then, $I(\mu) = (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S]$ is the L -hyperideal generated by μ .

Proof. Let $I(\mu) = (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S]$, then $I(\mu)$ is a nonempty L -subset of S containing μ . Moreover,

$$\begin{aligned} I(\mu) \circ \chi_S &= (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S] \circ \chi_S \\ &\subseteq (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S] \circ (\chi_S] \\ &\subseteq (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S) \circ (\chi_S] \\ &\subseteq (\mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S \cup \mu \circ \chi_S \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S \circ \chi_S] \\ &\subseteq (\mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S] \subseteq I(\mu) \end{aligned}$$

and also $I(\mu) = I(\mu)$. Hence, $I(\mu)$ is an L -right hyperideal. Similarly, $I(\mu)$ is an L -left hyperideal. Let now ν be an L -hyperideal such that $\mu \subseteq \nu$. Then,

$$I(\mu) = (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S) \subseteq (\nu \cup \chi_S \circ \nu \cup \nu \circ \chi_S \cup \chi_S \circ \nu \circ \chi_S) \subseteq (\nu) = \nu.$$

Therefore, $I(\mu) = (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S)$ is the L -hyperideal generated by μ . $\square$

Proposition 4.5. Let $(S, \circ, R)$ be an L -ordered L -semihypergroup. If μ is an L -left hyperideal and ν is an L -right hyperideal of S , then $(\mu \circ \nu)$ is an L -hyperideal of S .

Proof. Since μ and ν are nonempty L -subsets of S , $\bigvee_{z \in S} (\mu \circ \nu)(z) = \bigvee_{z, x, y \in S} \mu(x) \star \nu(y) \star (x \circ y)(z) = \bigvee_{x, y \in S} \mu(x) \star \nu(y) \star (\bigvee_{z \in S} (x \circ y)(z)) = \bigvee_{x, y \in S} \mu(x) \star \nu(y) = 1$, which implies that $\mu \circ \nu$ is nonempty. Thus, $(\mu \circ \nu)$ is also nonempty. In addition,

$$(\mu \circ \nu) \circ \chi_S = (\mu \circ \nu) \circ (\chi_S) \subseteq (\mu \circ \nu \circ \chi_S) \subseteq (\mu \circ \nu).$$

Similarly, $\chi_S \circ (\mu \circ \nu) \subseteq (\mu \circ \nu)$. Therefore, $(\mu \circ \nu)$ is an L -hyperideal of S . $\square$

Corollary 4.6. If $(S, \circ, R)$ is an L -ordered L -semihypergroup and μ, ν are L -hyperideals of S , then $(\mu \circ \nu)$ is an L -hyperideal of S .

Proof. This is a direct consequence of Proposition 4.5. $\square$

Lemma 4.7. Let $(S, \circ, R)$ be an L -ordered fuzzy semihypergroup and $\{\mu_i | i \in I\}$ a family of L -hyperideals, then $\bigcup_{i \in I} \mu_i$ is an L -hyperideal and $\bigcap_{i \in I} \mu_i$ is also an L -hyperideal when $\bigcap_{i \in I} \mu_i \in L^{S^*}$.

Proof. First, we show that $\bigcup_{i \in I} \mu_i$ is an L -hyperideal. In fact, it is obvious that $\bigcup_{i \in I} \mu_i$ is nonempty by $\bigcup_{i \in I} \mu_i \geq \mu_i$. Since every μ_i is an L -hyperideal, for all $x, y \in X$,

$$\left(\bigcup_{i \in I} \mu_i \right)(x) \star R(y, x) = \bigvee_{i \in I} \mu_i(x) \star R(y, x) \leq \bigvee_{i \in I} \mu_i(y) = \left(\bigcup_{i \in I} \mu_i \right)(y),$$

$$\begin{aligned} \left(\left(\bigcup_{i \in I} \mu_i \right) \circ \chi_S \right)(x) &= \bigvee_{s, t \in S} \left(\bigcup_{i \in I} \mu_i \right)(s) \star (s \circ t)(x) = \bigvee_{s, t \in S} \bigvee_{i \in I} \mu_i(s) \star (s \circ t)(x) = \bigvee_{i \in I} \bigvee_{s, t \in S} \mu_i(s) \star (s \circ t)(x) \\ &= \bigvee_{i \in I} (\mu_i \circ \chi_S)(x) \leq \bigvee_{i \in I} \mu_i(x) = \left(\bigcup_{i \in I} \mu_i \right)(x), \end{aligned}$$

and $\left(\chi_S \circ \left(\bigcup_{i \in I} \mu_i \right) \right)(x) \leq \bigvee_{i \in I} \mu_i(x)$ similarly.

Second, we show that $\bigcap_{i \in I} \mu_i$ is an L -hyperideal. Again, because every μ_i is an L -hyperideal, for all $x, y \in X$,

$$\left(\bigcap_{i \in I} \mu_i \right)(x) \star R(y, x) \leq \bigwedge_{i \in I} \mu_i(x) \star R(y, x) \leq \bigwedge_{i \in I} \mu_i(y) = \left(\bigcap_{i \in I} \mu_i \right)(y),$$

$$\begin{aligned} \left(\left(\bigcap_{i \in I} \mu_i \right) \circ \chi_S \right)(x) &= \bigvee_{s, t \in S} \left(\bigcap_{i \in I} \mu_i \right)(s) \star (s \circ t)(x) \leq \bigvee_{s, t \in S} \bigwedge_{i \in I} \mu_i(s) \star (s \circ t)(x) \\ &\leq \bigwedge_{i \in I} \bigvee_{s, t \in S} \mu_i(s) \star (s \circ t)(x) = \bigwedge_{i \in I} (\mu_i \circ \chi_S)(x) \leq \bigwedge_{i \in I} \mu_i(x) = \left(\bigcap_{i \in I} \mu_i \right)(x), \end{aligned}$$

and $\left(\chi_S \circ \left(\bigcap_{i \in I} \mu_i \right) \right)(x) \leq \bigvee_{i \in I} \mu_i(x)$ similarly. $\square$

Remark 4.8. In particular, for any two L -hyperideals μ, ν , $\mu \cap \nu \in L^{S^*}$. Hence, $\mu \cap \nu$ is an L -hyperideal.

Proof. In fact, for any $x, y, z \in S$, $\mu(z) \geq (\mu \circ \chi_S)(z) \geq (\mu \circ \nu)(z)$ and $\nu(z) \geq (\mu \circ \nu)(z)$ similarly. Therefore,

$$\bigvee_{z \in S} (\mu \cap \nu)(z) \geq \bigvee_{z \in S} (\mu \circ \nu)(z) = 1. \quad \square$$

For convenience, in this article we use the symbol μ^n to indicate: $\mu \circ \mu \circ \dots \circ \mu$ (n times), where $\mu \in L^{S^*}$.

Definition 4.9. A nonempty L -subset μ of an L -ordered L -semihypergroup $(S, \circ, R)$ is called idempotent if $\mu = (\mu^2)$.

