Research Article

Fan-Liang Li*

Left and right inverse eigenpairs problem with a submatrix constraint for the generalized centrosymmetric matrix

<https://doi.org/10.1515/math-2020-0020>

received June 20, 2019; accepted January 18, 2020

Abstract: Left and right inverse eigenpairs problem is a special inverse eigenvalue problem. There are many meaningful results about this problem. However, few authors have considered the left and right inverse eigenpairs problem with a submatrix constraint. In this article, we will consider the left and right inverse eigenpairs problem with the leading principal submatrix constraint for the generalized centrosymmetric matrix and its optimal approximation problem. Combining the special properties of left and right eigenpairs and the generalized singular value decomposition, we derive the solvability conditions of the problem and its general solutions. With the invariance of the Frobenius norm under orthogonal transformations, we obtain the unique solution of optimal approximation problem. We present an algorithm and numerical experiment to give the optimal approximation solution. Our results extend and unify many results for left and right inverse eigenpairs problem and the inverse eigenvalue problem of centrosymmetric matrices with a submatrix constraint.

Keywords: leading principal submatrix, submatrix constraint, generalized centrosymmetric matrix, left and right inverse eigenpairs, optimal approximation

MSC 2010: 65F18, 15A24

1 Introduction

Throughout this article we use some notations as follows. Let $C^{n \times m}$ be the set of all $n \times m$ complex matrices, $R^{n \times m}$ be the set of all $n \times m$ real matrices, $C^n = C^{n \times 1}$, $R^n = R^{n \times 1}$, R denote the set of all real numbers, $OR^{n \times n}$ denote the set of all $n \times n$ orthogonal matrices, $R(A)$, A^T , $r(A)$, $\text{tr}(A)$ and A^+ be the column space, the transpose, rank, trace and the Moore–Penrose generalized inverse of a matrix A , respectively. I_n is the identity matrix of size n . Let e_i be the i th column of I_n , and set $J_n = (e_n, \dots, e_1)$. For $A, B \in R^{n \times m}$, $\langle A, B \rangle = \text{tr}(B^T A)$ denotes the inner product of matrices A and B . The induced matrix norm is called the Frobenius norm, i.e. $\|A\| = \langle A, A \rangle^{1/2} = (\text{tr}(A^T A))^{1/2}$, then $R^{n \times m}$ is a Hilbert inner product space.

Generally, the left and right inverse eigenpairs problem is as follows: given partial left and right eigenpairs (eigenvalue and corresponding eigenvector) (y_j, y_j) , $j = 1, \dots, l$; (λ_i, x_i) , $i = 1, \dots, h$, and a special $n \times m$ matrix set S , to find $A \in S$ such that

$$\begin{cases} Ax_i = \lambda_i x_i, & i = 1, \dots, h, \\ y_j^T A = y_j y_j^T, & j = 1, \dots, l, \end{cases} \quad (1.1)$$

* **Corresponding author: Fan-Liang Li**, Institute of Mathematics and Physics, School of Sciences, Central South University of Forestry and Technology, Changsha, 410004, P. R. China; School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, 410114, Hunan, P. R. China, e-mail: lfl302@tom.com

where $h \leq n$ and $l \leq n$. If $X = (x_1, \dots, x_h)$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_h)$, $Y = (y_1, \dots, y_l)$, $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_l)$, then (1.1) is equivalent to

$$\begin{cases} AX = X\Lambda, \\ Y^T A = \Gamma Y^T. \end{cases} \quad (1.2)$$

This problem, which mainly arises in perturbation analysis of matrix eigenvalue and in recursive matters, has some practical applications in economic and scientific computation fields [1–3].

Many important results have been achieved on this problem associated with many kinds of matrix sets. Li et al. [4–9] have solved the left and right inverse eigenpairs problems for skew-centrosymmetric matrices, generalized centrosymmetric matrices, κ -persymmetric matrices, symmetrizable matrices, orthogonal matrices and κ -Hermitian matrices by using the special properties of eigenpairs of matrix. Zhang and Xie [10], Ouyang [11], Liang and Dia [12] and Yin and Huang [13] have, respectively, solved the left and right inverse eigenvalue problems for real matrices, semipositive subdefinite matrices, generalized reflexive and anti-reflexive matrices and (R, S) symmetric matrices with the special structure of matrix.

Arav et al. [2] and Loewy and Mehrmann [3] studied the recursive inverse eigenvalue problem which arises in the Leontief economic model. Namely, given eigenvalue λ_i of A_i , in which A_i is the i th leading principal submatrix of A , corresponding left eigenvector y_i and right eigenvector x_i of λ_i , construct a matrix $A \in C^{n \times m}$ such that

$$\begin{cases} A_i x_i = \lambda_i x_i, \\ y_i^T A_i = \lambda_i y_i^T, \end{cases} \quad i = 1, \dots, n.$$

This recursive inverse eigenvalue problem is a special case of the left and right inverse eigenvalue problem with the leading principal submatrix constraint. Few authors have considered the left and right inverse eigenpairs problem with a submatrix constraint. In this article, we will consider the left and right inverse eigenpairs problem with the leading principal submatrix constraint for the generalized centrosymmetric matrix, which has not been discussed.

Definition 1. Let κ be a real fixed product of disjoint transpositions and J be the associated permutation matrix. $A = (a_{ij}) \in R^{n \times m}$, if $a_{ij} = a_{\kappa(i)\kappa(j)}$ (or $a_{ij} = -a_{\kappa(i)\kappa(j)}$), then A is called the generalized centrosymmetric matrix (or generalized centro-skewsymmetric matrix), and $\text{GCSR}^{n \times m}$ (or $\text{GCSSR}^{n \times n}$) denote the set of all generalized centrosymmetric matrices (or the set of all generalized centro-skewsymmetric matrices).

From Definition 1, it is easy to derive the following conclusions.

- (1) $J^T = J$ and $J^2 = I_n$. Real matrices and centrosymmetric matrices are the special cases of generalized centrosymmetric matrices with $\kappa(i) = i$ and $\kappa(i) = n - i + 1$ or $J = I_n$ and $J = J_n$, respectively.
- (2) $A \in \text{GCSR}^{n \times n}$ if and only if $A = JAJ$ and $A \in \text{GCSSR}^{n \times n}$ if and only if $A = -JAJ$.
- (3) $R^{n \times n} = \text{GCSR}^{n \times n} \oplus \text{GCSSR}^{n \times n}$, where the notation $V_1 \oplus V_2$ stands for the orthogonal direct sum of linear subspaces V_1 and V_2 .