Proposition 4.10. Let $(S, \circ, R)$ be an L -ordered fuzzy semihypergroup. The following conditions are equivalent:

- (1) all L -hyperideals are idempotent;
- (2) for all L -hyperideals μ, ν , $\mu \cap \nu = (\mu \circ \nu)$;
- (3) for all $\mu, \nu \in L^{S^*}$, $I(\mu) \cap I(\nu) = (I(\mu) \circ I(\nu))$;
- (4) for all $\mu \in L^{S^*}$, $I(\mu) = ((I(\mu))^2)$;
- (5) for all $\mu \in L^{S^*}$, $\mu \subseteq (\chi_S \circ \mu \circ \chi_S \circ \mu \circ \chi_S)$.

Proof. (1) $\Rightarrow$ (2). Let μ, ν be L -hyperideals. By Lemma 4.3, $(\mu \circ \nu) \subseteq (\mu \circ \chi_S) \subseteq (\mu)$ and $(\mu \circ \nu) \subseteq (\chi_S \circ \nu) \subseteq (\nu)$, so $(\mu \circ \nu) \subseteq \mu \cap \nu$. On the other hand, by Remark 4.8, $\mu \cap \nu$ is an L -hyperideal. Therefore,

$$\mu \cap \nu = ((\mu \cap \nu)^2) = ((\mu \cap \nu) \circ (\mu \cap \nu)) \subseteq (\mu \circ \nu).$$

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Let $\mu \in L^{S^*}$. By assumptions and Lemma 4.3, we have

$$(I(\mu))^2 = ((I(\mu))^2) \circ I(\mu) = ((I(\mu))^2) \circ (I(\mu)) \subseteq ((I(\mu))^3),$$

and so

$$(I(\mu))^3 = (I(\mu))^2 \circ I(\mu) = (I(\mu))^2 \circ (I(\mu)) \subseteq ((I(\mu))^3) \circ (I(\mu)) \subseteq ((I(\mu))^4).$$

Furthermore, $((I(\mu))^4) \subseteq ((I(\mu))^5)$. Hence,

$$\begin{aligned} I(\mu) &= ((I(\mu))^2) \subseteq ((I(\mu))^3) \subseteq ((I(\mu))^4) \subseteq ((I(\mu))^5) \\ &= ((I(\mu))^5) \subseteq (\chi_S \circ I(\mu)) \subseteq (I(\mu)) = I(\mu), \end{aligned}$$

thus, $I(\mu) = ((I(\mu))^5)$. On the other hand,

$$\begin{aligned} (I(\mu))^3 &= (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S)^3 \\ &\subseteq ((\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S)^2) \circ (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S) \\ &\subseteq (\chi_S \circ \mu \cup \chi_S \circ \mu \circ \chi_S) \circ (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S) \\ &\subseteq ((\chi_S \circ \mu \cup \chi_S \circ \mu \circ \chi_S) \circ (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S)) \\ &\subseteq (\chi_S \circ \mu \circ \chi_S), \end{aligned}$$

thereby,

$$(I(\mu))^4 \subseteq (\chi_S \circ \mu \circ \chi_S) \circ (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S) \subseteq (\chi_S \circ \mu \circ \chi_S \circ \mu \cup \chi_S \circ \mu \circ \chi_S \circ \mu \circ \chi_S).$$

Thus, we have

$$(I(\mu))^5 \subseteq (\chi_S \circ \mu \circ \chi_S \circ \mu \cup \chi_S \circ \mu \circ \chi_S \circ \mu \circ \chi_S) \circ (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S) \subseteq (\chi_S \circ \mu \circ \chi_S \circ \mu \circ \chi_S).$$

Therefore,

$$\mu \subseteq I(\mu) = ((I(\mu))^5) \subseteq (\chi_S \circ \mu \circ \chi_S \circ \mu \circ \chi_S) = (\chi_S \circ \mu \circ \chi_S \circ \mu \circ \chi_S).$$

(5) $\Rightarrow$ (1). Let μ be an L -hyperideal. Then, $(\mu^2) = (\mu \circ \mu) \subseteq (\mu \circ \chi_S) \subseteq (\mu) = \mu$. Conversely,

$$\mu \subseteq (\chi_S \circ \mu \circ \chi_S \circ \mu \circ \chi_S) \subseteq (\mu \circ \chi_S \circ \mu) \subseteq (\mu \circ \mu) = (\mu^2). \quad \square$$

5 Several kinds of special L -hyperideals of L -ordered L -semihypergroups

In this section, we introduce the concepts of prime, weakly prime, and semiprime L -hyperideals of L -ordered L -semihypergroups and give their characterization. In particular, the relationships among these types of L -hyperideals are also discussed.

Definition 5.1. Let $(S, \circ, R)$ be an L -ordered L -semihypergroup and $\delta \in L^{S^*}$.

- (1) δ is prime if $\text{sub}(\mu \circ \nu, \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$ for all $\mu, \nu \in L^S$;
- (2) δ is weakly prime if $\text{sub}(\mu \circ \nu, \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$ for all L -hyperideals μ, ν ;
- (3) δ is semiprime if $\text{sub}(\mu \circ \mu, \delta) \leq \text{sub}(\mu, \delta)$ for all $\mu \in L^S$.

Remark 5.2. Let L be a frame, then

- (1) δ is prime if and only if $\text{sub}(x_\alpha \circ y_\beta, \delta) \leq (\alpha \rightarrow \delta(x)) \vee (\beta \rightarrow \delta(y))$ for all $x, y \in S$ and all $\alpha, \beta \in L$;
- (2) δ is semiprime if and only if $\text{sub}(x_\alpha \circ x_\alpha, \delta) \leq \alpha \rightarrow \delta(x)$ if and only if $\text{sub}(x \circ x, \delta) \leq \delta(x)$ for all $x \in S$ and all $\alpha \in L$.

Proof.

- (1) Let δ be prime, $x, y \in S$, and $\alpha, \beta \in L$. Then,

$$\text{sub}(x_\alpha \circ y_\beta, \delta) \leq \text{sub}(x_\alpha, \delta) \vee \text{sub}(y_\beta, \delta) = (\alpha \rightarrow \delta(x)) \vee (\beta \rightarrow \delta(y)).$$

Conversely, for all $\mu, \nu \in L^S$,

$$\begin{aligned} \text{sub}(\mu \circ \nu, \delta) &= \left(\bigwedge_{z \in S} (\mu \circ \nu)(z) \rightarrow \delta(z) \right) \\ &= \left(\bigwedge_{z \in S} \left(\bigvee_{x, y \in S} \mu(x) * \nu(y) * (x \circ y)(z) \right) \rightarrow \delta(z) \right) \\ &= \bigwedge_{z \in S} \bigwedge_{x, y \in S} (x_{\mu(x)} \circ y_{\nu(y)})(z) \rightarrow \delta(z) \\ &= \bigwedge_{x, y \in S} \text{sub}(x_{\mu(x)} \circ y_{\nu(y)}, \delta) \\ &\leq \bigwedge_{x, y \in S} (\mu(x) \rightarrow \delta(x)) \vee (\nu(y) \rightarrow \delta(y)) \\ &= \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta). \end{aligned}$$

- (2) We only show that δ is semiprime if and only if $\text{sub}(x \circ x, \delta) \leq \delta(x)$, for all $x \in S$. Let δ be semiprime and $x \in S$. Then, $\text{sub}(x \circ x, \delta) = \text{sub}(\chi_x \circ \chi_x, \delta) \leq \text{sub}(\chi_x, \delta) = \delta(x)$. As for the converse, for all $\mu \in L^S$, we get

$$\begin{aligned} \text{sub}(\mu \circ \mu, \delta) &= \bigwedge_{x, y, z \in S} \mu(y) * \mu(z) * (y \circ z)(x) \rightarrow \delta(x) \\ &= \bigwedge_{y, z \in S} \mu(y) * \mu(z) \rightarrow \left(\bigwedge_{x \in S} (y \circ z)(x) \rightarrow \delta(x) \right) \\ &= \bigwedge_{y, z \in S} \mu(y) * \mu(z) \rightarrow \text{sub}(y \circ z, \delta) \\ &\leq \bigwedge_{y \in S} \mu(y) \rightarrow \text{sub}(y \circ y, \delta) \\ &\leq \bigwedge_{y \in S} \mu(y) \rightarrow \delta(y) = \text{sub}(\mu, \delta). \end{aligned}$$

□

By Definition 5.1, every prime L -subset is weakly prime and semiprime. But the contrary may not be true. We can illustrate it by example.

Example 5.3. Let $S = \{s_1, s_2, s_3, s_4, s_5\}$, $L = \{0, 1\}$. We define $R: S \times S \rightarrow L$ as follows

$$(R_{ij}) = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $R_{ij} = R(s_i, s_j)$ for $i, j = 1, 2, 3, 4, 5$. By definition, one can easily see (S, R) be an L -poset. Let $\mu: S \rightarrow L$ be any L -subsets of S . There are $2^5 = 32$ choices for μ . We denote μ by a vector of dimension 5 as follows, where the i th coordinate is $\mu(s_i)$ for $i = 1, 2, 3, 4, 5$.

$$\begin{aligned} \mu_1 &= (0, 0, 0, 0, 0), \quad \mu_2 = (1, 0, 0, 0, 0), \quad \mu_3 = (0, 1, 0, 0, 0), \quad \mu_4 = (0, 0, 1, 0, 0), \\ \mu_5 &= (0, 0, 0, 1, 0), \quad \mu_6 = (0, 0, 0, 0, 1), \quad \mu_7 = (1, 1, 0, 0, 0), \quad \mu_8 = (1, 0, 1, 0, 0), \\ \mu_9 &= (1, 0, 0, 1, 0), \quad \mu_{10} = (1, 0, 0, 0, 1), \quad \mu_{11} = (0, 1, 1, 0, 0), \quad \mu_{12} = (0, 1, 0, 1, 0), \\ \mu_{13} &= (0, 1, 0, 0, 1), \quad \mu_{14} = (0, 0, 1, 1, 0), \quad \mu_{15} = (0, 0, 1, 0, 1), \quad \mu_{16} = (0, 0, 0, 1, 1), \\ \mu_{17} &= (1, 1, 1, 0, 0), \quad \mu_{18} = (1, 1, 0, 1, 0), \quad \mu_{19} = (1, 1, 0, 0, 1), \quad \mu_{20} = (1, 0, 1, 1, 0), \\ \mu_{21} &= (1, 0, 1, 0, 1), \quad \mu_{22} = (1, 0, 0, 1, 1), \quad \mu_{23} = (0, 1, 1, 1, 0), \quad \mu_{24} = (0, 1, 1, 0, 1), \\ \mu_{25} &= (0, 1, 0, 1, 1), \quad \mu_{26} = (0, 0, 1, 1, 1), \quad \mu_{27} = (1, 1, 1, 1, 0), \quad \mu_{28} = (1, 1, 1, 0, 1), \\ \mu_{29} &= (1, 1, 0, 1, 1), \quad \mu_{30} = (1, 0, 1, 1, 1), \quad \mu_{31} = (0, 1, 1, 1, 1), \quad \mu_{32} = (1, 1, 1, 1, 1). \end{aligned}$$

Moreover, we define $\circ: S \times S \rightarrow L^{S^*}$ as follows: $s_1 \circ s_i = s_2 \circ s_i = (1, 1, 0, 0, 0)$, $i = 1, 2, 3, 4, 5$, $s_3 \circ s_1 = s_3 \circ s_2 = s_4 \circ s_1 = s_4 \circ s_2 = s_5 \circ s_1 = s_5 \circ s_2 = (1, 1, 0, 0, 0)$, $s_3 \circ s_3 = s_3 \circ s_4 = s_4 \circ s_3 = s_5 \circ s_3 = s_5 \circ s_4 = (0, 0, 1, 0, 0)$, $s_3 \circ s_5 = s_4 \circ s_5 = s_5 \circ s_5 = (0, 0, 0, 0, 1)$, $s_4 \circ s_4 = (0, 0, 0, 1, 0)$. By careful calculation, we have that $(S, \circ, R)$ is an L -ordered L -semihypergroup. It is easy to verify that $\mu_7, \mu_{28}, \mu_{32} = \chi_S$ are all L -hyperideals of $(S, \circ, R)$. And we can show that μ_{17} is weakly prime and semiprime, but μ_{17} is not prime. In fact,

$$\begin{aligned} \mu_{17}(s_i) &= 1 \geq \text{sub}(s_i \circ s_i, \mu_{17}), \quad i = 1, 2, 3, 4, \\ \text{sub}(s_5 \circ s_5, \mu_{17}) &= (s_5 \circ s_5)(s_5) \rightarrow \mu_{17}(s_5) = 1 \rightarrow 0 = \mu_{17}(s_5), \end{aligned}$$

which implies that μ_{17} is semiprime. Moreover,

$$\begin{aligned} \mu_7 \circ \mu_{28} &= \mu_{28} \circ \mu_7 = (1, 1, 0, 0, 0) \subseteq \mu_{17}, \\ \mu_7 \circ \mu_{32} &= \mu_{32} \circ \mu_7 = (1, 1, 0, 0, 0) \subseteq \mu_{17}, \\ \mu_{32} \circ \mu_{28} &= \mu_{28} \circ \mu_{32} = (1, 1, 1, 0, 1) \not\subseteq \mu_{17}, \end{aligned}$$

and so

$$\begin{aligned} \text{sub}(\mu_{28} \circ \mu_7, \mu_{17}) &= \text{sub}(\mu_7 \circ \mu_{28}, \mu_{17}) = 1 = \text{sub}(\mu_7, \mu_{17}) \vee \text{sub}(\mu_{28}, \mu_{17}), \\ \text{sub}(\mu_{32} \circ \mu_7, \mu_{17}) &= \text{sub}(\mu_7 \circ \mu_{32}, \mu_{17}) = 1 = \text{sub}(\mu_7, \mu_{17}) \vee \text{sub}(\mu_{32}, \mu_{17}), \\ \text{sub}(\mu_{28} \circ \mu_{32}, \mu_{17}) &= \text{sub}(\mu_{32} \circ \mu_{28}, \mu_{17}) = 0 = \text{sub}(\mu_{32}, \mu_{17}) \vee \text{sub}(\mu_{28}, \mu_{17}). \end{aligned}$$

Therefore, μ_{17} is weakly prime. However, μ_{17} is not prime by

$$\text{sub}(s_5 \circ s_4, \mu_{17}) = 1 > \mu_{17}(s_5) \vee \mu_{17}(s_4) = 0.$$

Theorem 5.4. Let δ be an L -hyperideal of the L -ordered L -semihypergroup $(S, \circ, R)$. The following conditions are equivalent:

- (1) δ is prime;
- (2) for all L -right hyperideals μ and all L -left hyperideals ν , $\text{sub}(\mu \circ \nu, \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$;
- (3) for all $\mu, \nu \in L^{S^*}$, $\text{sub}(L(\mu) \circ R(\nu), \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1). Since

$$L(\mu) \circ R(\nu) = ((\mu \cup \chi_S \circ \mu] \circ (\nu \cup \nu \circ \chi_S]) \subseteq (\mu \circ \nu \cup \chi_S \circ \mu \circ \nu \cup \mu \circ \nu \circ \chi_S \cup \chi_S \circ \mu \circ \nu \circ \chi_S],$$

we have

$$\begin{aligned}
 \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta) &\geq \text{sub}(L(\mu) \circ R(\nu), \delta) \\
 &= \text{sub}((\mu \cup \chi_S \circ \mu] \circ (\nu \cup \nu \circ \chi_S], \delta) \\
 &\geq \text{sub}((\mu \circ \nu \cup \chi_S \circ \mu \circ \nu \cup \mu \circ \nu \circ \chi_S \cup \chi_S \circ \mu \circ \nu \circ \chi_S], \\
 &\quad (\delta \cup \chi_S \circ \delta \cup \delta \circ \chi_S \cup \chi_S \circ \delta \circ \chi_S]) \\
 &\geq \text{sub}(\mu \circ \nu, \delta) \wedge \text{sub}(\chi_S \circ \mu \circ \nu, \chi_S \circ \delta) \wedge \text{sub}(\mu \circ \nu \circ \chi_S, \delta \circ \chi_S) \\
 &\quad \wedge \text{sub}(\chi_S \circ \mu \circ \nu \circ \chi_S, \chi_S \circ \delta \circ \chi_S) \geq \text{sub}(\mu \circ \nu, \delta).
 \end{aligned}$$

Hence, δ is prime. $\square$

Theorem 5.5. Let δ be an L -hyperideal of the L -ordered L -semihypergroup $(S, \circ, R)$. The following conditions are equivalent:

- (1) δ is weakly prime;
- (2) for all $\mu, \nu \in L^{S^*}$, $\text{sub}((\mu \circ \chi_S \circ \nu], \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$;
- (3) for all $\mu, \nu \in L^{S^*}$, $\text{sub}(I(\mu) \circ I(\nu), \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$;
- (4) for all L -right hyperideals μ, ν , $\text{sub}(\mu \circ \nu, \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$;
- (5) for all L -left hyperideals μ, ν , $\text{sub}(\mu \circ \nu, \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$;
- (6) for all L -right hyperideals μ and all L -left hyperideals ν , $\text{sub}(\mu \circ \nu, \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$.

Proof. (1) $\Rightarrow$ (2). By Lemma 4.3, for all $\mu, \nu \in L^{S^*}$, $(\chi_S \circ \mu \circ \chi_S] \circ (\chi_S \circ \nu \circ \chi_S] \subseteq (\chi_S \circ \mu \circ \chi_S \circ \chi_S \circ \nu \circ \chi_S] \subseteq (\chi_S \circ (\mu \circ \chi_S \circ \nu) \circ \chi_S]$. Moreover, $((I(\mu))^2] \circ I(\mu) \subseteq ((I(\mu))^3] \subseteq ((\chi_S \circ \mu \circ \chi_S]) = (\chi_S \circ \mu \circ \chi_S]$ and $((I(\nu))^2] \circ I(\nu) \subseteq ((I(\nu))^3] \subseteq ((\chi_S \circ \nu \circ \chi_S]) = (\chi_S \circ \nu \circ \chi_S]$ by the proof of Proposition 4.10. Hence,

$$\begin{aligned}
 \text{sub}((\mu \circ \chi_S \circ \nu], \delta) &\leq \text{sub}(\chi_S \circ (\mu \circ \chi_S \circ \nu] \circ \chi_S, \chi_S \circ \delta \circ \chi_S) \\
 &= \text{sub}(\chi_S \circ (\mu \circ \chi_S \circ \nu] \circ \chi_S, (\chi_S \circ \delta \circ \chi_S]) \\
 &= \text{sub}(\chi_S \circ (\mu \circ \chi_S \circ \nu] \circ \chi_S, (\chi_S \circ \delta \circ \chi_S]) \\
 &\leq \text{sub}((\chi_S \circ \mu \circ \chi_S] \circ (\chi_S \circ \nu \circ \chi_S], (\chi_S \circ \delta \circ \chi_S]) * \text{sub}((\chi_S \circ \delta \circ \chi_S], \delta) \\
 &\leq \text{sub}((\chi_S \circ \mu \circ \chi_S] \circ (\chi_S \circ \nu \circ \chi_S], \delta) \\
 &\leq \text{sub}((\chi_S \circ \mu \circ \chi_S], \delta) \vee \text{sub}((\chi_S \circ \nu \circ \chi_S], \delta) \\
 &= [\text{sub}((\chi_S \circ \mu \circ \chi_S], \delta) * \text{sub}(((I(\mu))^2] \circ I(\mu), (\chi_S \circ \mu \circ \chi_S])] \\
 &\quad \vee [\text{sub}((\chi_S \circ \nu \circ \chi_S], \delta) * \text{sub}(((I(\nu))^2] \circ I(\nu), (\chi_S \circ \nu \circ \chi_S))] \\
 &\leq \text{sub}(((I(\mu))^2] \circ I(\mu), \delta) \vee \text{sub}(((I(\nu))^2] \circ I(\nu), \delta) \\
 &\leq \text{sub}(I(\mu), \delta) \vee \text{sub}(I(\nu), \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta).
 \end{aligned}$$

(2) $\Rightarrow$ (3). For all $\mu, \nu \in L^{S^*}$, it is easy to check that

$$(\mu \circ \chi_S \circ \nu] \subseteq ((\mu] \circ (\chi_S \circ \nu]) \subseteq (I(\mu) \circ I(\nu)).$$

Thus,

$$\begin{aligned}
 \text{sub}(I(\mu) \circ I(\nu), \delta) &= \text{sub}(I(\mu) \circ I(\nu), \delta) * \text{sub}((\mu \circ \chi_S \circ \nu], I(\mu) \circ I(\nu)) \\
 &\leq \text{sub}((I(\mu) \circ I(\nu)], (\delta]) * \text{sub}((\mu \circ \chi_S \circ \nu], (I(\mu) \circ I(\nu)]) \\
 &\leq \text{sub}((\mu \circ \chi_S \circ \nu], (\delta]) \\
 &= \text{sub}((\mu \circ \chi_S \circ \nu], \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta).
 \end{aligned}$$