Centrosymmetry, persymmetry and symmetry are three important symmetric structures of a square $n \times n$ matrix and have profound applications, such as engineering, statistics and so on [14–16]. There are many meaningful results about the inverse problem and the inverse eigenvalue problem of centrosymmetric matrices with a submatrix constraint. Peng et al. [17] and Bai [18] discussed the inverse problem and the inverse eigenvalue problem of centrosymmetric matrices with a principal submatrix constraint, respectively. Zhao et al. [19] studied least squares solutions to $AX = B$ for symmetric centrosymmetric matrices under a central principal submatrix constraint and the optimal approximation. The matrix inverse problem (or inverse eigenvalue problem) with a submatrix constraint is also called the matrix extension problem. Since de Boor and Golub [20] first put forward and considered the Jacobi matrix extension problem in 1978, many authors have studied the matrix extension problem and a series of meaningful results have been achieved [17–19, 21–26].

Assume (λ_i, x_i) , $i = 1, \dots, m$, be right eigenpairs of A ; (μ_j, y_j) , $j = 1, \dots, h$, be left eigenpairs of A . Let $X = (x_1, \dots, x_m) \in \mathbb{C}^{n \times m}$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m) \in \mathbb{C}^{m \times m}$; $Y = (y_1, \dots, y_h) \in \mathbb{C}^{n \times h}$, $\Gamma = \text{diag}(\mu_1, \dots, \mu_h) \in \mathbb{C}^{h \times h}$. The problems studied in this article can be described as follows.

Problem I. Given $X = (x_1, \dots, x_m) \in \mathbb{C}^{n \times m}$, $Y = (y_1, \dots, y_h) \in \mathbb{C}^{n \times h}$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m) \in \mathbb{C}^{m \times m}$, $\Gamma = \text{diag}(\mu_1, \dots, \mu_h) \in \mathbb{C}^{h \times h}$, $A_0 \in \mathbb{R}^{p \times p}$, $m \leq n$, $h \leq n$, $p \leq n$, find $A \in \text{GCSR}^{n \times n}$ such that

$$\begin{cases} AX = X\Lambda, \\ Y^T A = \Gamma Y^T, \end{cases} \quad A[1:p] = A_0,$$

where $A[1:p]$ denotes the $p \times p$ leading principal submatrix.

Problem II. Given $A^* \in \mathbb{R}^{n \times n}$, find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\| = \min_{\forall A \in S_E} \|A^* - A\|,$$

where S_E is the solution set of Problem I.

This article is organized as follows. In Section 2, we first study the special properties of eigenpairs and the structure of generalized centrosymmetric matrices. Then, we provide the solvability conditions for and the general solutions of Problem I. In Section 3, we first attest the existence and uniqueness theorem of Problem II and then present the unique approximation solution with the orthogonal invariance of the Frobenius norm. Finally, we provide an algorithm to compute the unique approximation solution. Some conclusions are provided in Section 4.

2 Solvability conditions of Problem I

Definition 2. Let $x \in \mathbb{C}^n$. If $Jx = x$ (or $Jx = -x$), then x is called the generalized symmetric (or generalized skew-symmetric) vector. Denote the set of all generalized symmetric (or generalized skew-symmetric) vectors by GC^n (or GSC^n).

Denote

$$P_1 = \frac{1}{2}(I_n + J), \quad P_2 = \frac{1}{2}(I_n - J).$$

Let $(u_1, u_2, \dots, u_{n-r})$ and $(u_{n-r+1}, u_{n-r+2}, \dots, u_n)$ are the orthonormal bases for $R(P_1)$ and $R(P_2)$, respectively, and are denoted as $K_1 = (u_1, u_2, \dots, u_{n-r})$, $K_2 = (u_{n-r+1}, u_{n-r+2}, \dots, u_n)$ and $K = (K_1, K_2)$. Combining Definitions 1 and 2, it is easy to derive the following equalities.

$$P_1 = K_1 K_1^T, \quad P_2 = K_2 K_2^T. \quad (2.1)$$

$$J = K_1 K_1^T - K_2 K_2^T = K \begin{pmatrix} I_{n-r} & 0 \\ 0 & I_r \end{pmatrix} K^T. \quad (2.2)$$

Combining conclusion (2) of Definition 1, (2.1) and (2.2), it is easy to derive the following lemma.

Lemma 1. $A \in GCSR^{n \times n}$ if and only if

$$A = K \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} K^T,$$

where $A_{11} = K_1^T A K_1 \in R^{(n-r) \times (n-r)}$, $A_{22} = K_2^T A K_2 \in R^{r \times r}$.

If $J = J_n$ and $n = 2k$, then

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ J_k & -J_k \end{pmatrix}.$$

If $J = J_n$, and $n = 2k + 1$, then

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ J_k & 0 & -J_k \end{pmatrix}.$$

Similarly, we have the following splitting of centrosymmetric matrices into smaller submatrices using K .

Lemma 2. [27] (1) If $A \in CSR^{2k \times 2k}$, then A can be written as

$$A = \begin{pmatrix} B & C J_k \\ J_k C & J_k B J_k \end{pmatrix} = K \begin{pmatrix} B + C & 0 \\ 0 & B - C \end{pmatrix} K^T, \quad B, C \in R^{k \times k}.$$

(2) If $A \in CSR^{(2k+1) \times (2k+1)}$, then A can be written as

$$A = \begin{pmatrix} B & p & C J_k \\ q^T & d & q^T J_k \\ J_k C & J_k p & J_k B J_k \end{pmatrix} = K \begin{pmatrix} B + C & \sqrt{2} p & 0 \\ \sqrt{2} q^T & d & 0 \\ 0 & 0 & B - C \end{pmatrix} K^T, \quad B, C \in R^{k \times k}, p, q \in R^k, d \in R,$$

where $CSR^{n \times n}$ denotes the set of all $n \times n$ centrosymmetric matrices, $k = \left\lfloor \frac{n}{2} \right\rfloor$ denotes the largest integer number that is not greater than $\frac{n}{2}$. In fact, Lemma 2 is a special result of Lemma 1 with $J = J_n$.

For a real matrix $A \in R^{n \times m}$, its complex right eigenpairs are conjugate pairs. That is, if $a + b\sqrt{-1}$ and $x + \sqrt{-1}y$ are one of its right eigenpairs, then $a - b\sqrt{-1}$ and $x - \sqrt{-1}y$ are one of its right eigenpairs. This implies $Ax = ax - by$ and $Ay = bx + ay$, i.e.,

$$A(x, y) = (x, y) \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Therefore, in Problem I, we can assume that $X \in R^{n \times m}$ and

$$\Lambda = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_{\bar{m}}) \in R^{m \times m},$$

where $\bar{\lambda}_i, i = 1, \dots, \bar{m}$ are real numbers or 2×2 real matrices, $\bar{m} \leq m$. $\bar{m} = m$ holds if and only if all right eigenvalues of A are real numbers. We can also prove that the complex left eigenpairs of A are conjugate pairs. Hence, in Problem I, we can also assume that $Y \in R^{n \times h}$ and

$$\Gamma = \text{diag}(\bar{\mu}_1, \dots, \bar{\mu}_{\bar{h}}) \in R^{h \times h},$$

where $\bar{\mu}_i, i = 1, \dots, \bar{h}$ are real numbers or 2×2 real matrices, $\bar{h} \leq h$. $\bar{h} = h$ holds if and only if all left eigenvalues of A are real numbers.

Let $A \in \text{GCSR}^{n \times n}$, if $Ax = \lambda x$, where λ is a number, $x \in C^n$, and $x \neq 0$, then we have

$$JAJx = \lambda Jx, \quad AJx = \lambda Jx.$$

Hence, we have $A(x \pm Jx) = \lambda(x \pm Jx)$. It is obvious that $x + Jx$ and $x - Jx$ is a generalized symmetric vector and a generalized skew-symmetric vector, respectively. If $a + b\sqrt{-1}$ and $x + \sqrt{-1}y$ are one of its right eigenpairs, then we have

$$A(x, y) = (x, y) \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \quad (2.3)$$

According to conclusion (2) of Definition 1, we have

$$AJ(x, y) = J(x, y) \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \quad (2.4)$$

Combining (2.3) and (2.4) implies

$$A[(x, y) \pm J(x, y)] = [(x, y) \pm J(x, y)] \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

It is easy to see that the columns of $(x, y) + J(x, y)$ (or $(x, y) - J(x, y)$) is a generalized symmetric vector (or generalized skew-symmetric vector). Hence, the right eigenvectors of the generalized centrosymmetric matrix can be expressed as generalized symmetric vectors or generalized skew-symmetric vectors. It is clear that the left eigenvectors of the generalized centrosymmetric matrix have the same properties as the right ones. According to the aforementioned analysis, in Problem I, we may assume as follows:

$$\begin{aligned} X &= (X_1, X_2) \in R^{n \times m}, X_1 = JX_1 \in R^{n \times m_1}, X_2 = -JX_2 \in R^{n \times (m-m_1)}, \\ Y &= (Y_1, Y_2) \in R^{n \times h}, Y_1 = JY_1 \in R^{n \times h_1}, Y_2 = -JY_2 \in R^{n \times (h-h_1)}, \end{aligned} \quad (2.5)$$

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \Lambda_1 = \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_{m_1}) \in R^{m_1 \times m_1}, \Lambda_2 = \text{diag}(\bar{\lambda}_{m_1+1}, \dots, \bar{\lambda}_m) \in R^{(m-m_1) \times (m-m_1)},$$

$$\Gamma = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}, \Gamma_1 = \text{diag}(\bar{\mu}_1, \dots, \bar{\mu}_{h_1}) \in R^{h_1 \times h_1}, \Gamma_2 = \text{diag}(\bar{\mu}_{h_1+1}, \dots, \bar{\mu}_h) \in R^{(h-h_1) \times (h-h_1)}.$$

Lemma 3. [6] If X, Λ, Y, Γ are given by (2.5), then $(AX = X\Lambda, Y^T A = \Gamma Y^T)$ has a solution in $\text{GCSR}^{n \times n}$ if and only if

$$Y_1^T X_1 \Lambda_1 = \Gamma_1 Y_1^T X_1, X_1 \Lambda_1 = X_1 \Lambda_1 X_1^+ X_1, Y_1 \Gamma_1 = Y_1 \Gamma_1 Y_1^+ Y_1, \quad (2.6)$$

$$Y_2^T X_2 \Lambda_2 = \Gamma_2 Y_2^T X_2, X_2 \Lambda_2 = X_2 \Lambda_2 X_2^+ X_2, Y_2 \Gamma_2 = Y_2 \Gamma_2 Y_2^+ Y_2. \quad (2.7)$$

Moreover, its general solution can be expressed as

$$A = A_{10} + EFG, \quad \forall F \in GCSR^{n \times n}, \quad (2.8)$$

where

$$A_{10} = X_1 A_1 X_1^+ + (Y_1^T)^+ F_1 Y_1^T (I_n - X_1 X_1^+) + X_2 A_2 X_2^+ + (Y_2^T)^+ F_2 Y_2^T (I_n - X_2 X_2^+) \in GCSR^{n \times n}, \quad (2.9)$$

$$E = I_n - Y_1 Y_1^+ - Y_2 Y_2^+ \in GCSR^{n \times n}, \quad G = I_n - X_1 X_1^+ - X_2 X_2^+ \in GCSR^{n \times n}. \quad (2.10)$$

Combining Lemmas 1 and 3, it is easy to prove that A in (2.8) can be expressed as

$$A = K \begin{pmatrix} A_{110} + E_1 F_1 G_1 & 0 \\ 0 & A_{210} + E_2 F_2 G_2 \end{pmatrix} K^T, \quad (2.11)$$

where A_{110}, A_{210} are denoted by A_{10}, E_1, E_2 are denoted by E, G_1, G_2 are denoted by G , and $A_{110}, E_1, G_1 \in R^{(n-k) \times (n-k)}, A_{210}, E_2, G_2 \in R^{k \times k}$, for any $F_1 \in R^{(n-k) \times (n-k)}, F_2 \in R^{k \times k}$.