(3) $\Rightarrow$ (4). For all L -right hyperideals μ, ν , we have

$$I(\mu) = (\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S] \subseteq (\mu \cup \chi_S \circ \mu],$$

and $I(\nu) \subseteq (\nu \cup \chi_S \circ \nu]$ similarly. Then,

$$\begin{aligned}
I(\mu) \circ I(\nu) &\subseteq (\mu \cup \chi_S \circ \mu] \circ (\nu \cup \chi_S \circ \nu] \\
&\subseteq ((\mu \cup \chi_S \circ \mu) \circ (\nu \cup \chi_S \circ \nu)] \\
&\subseteq (\mu \circ \nu \cup \mu \circ \chi_S \circ \nu \cup \chi_S \circ \mu \circ \nu \cup \chi_S \circ \mu \circ \chi_S \circ \nu] \\
&\subseteq (\mu \circ \nu \cup \chi_S \circ \mu \circ \nu],
\end{aligned}$$

and so

$$\begin{aligned}
\text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta) &\geq \text{sub}(I(\mu) \circ I(\nu), \delta) \\
&\geq \text{sub}((\mu \circ \nu \cup \chi_S \circ \mu \circ \nu], \delta) \\
&\geq \text{sub}((\mu \circ \nu \cup \chi_S \circ \mu \circ \nu], \delta \cup \chi_S \circ \delta) * \text{sub}(\delta \cup \chi_S \circ \delta, \delta) \\
&\geq \text{sub}(\mu \circ \nu, \delta) * \text{sub}(\chi_S \circ \mu \circ \nu, \chi_S \circ \delta) \\
&= \text{sub}(\mu \circ \nu, \delta).
\end{aligned}$$

(3) $\Rightarrow$ (5). Similar to (3) $\Rightarrow$ (4).

(3) $\Rightarrow$ (6). Let μ be L -right hyperideals, ν be L -left hyperideals, and $x, y \in S$. Since

$$(\mu \cup \chi_S \circ \mu](x) = \bigvee_{y \in S} (\mu \cup \chi_S \circ \mu)(y) * R(x, y) \geq \bigvee_{y \in S} \mu(y) * R(x, y) = \mu(x),$$

and

$$(\nu \cup \nu \circ \chi_S](y) = \bigvee_{x \in S} (\nu \cup \nu \circ \chi_S)(x) * R(y, x) \geq \bigvee_{x \in S} \nu(x) * R(y, x) = \nu(y),$$

$I(\mu) \subseteq (\mu \cup \chi_S \circ \mu]$ and $I(\nu) \subseteq (\nu \cup \nu \circ \chi_S]$. Thus,

$$I(\mu) \circ I(\nu) \subseteq (\mu \cup \chi_S \circ \mu] \circ (\nu \cup \nu \circ \chi_S] \subseteq (\mu \circ \nu \cup \chi_S \circ \mu \circ \nu \cup \mu \circ \nu \circ \chi_S \cup \chi_S \circ \mu \circ \nu \circ \chi_S].$$

Therefore,

$$\begin{aligned}
\text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta) &\geq \text{sub}(I(\mu) \circ I(\nu), \delta) \\
&\geq \text{sub}((\mu \circ \nu \cup \chi_S \circ \mu \circ \nu \cup \mu \circ \nu \circ \chi_S \cup \chi_S \circ \mu \circ \nu \circ \chi_S], (\delta]) \\
&\geq \text{sub}((\mu \circ \nu \cup \chi_S \circ \mu \circ \nu \cup \mu \circ \nu \circ \chi_S \cup \chi_S \circ \mu \circ \nu \circ \chi_S], \\
&\quad (\delta \cup \chi_S \circ \delta \cup \delta \circ \chi_S \cup \chi_S \circ \delta \circ \chi_S]) * \text{sub}((\delta \cup \chi_S \circ \delta \cup \delta \circ \chi_S \cup \chi_S \circ \delta \circ \chi_S], (\delta]) \\
&= \text{sub}((\mu \circ \nu \cup \chi_S \circ \mu \circ \nu \cup \mu \circ \nu \circ \chi_S \cup \chi_S \circ \mu \circ \nu \circ \chi_S], \\
&\quad (\delta \cup \chi_S \circ \delta \cup \delta \circ \chi_S \cup \chi_S \circ \delta \circ \chi_S]) \\
&\geq \text{sub}(\mu \circ \nu \cup \chi_S \circ \mu \circ \nu \cup \mu \circ \nu \circ \chi_S \cup \chi_S \circ \mu \circ \nu \circ \chi_S, \delta \cup \chi_S \circ \delta \cup \delta \circ \chi_S \cup \chi_S \circ \delta \circ \chi_S) \\
&\geq \text{sub}(\mu \circ \nu, \delta).
\end{aligned}$$

(4) $\Rightarrow$ (1), (5) $\Rightarrow$ (1) and (6) $\Rightarrow$ (1) are obvious. □

Theorem 5.6. Let δ be an L -hyperideal of the L -ordered L -semihypergroup $(S, \circ, R)$. Then, δ is weakly prime if and only if for all L -hyperideals μ, ν ,

$$\text{sub}((\mu \circ \nu] \cap (\nu \circ \mu], \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta).$$

Proof. Let δ be weakly prime. By Lemma 4.3, we have that $(\mu \circ \nu]$, $(\nu \circ \mu]$ are L -hyperideals. And so,

$$\begin{aligned}
\text{sub}((\mu \circ \nu] \cap (\nu \circ \mu], \delta) &= \text{sub}((\mu \circ \nu] \cap (\nu \circ \mu], \delta) * \text{sub}((\mu \circ \nu] \circ (\nu \circ \mu], (\mu \circ \nu] \cap (\nu \circ \mu]) \\
&\leq \text{sub}((\mu \circ \nu] \circ (\nu \circ \mu], \delta) \\
&\leq \text{sub}((\mu \circ \nu], \delta) \vee \text{sub}((\nu \circ \mu], \delta) \\
&\leq \text{sub}(\mu \circ \nu, \delta) \vee \text{sub}(\nu \circ \mu, \delta) \\
&\leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta).
\end{aligned}$$

Conversely, suppose that for all L -hyperideals μ, ν ,

$$\text{sub}((\mu \circ \nu] \cap (\nu \circ \mu], \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta).$$

Then,

$$\begin{aligned} \text{sub}(\mu \circ \nu, \delta) &\leq \text{sub}((\mu \circ \nu], (\delta]) = \text{sub}((\mu \circ \nu], \delta) \\ &= \text{sub}((\mu \circ \nu], \delta) * \text{sub}((\mu \circ \nu] \cap (\nu \circ \mu], (\mu \circ \nu]) \\ &\leq \text{sub}(\mu \circ \nu] \cap (\nu \circ \mu], \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta). \end{aligned} \quad \square$$

Let S be a nonempty set and $\mathcal{F} \subseteq L^S$. $\mathcal{F}$ is called a weak chain, if $\text{sub}(\nu, \mu) \vee \text{sub}(\mu, \nu) = 1$ for all $\mu, \nu \in \mathcal{F}$.

Theorem 5.7. *Let $(S, \circ, R)$ be an L -ordered L -semihypergroup. Then, all L -hyperideals are weakly prime if and only if they are idempotent and they form a weak chain.*

Proof. Let all L -hyperideals of $(S, \circ, R)$ be weakly prime and μ, ν be L -hyperideals. By the assumption and Lemma 4.3, we have $(\mu \circ \nu]$ is a weakly prime L -hyperideal and $\mu \circ \nu \subseteq (\mu \circ \nu]$. Then,

$$1 = \text{sub}((\mu \circ \nu] \cap (\nu \circ \mu], (\mu \circ \nu]) \leq \text{sub}(\mu, (\mu \circ \nu]) \vee \text{sub}(\nu, (\mu \circ \nu]) \leq \text{sub}(\mu, \nu) \vee \text{sub}(\nu, \mu),$$

which implies that $\text{sub}(\mu, \nu) \vee \text{sub}(\nu, \mu) = 1$. Moreover, by the assumption again we have $(\mu^2] = (\mu \circ \mu] \subseteq (\mu \circ \chi_S] \subseteq (\mu] = \mu$, and $1 = \text{sub}((\mu^2] \cap (\mu^2], (\mu^2]) \leq \text{sub}(\mu, (\mu^2]) \leq 1$, which means that $\mu = (\mu^2]$.