Denote

$$\begin{aligned} (I_p, 0) K_1 E_1 &= \bar{E}_1, (I_p, 0) K_2 E_2 = \bar{E}_2, G_1 K_1^T (I_p, 0)^T = \bar{G}_1, G_2 K_2^T (I_p, 0)^T = \bar{G}_2, \\ A_0 - (I_p, 0) K_1 A_{110} K_1^T (I_p, 0)^T - (I_p, 0) K_2 A_{210} K_2^T (I_p, 0)^T &= \bar{A}_0. \end{aligned} \quad (2.12)$$

Combining (2.11) and (2.12), $A[1:p] = A_0$ if and only if the following equation holds.

$$\bar{E}_1 F_1 \bar{G}_1 + \bar{E}_2 F_2 \bar{G}_2 = \bar{A}_0. \quad (2.13)$$

Suppose that the generalized singular value decomposition (GSVD) of matrix pairs $(\bar{E}_1^T, \bar{E}_2^T)$ is as follows:

$$\bar{E}_1^T = Q_1 \Sigma_1 S^T, \quad \bar{E}_2^T = Q_2 \Sigma_2 S^T, \quad (2.14)$$

where $Q_1 \in OR^{(n-k) \times (n-k)}, Q_2 \in OR^{k \times k}, S \in R^{p \times p}$ is nonsingular, and

$$\Sigma_1 = \begin{pmatrix} I_{r_1} & 0 & 0 & 0 \\ 0 & \Theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Theta_2 & 0 & 0 \\ 0 & 0 & I_{s_1-r_1-t_1} & 0 \end{pmatrix} \quad (2.15)$$

$r_1 \quad t_1 \quad s_1-r_1-t_1 \quad p-s_1 \qquad r_1 \quad t_1 \quad s_1-r_1-t_1 \quad p-s_1$

with $s_1 = r(\bar{E}_1, \bar{E}_2)^T, r_1 = s_1 - r(\bar{E}_2^T), t_1 = r(\bar{E}_1^T) + r(\bar{E}_2^T) - s_1, \Theta_1 = \text{diag}(\gamma_1, \dots, \gamma_{t_1}), \Theta_2 = \text{diag}(\delta_1, \dots, \delta_{t_1})$, with $1 \geq \gamma_1 \geq \dots \geq \gamma_{t_1} > 0, 0 < \delta_1 \leq \dots \leq \delta_{t_1} \leq 1, \gamma_i^2 + \delta_i^2 = 1, i = 1, \dots, t_1$.

Suppose that the GSVD of matrix pairs $(\bar{G}_1, \bar{G}_2)$ is

$$\bar{G}_1 = P_1 \Sigma_3 W^T, \quad \bar{G}_2 = P_2 \Sigma_4 W^T, \quad (2.16)$$

where $P_1 \in OR^{(n-k) \times (n-k)}, P_2 \in OR^{k \times k}, W \in R^{p \times p}$ is nonsingular, and

$$\Sigma_3 = \begin{pmatrix} I_{r_1} & 0 & 0 & 0 \\ 0 & \Theta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Theta_4 & 0 & 0 \\ 0 & 0 & I_{s_2-r_2-t_2} & 0 \end{pmatrix} \quad (2.17)$$

$r_2 \quad t_2 \quad s_2-r_2-t_2 \quad p-s_2 \qquad r_2 \quad t_2 \quad s_2-r_2-t_2 \quad p-s_2$

with $s_2 = r(\bar{G}_2)$, $r_2 = s_2 - r(\bar{G}_2)$, $t_2 = r(\bar{G}_1) + r(\bar{G}_2) - s_2$, $\Theta_3 = \text{diag}(\alpha_1, \dots, \alpha_{t_2})$, $\Theta_4 = \text{diag}(\beta_1, \dots, \beta_{t_2})$, with $1 \geq \alpha_{t_2} \geq \dots \geq \alpha_1 > 0$, $0 < \beta_1 \leq \dots \leq \beta_{t_2} \leq 1$, $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, \dots, t_2$, where r_1 , t_1 , $s_1 - r_1 - t_1$, $p - s_1$, r_2 , t_2 , $s_2 - r_2 - t_2$, $p - s_2$ denote the number of columns of the sub-block-matrix of Σ_1 , Σ_2 , Σ_3 and Σ_4 .

Combining (2.14) and (2.16) implies that (2.13) can be written as

$$\Sigma_1^T Q_1^T F_1 P_1 \Sigma_3 + \Sigma_2^T Q_2^T F_2 P_2 \Sigma_4 = S^{-1} \bar{A}_0 W^{-T}. \quad (2.18)$$

Partition $Q_1^T F_1 P_1$, $Q_2^T F_2 P_2$, $S^{-1} \bar{A}_0 W^{-T}$ into the following form:

$$Q_1^T F_1 P_1 = \begin{pmatrix} F_{111} & F_{112} & F_{113} \\ F_{121} & F_{122} & F_{123} \\ F_{131} & F_{132} & F_{133} \end{pmatrix}, \quad Q_2^T F_2 P_2 = \begin{pmatrix} F_{211} & F_{212} & F_{213} \\ F_{221} & F_{222} & F_{223} \\ F_{231} & F_{232} & F_{233} \end{pmatrix}, \quad (2.19)$$

$$S^{-1} \bar{A}_0 W^{-T} = \begin{pmatrix} A_{011} & A_{012} & A_{013} & A_{014} \\ A_{021} & A_{022} & A_{023} & A_{024} \\ A_{031} & A_{032} & A_{033} & A_{034} \\ A_{041} & A_{042} & A_{043} & A_{044} \end{pmatrix}. \quad (2.20)$$

Combining (2.19) and (2.20) implies that (2.18) can be written as

$$\begin{pmatrix} F_{111} & F_{112} \Theta_3 & 0 & 0 \\ \Theta_1 F_{121} & \Theta_1 F_{122} \Theta_3 + \Theta_2 F_{222} \Theta_4 & \Theta_2 F_{223} & 0 \\ 0 & F_{232} \Theta_4 & F_{233} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{011} & A_{012} & A_{013} & A_{014} \\ A_{021} & A_{022} & A_{023} & A_{024} \\ A_{031} & A_{032} & A_{033} & A_{034} \\ A_{041} & A_{042} & A_{043} & A_{044} \end{pmatrix}. \quad (2.21)$$

Combining Lemma 3 and (2.11)–(2.21) derives the following theorem.