Conversely, let all L -hyperideals of $(S, \circ, R)$ be idempotent and they form a weak chain. Then, for L -hyperideals μ, ν, δ , we have

$$\begin{aligned} \text{sub}(\mu \circ \nu, \delta) &\rightarrow (\text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)) \\ &\geq (\text{sub}(\mu \circ \nu, \delta) \rightarrow \text{sub}(\mu, \delta)) \vee (\text{sub}(\mu \circ \nu, \delta) \rightarrow \text{sub}(\nu, \delta)) \\ &\geq \text{sub}(\mu, \mu \circ \nu) \vee \text{sub}(\nu, \mu \circ \nu) = \text{sub}((\mu], (\mu \circ \nu]) \vee \text{sub}((\nu], (\mu \circ \nu]) \\ &= \text{sub}(\mu, \mu \cap \nu) \vee \text{sub}(\nu, \mu \cap \nu) = \text{sub}(\mu, \nu) \vee \text{sub}(\nu, \mu) = 1, \end{aligned}$$

which implies $\text{sub}(\mu \circ \nu, \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$. $\square$

Theorem 5.8. *Let $(S, \circ, R)$ be an L -ordered L -semihypergroup and δ an L -hyperideal. Then, δ is prime if and only if δ is weakly prime and semiprime. In particular, the prime L -hyperideal is consistent with the weakly prime L -hyperideal in the commutative L -ordered L -semihypergroups.*

Proof. Let δ be an prime L -hyperideal, then it is obvious that δ is weakly prime and semiprime.

Conversely, let δ be weakly prime and semiprime and μ, ν L -hyperideals. Then,

$$(\nu \circ \chi_S \circ \mu] \circ (\nu \circ \chi_S \circ \mu] \subseteq (\nu \circ \chi_S \circ \mu \circ \nu \circ \chi_S \circ \mu] \subseteq (\chi_S \circ (\mu \circ \nu) \circ \chi_S]$$

and

$$(\chi_S \circ \nu \circ \chi_S] \circ (\chi_S \circ \mu \circ \chi_S] \subseteq \chi_S \circ \nu \circ \chi_S \circ \chi_S \circ \mu \circ \chi_S] \subseteq (\chi_S \circ (\nu \circ \chi_S \circ \mu) \circ \chi_S].$$

Since $(\chi_S \circ \nu \circ \chi_S], (\chi_S \circ \mu \circ \chi_S]$ are L -hyperideals and δ is weakly prime and semiprime,

$$\begin{aligned}
\text{sub}(\mu \circ \nu, \delta) &\leq \text{sub}((\chi_S \circ (\mu \circ \nu) \circ \chi_S], (\chi_S \circ \delta \circ \chi_S]) \\
&= \text{sub}((\chi_S \circ (\mu \circ \nu) \circ \chi_S], (\chi_S \circ \delta \circ \chi_S]) \\
&\quad * \text{sub}((\nu \circ \chi_S \circ \mu] \circ (\nu \circ \chi_S \circ \mu], (\chi_S \circ (\mu \circ \nu) \circ \chi_S]) \\
&\leq \text{sub}((\nu \circ \chi_S \circ \mu] \circ (\nu \circ \chi_S \circ \mu], (\chi_S \circ \delta \circ \chi_S]) \\
&\leq \text{sub}((\nu \circ \chi_S \circ \mu] \circ (\nu \circ \chi_S \circ \mu], \delta) \leq \text{sub}((\nu \circ \chi_S \circ \mu], \delta) \\
&\leq \text{sub}((\chi_S \circ (\nu \circ \chi_S \circ \mu] \circ \chi_S], (\chi_S \circ \delta \circ \chi_S]) \\
&= \text{sub}((\chi_S \circ (\nu \circ \chi_S \circ \mu] \circ \chi_S], (\chi_S \circ \delta \circ \chi_S]) \\
&\quad * \text{sub}((\chi_S \circ \nu \circ \chi_S] \circ (\chi_S \circ \mu \circ \chi_S], (\chi_S \circ (\nu \circ \chi_S \circ \mu) \circ \chi_S]) \\
&\leq \text{sub}((\chi_S \circ \nu \circ \chi_S] \circ (\chi_S \circ \mu \circ \chi_S], (\chi_S \circ \delta \circ \chi_S]) \\
&\leq \text{sub}((\chi_S \circ \nu \circ \chi_S] \circ (\chi_S \circ \mu \circ \chi_S], \delta) \\
&\leq \text{sub}((\chi_S \circ \mu \circ \chi_S], \delta) \vee \text{sub}((\chi_S \circ \nu \circ \chi_S], \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta) \\
&\quad (\text{Similar to the proof of (1)} \Rightarrow \text{(2) in Theorem 5.5).}
\end{aligned}$$

Therefore, δ is prime.

In particular, suppose that $(S, \circ, R)$ is commutable. Then, every weakly prime L -hyperideal is prime. In fact, let δ be a weakly prime L -hyperideal and μ, ν L -hyperideals, then

$$\begin{aligned}
&\text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta) \\
&\geq \text{sub}(I(\mu) \circ I(\nu), \delta) \\
&= \text{sub}((\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S] \circ (\nu \cup \chi_S \circ \nu \cup \nu \circ \chi_S \cup \chi_S \circ \nu \circ \chi_S], \delta) \\
&\geq \text{sub}((\mu \cup \chi_S \circ \mu \cup \mu \circ \chi_S \cup \chi_S \circ \mu \circ \chi_S) \circ (\nu \cup \chi_S \circ \nu \cup \nu \circ \chi_S \cup \chi_S \circ \nu \circ \chi_S)], \delta) \\
&\geq \text{sub}((\mu \circ \nu \cup \chi_S \circ \mu \circ \nu], \delta) \\
&\geq \text{sub}((\mu \circ \nu \cup \chi_S \circ \mu \circ \nu], (\delta \cup \chi_S \circ \delta]) * \text{sub}((\delta \cup \chi_S \circ \delta], (\delta]) \\
&\geq \text{sub}((\mu \circ \nu, \delta).
\end{aligned}$$

Therefore, δ is prime by Theorem 5.4. $\square$

Definition 5.9. Let $(S, \circ, R)$ be an L -ordered L -semihypergroup. $(S, \circ, R)$ is called intra-regular if for all $x \in S$, $(\chi_S \circ x^2 \circ \chi_S](x) = 1$.

Example 5.10. Let $S = \{s_1, s_2, s_3, s_4\}$, $L = \{0, 1\}$. We define $R: S \times S \rightarrow L$ as follows:

$$(R_{ij}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $R_{ij} = R(s_i, s_j)$ for $i, j = 1, 2, 3, 4$. By definition, one can easily check that (S, R) is an L -poset. Moreover, we define $\circ: S \times S \rightarrow L^{S^*}$ as follows: $s_1 \circ s_1 = (1, 0, 0, 0)$, $s_1 \circ s_2 = (1, 1, 0, 0)$, $s_1 \circ s_3 = (1, 0, 1, 0)$, $s_1 \circ s_4 = (1, 1, 1, 1)$, $s_2 \circ s_1 = (0, 1, 0, 0)$, $s_2 \circ s_2 = (0, 1, 0, 0)$, $s_2 \circ s_3 = (0, 1, 0, 1)$, $s_2 \circ s_4 = (0, 1, 0, 1)$, $s_3 \circ s_1 = s_3 \circ s_3 = (0, 0, 1, 0)$, $s_3 \circ s_2 = s_3 \circ s_4 = (0, 0, 1, 1)$, $s_4 \circ s_1 = s_4 \circ s_2 = s_4 \circ s_3 = s_4 \circ s_4 = (0, 0, 0, 1)$. By simple calculation, we have that $(S, \circ, R)$ is an L -ordered L -semihypergroup. Moreover, it is not difficult to check that for all $x \in S$, $1 = (x \circ x)(x) = \chi_x(x) = x^4(x) \subseteq (\chi_S \circ x^2 \circ \chi_S)(x) \subseteq (\chi_S \circ x^2 \circ \chi_S](x) \leq 1$, and so $(\chi_S \circ x^2 \circ \chi_S](x) = 1$. Therefore, $(S, \circ, R)$ is intra-regular.

Lemma 5.11. Let $(S, \circ, R)$ be an L -ordered L -semihypergroup and all L -hyperideals of it semiprime. Then, the following conditions hold:

- (1) for all $\mu \in L^{S^*}$, $I(\mu) = (\chi_S \circ \mu \circ \chi_S]$;
- (2) for all $\mu, \nu \in L^{S^*}$, $I(\mu) \cap I(\nu) = I(\mu \circ \nu)$.

Proof.

- (1) Let $\mu \in L^{S^*}$, then it is obvious $(\chi_S \circ \mu \circ \chi_S] \subseteq I(\mu)$. Since $\mu^2 \circ \mu^2 = \mu^2 \circ \mu \circ \mu \subseteq (\chi_S \circ \mu \circ \chi_S]$ and $(\chi_S \circ \mu \circ \chi_S]$ is an L -hyperideal, $\mu^2 \subseteq (\chi_S \circ \mu \circ \chi_S]$ and so $1 = \text{sub}(\mu^2, (\chi_S \circ \mu \circ \chi_S]) \leq \text{sub}(\mu^2, (\chi_S \circ \mu \circ \chi_S]) \leq \text{sub}(\mu, (\chi_S \circ \mu \circ \chi_S]) \leq 1$. Then, $I(\mu) \subseteq (\chi_S \circ \mu \circ \chi_S]$. Therefore, $I(\mu) = (\chi_S \circ \mu \circ \chi_S]$.
- (2) Let $\mu, \nu \in L^{S^*}$. Then, $\mu \circ \nu \subseteq I(\mu) \circ I(\nu) \subseteq I(\mu) \circ \chi_S \subseteq I(\mu)$ and $\mu \circ \nu \subseteq I(\mu) \circ I(\nu) \subseteq \chi_S \circ I(\nu) \subseteq I(\nu)$, and so $I(\mu \circ \nu) \subseteq I(\mu) \cap I(\nu)$ by Lemma 4.7. On the other hand, we have

$$(\nu \circ \chi_S \circ \chi_S \circ \mu)^2 = \nu \circ \chi_S \circ \chi_S \circ \mu \circ \nu \circ \chi_S \circ \chi_S \circ \mu \subseteq \chi_S \circ (\mu \circ \nu) \circ \chi_S \subseteq (\chi_S \circ (\mu \circ \nu) \circ \chi_S],$$

and

$$\begin{aligned} 1 &= \text{sub}((\nu \circ \chi_S \circ \chi_S \circ \mu)^2, (\chi_S \circ (\mu \circ \nu) \circ \chi_S]) \\ &\leq \text{sub}(\nu \circ \chi_S \circ \chi_S \circ \mu, (\chi_S \circ (\mu \circ \nu) \circ \chi_S]) \\ &\leq \text{sub}(\chi_S \circ \nu \circ \chi_S \circ \chi_S \circ \mu \circ \chi_S, (\chi_S \circ (\mu \circ \nu) \circ \chi_S]) \\ &\leq 1, \end{aligned}$$

which implies that $\text{sub}(\chi_S \circ \nu \circ \chi_S \circ \chi_S \circ \mu \circ \chi_S, (\chi_S \circ (\mu \circ \nu) \circ \chi_S]) = 1$. So,

$$\begin{aligned} (I(\mu) \cap I(\nu))(t) &= ((\chi_S \circ \mu \circ \chi_S] \cap (\chi_S \circ \nu \circ \chi_S])(t) \\ &= \text{sub}(\chi_t, (\chi_S \circ \mu \circ \chi_S]) * \text{sub}(\chi_t, (\chi_S \circ \nu \circ \chi_S]) \\ &\leq \text{sub}(\chi_t \circ \chi_t, \chi_t \circ (\chi_S \circ \mu \circ \chi_S]) * \text{sub}(\chi_t \circ (\chi_S \circ \mu \circ \chi_S], (\chi_S \circ \nu \circ \chi_S] \circ (\chi_S \circ \mu \circ \chi_S]) \\ &\leq \text{sub}(\chi_t \circ \chi_t, \chi_t \circ (\chi_S \circ \mu \circ \chi_S]) * \text{sub}(\chi_t \circ (\chi_S \circ \mu \circ \chi_S], (\chi_S \circ \nu \circ \chi_S \circ \chi_S \circ \mu \circ \chi_S]) \\ &= \text{sub}(\chi_t \circ \chi_t, \chi_t \circ (\chi_S \circ \mu \circ \chi_S]) * \text{sub}(\chi_t \circ (\chi_S \circ \mu \circ \chi_S], (\chi_S \circ \nu \circ \chi_S \circ \chi_S \circ \mu \circ \chi_S]) \\ &\quad * \text{sub}((\chi_S \circ \nu \circ \chi_S \circ \chi_S \circ \mu \circ \chi_S], (\chi_S \circ (\mu \circ \nu) \circ \chi_S]) \\ &\leq \text{sub}(\chi_t \circ \chi_t, \chi_t \circ (\chi_S \circ \mu \circ \chi_S]) * \text{sub}(\chi_t \circ (\chi_S \circ \mu \circ \chi_S], (\chi_S \circ (\mu \circ \nu) \circ \chi_S]) \\ &\leq \text{sub}(\chi_t \circ \chi_t, (\chi_S \circ (\mu \circ \nu) \circ \chi_S]) \\ &= (\chi_S \circ (\mu \circ \nu) \circ \chi_S](t) = I(\mu \circ \nu)(t). \end{aligned}$$

Therefore, $I(\mu) \cap I(\nu) = I(\mu \circ \nu)$. □

Lemma 5.12. Let L be a frame and $(S, \circ, R)$ an L -ordered L -semihypergroup. Then, $(S, \circ, R)$ is intra-regular if and only if all L -hyperideals of it are semiprime.