Theorem 1. If X , Λ , Y , Γ are given by (2.5) and given $A_0 \in R^{p \times p}$, then Problem I has a solution in $GCSR^{n \times n}$ if and only if (2.6), (2.7) and the following equations hold:

$$A_{013} = 0, A_{014} = 0, A_{024} = 0, A_{031} = 0, A_{034} = 0, A_{041} = 0, A_{042} = 0, A_{043} = 0, A_{044} = 0. \quad (2.22)$$

Moreover, the general solution is

$$A = K \begin{pmatrix} A_{110} + E_1 F_1 G_1 & 0 \\ 0 & A_{210} + E_2 F_2 G_2 \end{pmatrix} K^T, \quad (2.23)$$

where A_{110} , E_1 , G_1 , A_{210} , E_2 , G_2 are denoted by (2.11), and

$$F_1 = Q_1 \begin{pmatrix} A_{011} & A_{012} \Theta_3^{-1} & F_{113} \\ \Theta_1^{-1} A_{021} & \Theta_1^{-1} (A_{022} - \Theta_2 F_{222} \Theta_4) \Theta_3^{-1} & F_{123} \\ F_{131} & F_{132} & F_{133} \end{pmatrix} P_1^T, \quad (2.24)$$

$$F_2 = Q_2 \begin{pmatrix} F_{211} & F_{212} & F_{213} \\ F_{221} & F_{222} & \Theta_2^{-1} A_{023} \\ F_{231} & A_{032} \Theta_4^{-1} & A_{033} \end{pmatrix} P_2^T \quad (2.25)$$

where $F_{113}, F_{123}, F_{131}, F_{133}, F_{211}, F_{212}, F_{213}, F_{221}, F_{222}$ and F_{231} are the arbitrary matrices.

3 An expression of the solution of Problem II

From (2.23), it is easy to prove that the solution set S_E of Problem I is a nonempty closed convex set if Problem I has a solution in $\text{GCSR}^{n \times n}$. We claim that for any given $A^* \in R^{n \times n}$, there exists a unique optimal approximation for Problem II.

Combining (2.8)–(2.11) and Lemma 1, (2.23) can be written as

$$A = A_{10} + EFG, F = K \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} K^T, \quad (3.1)$$

where E and G are denoted by (2.10), F_1 and F_2 are denoted by (2.24) and (2.25), respectively.

According to conclusion (3) of Definition 1, for any $A^* \in R^{n \times n}$, there exist only one $A_1^* \in \text{GCSR}^{n \times n}$ and only one $A_2^* \in \text{GCSSR}^{n \times n}$ which satisfy

$$A^* = A_1^* + A_2^*, \quad (3.2)$$

where

$$A_1^* = \frac{1}{2}(A^* + JA^*J), \quad A_2^* = \frac{1}{2}(A^* - JA^*J). \quad (3.3)$$

According to Lemma 1, A_1^* can be written as

$$A_1^* = K \begin{pmatrix} A_{11}^* & 0 \\ 0 & A_{22}^* \end{pmatrix} K^T, \quad (3.4)$$

where $A_{11}^* \in R^{(n-k) \times (n-k)}$, $A_{22}^* \in R^{n \times n}$ are given by A_1^* . Denote

$$Q_1^T A_{11}^* P_1 = \begin{pmatrix} A_{111}^* & A_{112}^* & A_{113}^* \\ A_{121}^* & A_{122}^* & A_{123}^* \\ A_{131}^* & A_{132}^* & A_{133}^* \end{pmatrix}, \quad Q_2^T A_{22}^* P_2 = \begin{pmatrix} A_{211}^* & A_{212}^* & A_{213}^* \\ A_{221}^* & A_{222}^* & A_{223}^* \\ A_{231}^* & A_{232}^* & A_{233}^* \end{pmatrix}. \quad (3.5)$$

Theorem 2. Given X, Y, Λ, Γ according to (2.5) and A_0 . If they satisfy the conditions of Theorem 1, and given $A^* \in R^{n \times n}$, then Problem II has the unique solution $\hat{A}$. Moreover, $\hat{A}$ can be expressed as

$$\hat{A} = A_{10} + E\hat{F}G, \quad (3.6)$$

where A_{10} , E , G are denoted by (2.9) and (2.10) with

$$\hat{F} = K \begin{pmatrix} \hat{F}_1 & 0 \\ 0 & \hat{F}_2 \end{pmatrix} K^T, \quad (3.7)$$

where

$$\hat{F}_1 = Q_1 \begin{pmatrix} A_{011} & A_{012} \Theta_3^{-1} & A_{113}^* \\ \Theta_1^{-1} A_{021} & \Theta_1^{-1} (A_{022} - \Theta_2 A_{222}^* \Theta_4) \Theta_3^{-1} & A_{123}^* \\ A_{131}^* & A_{132}^* & A_{133}^* \end{pmatrix} P_1^T, \quad (3.8)$$

$$\hat{F}_2 = Q_2 \begin{pmatrix} A_{211}^* & A_{212}^* & A_{213}^* \\ A_{221}^* & A_{222}^* & \Theta_2^{-1} A_{023} \\ A_{231}^* & A_{032} \Theta_4^{-1} & A_{033} \end{pmatrix} P_2^T. \quad (3.9)$$

Proof. Combining Theorem 1 and (3.2), for any $A \in S_E$, we have

$$\|A^* - A\|^2 = \|A_1^* - A\|^2 + \|A_2^*\|^2 = \|A_1^* - A_{10} - EFG\|^2 + \|A_2^*\|^2.$$

According to (2.10), it is easy to prove that E , F are orthogonal projection matrices. Hence, there exist orthogonal projection matrices $\bar{E}$, $\bar{G}$ which satisfy

$$\bar{E} + E = I_n, \bar{E}E = 0; \quad \bar{G} + G = I_n, \bar{G}G = 0. \quad (3.10)$$

From this, we have

$$\begin{aligned} \|A^* - A\|^2 &= \|(\bar{E} + E)(A_1^* - A_{10}) - EFG\|^2 + \|A_2^*\|^2 \\ &= \|\bar{E}(A_1^* - A_{10})\|^2 + \|E(A_1^* - A_{10}) - EFG\|^2 + \|A_2^*\|^2 \\ &= \|\bar{E}(A_1^* - A_{10})\|^2 + \|E(A_1^* - A_{10})(G + \bar{G}) - EFG\|^2 + \|A_2^*\|^2 \\ &= \|\bar{E}(A_1^* - A_{10})\|^2 + \|E(A_1^* - A_{10})G - EFG\|^2 + \|E(A_1^* - A_{10})\bar{G}\|^2 + \|A_2^*\|^2. \end{aligned}$$

This implies that

$$\min_{\text{for any } A \in S_E} \|A^* - A\| \Leftrightarrow \min_{F \text{ denoted by (3.1)}} \|E(A_1^* - A_{10})G - EFG\|.$$

According to (2.9) and (2.10), it is easy to prove $EA_{10}G = 0$. Hence, we have

$$\min_{\text{for any } A \in S_E} \|A^* - A\| \Leftrightarrow \min_{F \text{ denoted by (3.1)}} \|E(EA_1^*G - EFG)\|.$$

It is clear that if $\hat{F} = A_1^* + \bar{E}\bar{F}\bar{G}$, for any $\bar{F} \in GCSR^{n \times n}$, then

$$\|EA_1^*G - E\hat{F}G\| = 0. \quad (3.11)$$

Combining Lemma 1, (2.11) and (3.10), we have

$$\bar{E} = K \begin{pmatrix} I_{n-k} - E_1 & 0 \\ 0 & I_k - E_2 \end{pmatrix} K^T, \quad \bar{G} = K \begin{pmatrix} I_{n-k} - G_1 & 0 \\ 0 & I_k - G_2 \end{pmatrix} K^T,$$

where E_1, E_2, G_1, G_2 are denoted by (2.11).

$$\bar{F} = K \begin{pmatrix} \bar{F}_1 & 0 \\ 0 & \bar{F}_2 \end{pmatrix} K^T, \bar{F}_1 \in R^{(n-k) \times (n-k)}, \bar{F}_2 \in R^{k \times k}.$$

Denote

$$Q_1^T (I_{n-k} - E_1) \bar{F}_1 (I_{n-k} - G_1) P_1 = \begin{pmatrix} \bar{F}_{111} & \bar{F}_{112} & \bar{F}_{113} \\ \bar{F}_{121} & \bar{F}_{122} & \bar{F}_{123} \\ \bar{F}_{131} & \bar{F}_{132} & \bar{F}_{133} \end{pmatrix}, \quad Q_2^T (I_k - E_2) \bar{F}_2 (I_k - G_2) P_2 = \begin{pmatrix} \bar{F}_{211} & \bar{F}_{212} & \bar{F}_{213} \\ \bar{F}_{221} & \bar{F}_{222} & \bar{F}_{223} \\ \bar{F}_{231} & \bar{F}_{232} & \bar{F}_{233} \end{pmatrix}.$$

Combining (3.1) and (3.5), we have

$$\begin{pmatrix} A_{011} & A_{012} \Theta_3^{-1} & F_{113} \\ \Theta_1^{-1} A_{021} & \Theta_1^{-1} (A_{022} - \Theta_2 F_{222} \Theta_4) \Theta_3^{-1} & F_{123} \\ F_{131} & F_{132} & F_{133} \end{pmatrix} = \begin{pmatrix} A_{111}^* & A_{112}^* & A_{113}^* \\ A_{121}^* & A_{122}^* & A_{123}^* \\ A_{131}^* & A_{132}^* & A_{133}^* \end{pmatrix} + \begin{pmatrix} \bar{F}_{111} & \bar{F}_{112} & \bar{F}_{113} \\ \bar{F}_{121} & \bar{F}_{122} & \bar{F}_{123} \\ \bar{F}_{131} & \bar{F}_{132} & \bar{F}_{133} \end{pmatrix}, \quad (3.12)$$

$$\begin{pmatrix} F_{211} & F_{212} & F_{213} \\ F_{221} & F_{222} & \Theta_2^{-1} A_{023} \\ F_{231} & A_{032} \Theta_4^{-1} & A_{033} \end{pmatrix} = \begin{pmatrix} A_{211}^* & A_{212}^* & A_{213}^* \\ A_{221}^* & A_{222}^* & A_{223}^* \\ A_{231}^* & A_{232}^* & A_{233}^* \end{pmatrix} + \begin{pmatrix} \bar{F}_{211} & \bar{F}_{212} & \bar{F}_{213} \\ \bar{F}_{221} & \bar{F}_{222} & \bar{F}_{223} \\ \bar{F}_{231} & \bar{F}_{232} & \bar{F}_{233} \end{pmatrix}. \quad (3.13)$$

(3.12) and (3.13) imply that if $\bar{F}$ satisfies the following conditions, then we can also obtain (3.11).

$$\begin{aligned} \bar{F}_{111} &= A_{011} - A_{111}^*, \bar{F}_{112} = A_{012} \Theta_3^{-1} - A_{112}^*, \bar{F}_{113} = 0, \bar{F}_{121} = \Theta_1^{-1} A_{021} - A_{121}^*, \\ \bar{F}_{122} &= \Theta_1^{-1} (A_{022} - \Theta_2 F_{222} \Theta_4) \Theta_3^{-1} - A_{122}^*, \bar{F}_{123} = 0, \bar{F}_{131} = 0, \bar{F}_{132} = 0, \bar{F}_{133} = 0. \\ \bar{F}_{211} &= 0, \bar{F}_{212} = 0, \bar{F}_{213} = 0, \bar{F}_{221} = 0, \bar{F}_{222} = F_{222} - A_{222}^*, \\ \bar{F}_{223} &= \Theta_2^{-1} A_{023} - A_{223}^*, \bar{F}_{231} = 0, \bar{F}_{232} = A_{032} \Theta_4^{-1} - A_{232}^*, \bar{F}_{233} = A_{033} - A_{233}^*. \end{aligned} \quad (3.14)$$