Proof. Let $(S, \circ, R)$ be intra-regular, δ an L -hyperideal, and $x \in S$. By the assumption, $(\chi_S \circ x \circ x \circ \chi_S](x) = 1$, and so

$$\begin{aligned} \text{sub}(x \circ x, \delta) &\leq \text{sub}(\chi_S \circ x \circ x \circ \chi_S, \chi_S \circ \delta \circ \chi_S) \\ &\leq \text{sub}((\chi_S \circ x \circ x \circ \chi_S], (\chi_S \circ \delta \circ \chi_S]) \\ &= \text{sub}((\chi_S \circ x \circ x \circ \chi_S], (\chi_S \circ \delta \circ \chi_S]) \wedge \text{sub}((\chi_S \circ \delta \circ \chi_S], \delta) \\ &\leq \text{sub}((\chi_S \circ x \circ x \circ \chi_S], \delta) \\ &\leq (\chi_S \circ x \circ x \circ \chi_S](x) \rightarrow \delta(x) = \delta(x). \end{aligned}$$

Therefore, δ is semiprime.

Conversely, let all L -hyperideals of $(S, \circ, R)$ be semiprime and $x \in S$. Since $x^2 \subseteq I(x^2)$, $1 = \text{sub}(x^2, I(x^2)) \leq \text{sub}(\chi_x, I(x^2))$, $\chi_x \subseteq I(x^2) = (x^2 \cup \chi_S \circ x^2 \cup x^2 \circ \chi_S \cup \chi_S \circ x^2 \circ \chi_S] \leq [x^2] \vee (\chi_S \circ x^2] \vee (x^2 \circ \chi_S] \vee (\chi_S \circ x^2 \circ \chi_S]$ by the assumption. Moreover,

$$\begin{aligned}
(\chi_S \circ x^2 \circ \chi_S](x) &\geq (x^4](x) = ((x^2] \circ (x^2)](x) \\
&= \bigvee_{t,s,b \in S} (x^2](t) \wedge (x^2](s) \wedge (t \circ s)(b) \wedge R(x, b) \\
&\geq \bigvee_{b \in S} (x^2](x) \wedge (x^2](x) \wedge (x \circ \mu)(b) \wedge R(x, b) \\
&= (x^2](x) \wedge (\bigvee_{b \in S} (x \circ x)(b) \wedge R(x, b)) = (x^2](x),
\end{aligned}$$

$$\begin{aligned}
(\chi_S \circ x^2 \circ \chi_S](x) &\geq (\chi_S \circ (\chi_S \circ x^2) \circ x](x) = ((\chi_S] \circ (\chi_S \circ x^2] \circ x)](x) \geq (\chi_S \circ (\chi_S \circ x^2] \circ x](x) \\
&= \bigvee_{y \in S} (\chi_S \circ (\chi_S \circ x^2] \circ x)(y) \wedge R(x, y) \\
&= \bigvee_{y,a,b \in S} (a \circ b)(y) \wedge ((\chi_S \circ x^2] \circ x)(b) \wedge R(x, y) \\
&= \bigvee_{y,a,b,c \in S} (a \circ b)(y) \wedge (\chi_S \circ x^2](c) \wedge (c \circ x)(b) \wedge R(x, y) \\
&= \bigvee_{y,a,b,c,d,e,s \in S} (a \circ b)(y) \wedge (d \circ s)(e) \wedge (x^2](s) \wedge R(c, e) \wedge (c \circ x)(b) \wedge R(x, y) \\
&\geq \bigvee_{y,a,b \in S} (a \circ b)(y) \wedge (a \circ b)(y) \wedge (x \circ x)(b) \wedge R(x, y) \wedge (x \circ x)(b) \wedge R(x, y) \\
&= \bigvee_{y,a,b \in S} (a \circ b)(y) \wedge (x \circ x)(b) \wedge R(x, y) = (\chi_S \circ x^2](x).
\end{aligned}$$

Similarly, we can obtain $(\chi_S \circ x^2 \circ \chi_S](x) \geq (x^2 \circ \chi_S](x)$. Therefore, $x \in (\chi_S \circ x^2 \circ \chi_S]$, and so $(S, \circ, R)$ is intra-regular. $\square$

Theorem 5.13. Let L be a frame and $(S, \circ, R)$ an L -ordered L -semihypergroup. Then, all L -hyperideals are prime if and only if they form a weak chain and $(S, \circ, R)$ is intra-regular.

Proof. Let each L -hyperideal be prime, then each L -hyperideal is weakly prime. By Theorem 5.7, they form a weak chain. And each L -hyperideal is also semiprime. Therefore, it can be seen from Lemma 5.12 that $(S, \circ, R)$ is intra-regular.

Conversely, suppose $(S, \circ, R)$ is intra-regular and all L -hyperideals form a weak chain. Then, all L -hyperideals are prime. In fact, let δ be an L -hyperideal, μ an L -right hyperideal, and ν an L -left hyperideal. By Lemma 5.12, each L -hyperideal is semiprime, thus by Lemma 5.11, we have

$$\begin{aligned}
\text{sub}(\mu \circ \nu, \delta) &\rightarrow (\text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)) \\
&= \text{sub}(\mu \circ \nu, \delta) \rightarrow (\text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)) \\
&\geq \text{sub}(\mu \circ \nu, \delta) \rightarrow (\text{sub}(I(\mu), \delta) \vee \text{sub}(I(\nu), \delta)) \\
&\geq \text{sub}(I(\mu \circ \nu), I(\delta)) \rightarrow (\text{sub}(I(\mu), \delta) \vee \text{sub}(I(\nu), \delta)) \\
&= \text{sub}(I(\mu \circ \nu), \delta) \rightarrow (\text{sub}(I(\mu), \delta) \vee \text{sub}(I(\nu), \delta)) \\
&\geq (\text{sub}(I(\mu \circ \nu), \delta) \rightarrow \text{sub}(I(\mu), \delta)) \vee (\text{sub}(I(\mu \circ \nu), \delta) \rightarrow \text{sub}(I(\nu), \delta)) \\
&\geq \text{sub}(I(\mu \circ \nu), I(\mu)) \vee \text{sub}(I(\mu \circ \nu), I(\nu)) \\
&= \text{sub}(I(\mu) \cap I(\nu), I(\mu)) \vee \text{sub}(I(\mu) \cap I(\nu), I(\nu)) \\
&= \text{sub}(I(\nu), I(\mu)) \vee \text{sub}(I(\mu), I(\nu)) = 1,
\end{aligned}$$

which implies that $\text{sub}(\mu \circ \nu, \delta) \leq \text{sub}(\mu, \delta) \vee \text{sub}(\nu, \delta)$. Therefore, δ is prime. $\square$

6 Conclusions

In this study, we first introduced the concept of an L -ordered L -semihypergroup by combining an L -order and an L -semihypergroup. Then, we presented the relationship between it and an ordered semihypergroup.

Finally, we defined and investigated L -hyperideals. In particular, we studied prime, weakly prime, and semiprime L -hyperideals of L -ordered L -semihypergroups and established the relationships among them. Following this study, we will investigate the concepts of L -order-congruences and L -pseudoorders on an L -ordered L -semihypergroup and give out homomorphism theorems of L -ordered L -semihypergroups by L -pseudoorders.

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