According to (3.14), we have

$$\begin{aligned} F_{113} &= A_{113}^*, F_{123} = A_{123}^*, F_{131} = A_{131}^*, F_{132} = A_{132}^*, F_{133} = A_{133}^*, \\ F_{211} &= A_{211}^*, F_{212} = A_{212}^*, F_{213} = A_{213}^*, F_{221} = A_{221}^*, F_{222} = A_{222}^*, F_{231} = A_{231}^*. \end{aligned} \quad (3.15)$$

(3.15) gives the results of Theorem 2. □

Algorithm

1. Input A^*, A_0 , and input X, Y, Λ, Γ according to (2.5).
2. Compute $Y_1^T X_1 \Lambda_1, \Gamma_1 Y_1^T X_1, X_1 \Lambda_1, X_1 \Lambda_1 X_1^+ X_1, Y_1 \Gamma_1, Y_1 \Gamma_1 Y_1^+ Y_1, Y_2^T X_2 \Lambda_2, \Gamma_2 Y_2^T X_2, X_2 \Lambda_2, X_2 \Lambda_2 X_2^+ X_2, Y_2 \Gamma_2, Y_2 \Gamma_2 Y_2^+ Y_2$. If (2.6) and (2.7) hold, then go to 3; otherwise stop.
3. Compute A_{10}, E, G according to (2.9) and (2.10), and compute $A_{110}, A_{210}, E_1, E_2, G_1, G_2$ according to (2.11).
4. Compute $\bar{E}_1, \bar{E}_2, \bar{G}_1, \bar{G}_2, \bar{A}_0$ according to (2.12).
5. Compute the GSVD of matrix pairs $(\bar{E}_1^T, \bar{E}_2^T)$ and $(\bar{G}_1, \bar{G}_2)$, respectively.
6. Partition $S^{-1} \bar{A}_0 W^{-T}$ according to (2.20). If (2.22) holds, go to 7; otherwise stop.
7. Compute A_1^* according to (3.3).
8. Compute A_{11}^*, A_{22}^* according to (3.4), and partition $Q_1^T A_{11}^* P_1, Q_2^T A_{22}^* P_2$ according to (3.5).
9. Compute $\hat{F}_1$ and $\hat{F}_2$ according to (3.8) and (3.9), respectively.
10. Compute $\hat{F}$ according to (3.7).
11. Calculate $\hat{A}$ according to (3.6).

Example ($n = 10, h = 6, l = 2, p = 3$).

Give J and choose a random matrix A in GCSR $^{10 \times 10}$ as follows.

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0.9129 & 0.3401 & 0.3762 & 0.1782 & 0.8459 & 0.2490 & 0.3496 & 0.4997 & 0.8081 & 0.2734 \\ 0.4842 & 0.8901 & 0.4871 & 0.6404 & 0.3066 & 0.4359 & 0.7860 & 0.2541 & 0.7008 & 0.4527 \\ 0.4296 & 0.6955 & 0.6290 & 0.3674 & 0.7706 & 0.3658 & 0.6734 & 0.7211 & 0.2705 & 0.9054 \\ 0.3787 & 0.5113 & 0.2782 & 0.6369 & 0.3789 & 0.6256 & 0.7456 & 0.3834 & 0.6683 & 0.5673 \\ 0.9355 & 0.4189 & 0.4208 & 0.5631 & 0.7648 & 0.3268 & 0.6183 & 0.2866 & 0.4675 & 0.3522 \\ 0.4675 & 0.3522 & 0.2866 & 0.6183 & 0.3268 & 0.7648 & 0.5631 & 0.4208 & 0.9355 & 0.4189 \\ 0.6683 & 0.5673 & 0.3834 & 0.7456 & 0.6256 & 0.3789 & 0.6369 & 0.2782 & 0.3787 & 0.5113 \\ 0.2705 & 0.9054 & 0.7211 & 0.6734 & 0.3658 & 0.7706 & 0.3674 & 0.6290 & 0.4296 & 0.6955 \\ 0.8081 & 0.2734 & 0.4997 & 0.3496 & 0.2490 & 0.8459 & 0.1782 & 0.3762 & 0.9129 & 0.3401 \\ 0.7008 & 0.4527 & 0.2541 & 0.2541 & 0.4359 & 0.3066 & 0.6404 & 0.4871 & 0.4842 & 0.8901 \end{pmatrix}.$$

Compute the eigenvalues and the right eigenvectors of A , choose partial eigenpairs of A and obtain X_1 , X_2 , Λ_1 , Λ_2 according to (2.5).

$$X_1 = \begin{pmatrix} 0.2852 & -0.2264 & -0.2136 \\ 0.3242 & 0.0454 & 0.0978 \\ 0.3553 & 0.5749 & 0 \\ 0.3076 & -0.0458 & 0.1363 \\ 0.3044 & -0.2003 & 0.0148 \\ 0.3044 & -0.2003 & 0.0148 \\ 0.3076 & -0.0458 & 0.1363 \\ 0.3553 & 0.5749 & 0 \\ 0.2852 & -0.2264 & -0.2136 \\ 0.3242 & 0.0454 & 0.0978 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0.3432 & 0.1016 & -0.0780 \\ -0.0483 & 0.5103 & 0 \\ 0.3278 & -0.4345 & 0.1443 \\ -0.2424 & 0.0393 & -0.0108 \\ 0.4623 & -0.1051 & 0.0290 \\ -0.4623 & 0.1051 & -0.0290 \\ 0.2424 & -0.0393 & 0.0108 \\ -0.3278 & 0.4345 & -0.1443 \\ -0.3432 & -0.1016 & 0.0780 \\ 0.0483 & -0.5103 & 0 \end{pmatrix},$$

$$\Lambda_1 = \begin{pmatrix} 5.2477 & 0 & 0 \\ 0 & 0.7218 & 0.2883 \\ 0 & -0.2883 & 0.7128 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0.9026 & 0 & 0 \\ 0 & 0.2113 & 0.0948 \\ 0 & -0.0948 & 0.2113 \end{pmatrix}.$$

Compute the eigenvalues and the right eigenvectors of A^T , choose partial eigenpairs of A^T and obtain Y_1 , Y_2 , Γ_1 , Γ_2 according to (2.5).

$$Y_1 = \begin{pmatrix} -0.1898 \\ -0.4155 \\ 0.1347 \\ -0.0559 \\ 0.5197 \\ 0.5197 \\ -0.0559 \\ 0.1347 \\ -0.1898 \\ -0.4155 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0.3865 \\ 0.0860 \\ 0.2930 \\ 0.4379 \\ -0.2560 \\ 0.2560 \\ -0.4379 \\ -0.2930 \\ -0.3865 \\ -0.0860 \end{pmatrix}, \quad \Gamma_1 = (0.2847), \quad \Gamma_2 = (-0.4610).$$

It is clear that (2.6) and (2.7) hold. Input A_0 is

$$A_0 = \begin{pmatrix} 0.9129 & 0.3401 & 0.3762 \\ 0.4842 & 0.8901 & 0.4871 \\ 0.4296 & 0.6955 & 0.6290 \end{pmatrix}.$$

We can also prove that (2.22) holds. For a given matrix

$$A^* = \begin{pmatrix} -0.4326 & -0.1867 & 0.2944 & -0.3999 & -1.6041 & -1.0106 & 0.0000 & 0.5689 & 0.6232 & 0.3899 \\ -1.6656 & 0.7258 & -1.3362 & 0.6900 & 0.2573 & 0.6145 & -0.3179 & -0.2556 & 0.7990 & 0.08808 \\ 0.1253 & -0.5883 & 0.7143 & 0.8156 & -1.0565 & 0.5077 & 1.0950 & -0.3775 & 0.9409 & -0.6355 \\ 0.2877 & 2.1832 & 1.6236 & 0.7119 & 1.4151 & 1.6924 & -1.8740 & -0.2959 & -0.9921 & -0.5596 \\ -1.1465 & -0.1364 & -0.6918 & 1.2902 & -0.8051 & 0.5913 & 0.4282 & -1.4751 & 0.2120 & 0.4437 \\ 1.1909 & 0.1139 & 0.8580 & 0.6686 & 0.5287 & -0.6436 & 0.8956 & -0.2340 & 0.2379 & -0.9499 \\ 1.1892 & 1.0668 & 1.2540 & 1.1908 & 0.2193 & 0.3803 & 0.7310 & 0.1184 & -1.0078 & 0.7812 \\ -0.0376 & 0.0593 & -1.5937 & -1.2025 & -0.9219 & -1.0091 & 0.5779 & 0.3148 & -0.7420 & 0.5690 \\ 0.3273 & -0.0956 & -1.4410 & -0.0198 & -2.1707 & -0.0195 & 0.0403 & 1.4435 & 1.0823 & -0.8217 \\ 0.1746 & -0.8323 & 0.5711 & -0.1567 & -0.0592 & -0.0482 & 0.6771 & -0.3510 & -0.1315 & -0.2656 \end{pmatrix},$$

by Algorithm, the unique solution of Problem II is

$$\hat{A} = \begin{pmatrix} 0.9266 & 0.3432 & 0.3968 & -0.0394 & 0.6860 & 0.3670 & 0.6008 & 0.4813 & 0.8063 & 0.2759 \\ 0.4930 & 0.8920 & 0.4941 & 0.9728 & 0.5025 & 0.2942 & 0.4168 & 0.2558 & 0.6723 & 0.4423 \\ 0.4356 & 0.6968 & 0.6337 & 0.3239 & 0.7364 & 0.3935 & 0.7209 & 0.7153 & 0.2671 & 0.9055 \\ 0.1859 & 0.7142 & 0.3284 & 0.3009 & 0.9766 & 1.0656 & -0.0199 & 0.39999 & 0.4142 & 0.7545 \\ 0.9811 & 0.4541 & 0.4843 & 0.5740 & 0.7820 & 0.4512 & 0.4709 & 0.2349 & 0.3621 & 0.3539 \\ 0.3621 & 0.3539 & 0.2349 & 0.4709 & 0.4512 & 0.7820 & 0.5740 & 0.4843 & 0.9811 & 0.4541 \\ 0.4142 & 0.7545 & 0.3999 & -0.0199 & 0.0656 & 0.9766 & 0.3009 & 0.3284 & 0.1859 & 0.7142 \\ 0.2671 & 0.9055 & 0.7153 & 0.7209 & 0.3935 & 0.7364 & 0.3239 & 0.6337 & 0.4356 & 0.6968 \\ 0.8063 & 0.2759 & 0.4813 & 0.6008 & 0.3670 & 0.6860 & -0.0394 & 0.3868 & 0.9266 & 0.3432 \\ 0.6723 & 0.4423 & 0.2558 & 0.4168 & 0.2942 & 0.5025 & 0.9728 & 0.4941 & 0.4930 & 0.8920 \end{pmatrix}.$$

4 Conclusion

In this article, we have obtained the necessary and sufficient conditions and associated general solutions of Problem I (Theorem 1). For given matrix $A^* \in R^{n \times n}$, the unique optimal approximation solution of Problem II has been derived (Theorem 2). Our results extend and unify many results for left and right inverse eigenpairs problem, the inverse problem and the inverse eigenvalue problem of centrosymmetric matrices with a submatrix constraint, which is the first motivation of this work. For instance, in Problem I, if $Y = 0$, then this problem becomes Problem I in [17]; in Problem I, if $p = 0$, this problem becomes Problem I in [4–13].

The left and right eigenpairs of a real matrix are not all real eigenpairs, and its complex eigenpairs are all conjugate pairs. Hence, the supposition for Problem I in [4–7,10,11] is not suitable. In this article, we derive the suitable supposition for Problem I (X, Y, A, Γ are given by (2.5)), which is another motivation of this work.

Acknowledgements: This research was supported by the Natural Science Foundation of Hunan Province (2015JJ4090) and by Scientific Fund of Hunan Provincial Education Department of China (Grant no 13C1139). The authors are grateful to the anonymous reviewer and Dr. Justyna Zuk for their valuable comments and careful reading of the original manuscript of this article.

Conflicts of interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

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