Research Article

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The core inverse and constrained matrix approximation problem

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Abstract: In this article, we study the constrained matrix approximation problem in the Frobenius norm by using the core inverse:

$$\|Mx - b\|_F = \min \quad \text{subject to } x \in \mathcal{R}(M),$$

where $M \in \mathbb{C}_n^{\text{CM}}$. We get the unique solution to the problem, provide two Cramer's rules for the unique solution and establish two new expressions for the core inverse.

Keywords: core inverse, Cramer's rule, constrained matrix approximation problem

MSC 2010: 15A24, 15A29, 15A57

1 Introduction

Let M^* , $\mathcal{R}(M)$ and $\mathcal{N}(M)$ stand for the *conjugate transpose*, *range space* and *null space* of $M \in \mathbb{C}^{m \times n}$, respectively. The symbol $M(i \rightarrow b)$ denotes a matrix from M by replacing the i th column of M by $b \in \mathbb{C}^n$. The symbol e_i denotes the i th column of I_n in which $1 \leq i \leq n$. The *Moore-Penrose inverse* of M is the unique matrix $X \in \mathbb{C}^{n \times m}$ satisfying the relations: $MXM = M$, $XXM = X$, $(MX)^* = MX$ and $(XM)^* = XM$ and is denoted by $X = M^\dagger$ [1–3].

Let $M \in \mathbb{C}^{n \times n}$ be singular. The smallest positive integer k for which $\text{rk}(M^{k+1}) = \text{rk}(M^k)$ is called the *index* of M and is denoted by $\text{Ind}(M)$. The index of a non-singular matrix is 0 and the index of a null matrix is 1. Furthermore,

$$\mathbb{C}_n^{\text{CM}} = \{M | \text{Ind}(M) \leq 1, M \in \mathbb{C}^{n \times n}\}. \quad (1.1)$$

Let $M \in \mathbb{C}^{n \times n}$ with $\text{Ind}(M) = k$. A matrix X is the *Drazin inverse* of M if $MXM^k = M^k$, $XXM = X$ and $MX = XM$. We write $X = M^D$ for the Drazin inverse of M . In particular, when $M \in \mathbb{C}_n^{\text{CM}}$, the matrix X is the *group inverse* of M and is denoted by $X = M^\#$ [1–3].

The *core inverse* of $M \in \mathbb{C}_n^{\text{CM}}$ is defined as the unique matrix $X \in \mathbb{C}^{n \times n}$ satisfying the equations: $MXM = M$, $MX^2 = X$ and $(MX)^* = MX$ and is denoted by $X = M^\oplus$ [4,5]. It is noteworthy that the core inverse is a “least squares” inverse [6,7]. Moreover, it is proved that $M^\oplus = M^\#MM^\dagger$ [4].

Recently, the relevant conclusions of the core inverse are very rich. In [7–10], generalizations of core inverse are introduced, for example, the core-EP inverse and the weak group inverse. In [11–15], their algebraic properties and calculating methods are studied. In [16,17], the studying of them is extended to

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some new fields, for example, ring and operator. Moreover, those inverses are used to study partial orders in [4,5,10,18,19].

Consider the following equation:

$$Mx = b. \quad (1.2)$$

Let $M \in \mathbb{C}^{n \times n}$ with $\text{Ind}(M) = k$ and $b \in \mathcal{R}(M^k)$. Campbell and Meyer [20] show that $x = M^D b$ is the unique solution of (1.2) with respect to $x \in \mathcal{R}(M^k)$. Wei [21] gets the minimal P -norm solution of (1.2), where P is nonsingular, $P^{-1}MP$ is the Jordan canonical form of M and $\|x\|_p = \|P^{-1}x\|_2$. Furthermore, let $M \in \mathbb{C}^{m \times n}$. Wei [22] considered the unique solution of

$$WMWx = b \quad \text{subject to} \quad x \in \mathcal{R}((MW)^{k_1}),$$

where $W \in \mathbb{C}^{n \times m}$, $k_1 = \text{Ind}(MW)$, $k_2 = \text{Ind}(WM)$ and $b \in \mathcal{R}((WM)^{k_2})$. More results of (1.2) under some certain conditions can be found in [3,21,23–27].

It is well known that $b \in \mathcal{R}(M)$ if and only if (1.2) is solvable. Let $b \in \mathcal{R}(M)$ and the index of M is 1, then $x = M^\# b$ is the unique solution with $x \in \mathcal{R}(M)$ [20]. It follows from $M^\oplus = M^\# M M^\dagger$ that $M^\# b = M^\oplus b$ [28]. Furthermore, the unique solution $x = M^\oplus b$ is given by Cramer's rule [28, Theorem 3.3].

When $b \notin \mathcal{R}(M)$, (1.2) is unsolvable, yet, it has least-squares solutions. Motivated by the aforementioned works, it is natural to consider the least-squares solutions of (1.2) under the certain condition $x \in \mathcal{R}(M)$, i.e.,

$$\|Mx - b\|_F = \min \quad \text{subject to} \quad x \in \mathcal{R}(M), \quad (1.3)$$

where $M \in \mathbb{C}_n^{\text{CM}}$, $\text{rk}(M) = r < n$ and $b \in \mathbb{C}^n$.

2 Preliminaries

Lemma 2.1. [1] Let $M \in \mathbb{C}^{n \times n}$ be idempotent. Then, $M = P_{\mathcal{R}(M), \mathcal{N}(M)}$ with $\mathcal{R}(M) \oplus \mathcal{N}(M) = \mathbb{C}^n$. In contrast, if $\mathbf{F} \oplus \mathbf{G} = \mathbb{C}^n$, then there exists an idempotent $P_{\mathbf{F}, \mathbf{G}}$ such that $\mathcal{R}(P_{\mathbf{F}, \mathbf{G}}) = \mathbf{F}$ and $\mathcal{N}(P_{\mathbf{F}, \mathbf{G}}) = \mathbf{G}$.

Furthermore, $I_n - P_{\mathbf{F}, \mathbf{G}} = P_{\mathbf{G}, \mathbf{F}}$.

Lemma 2.2. [3] Let $M \in \mathbb{C}^{n \times n}$. Then, $\text{Ind}(M) = k$ if and only if

$$\mathcal{R}(M^k) \oplus \mathcal{N}(M^k) = \mathbb{C}^n. \quad (2.1)$$

Lemma 2.3. [3] Let $MXM = M$ and $XXM = X$. Then,

$$XM = P_{\mathcal{R}(X), \mathcal{N}(M)} \quad \text{and} \quad MX = P_{\mathcal{R}(M), \mathcal{N}(X)}.$$

Lemma 2.4. [3] Let $\mathbf{F} \oplus \mathbf{G} = \mathbb{C}^n$. Then,

- (1) $P_{\mathbf{F}, \mathbf{G}} M = M \Leftrightarrow \mathcal{R}(M) \subseteq \mathbf{F}$;
- (2) $MP_{\mathbf{F}, \mathbf{G}} = M \Leftrightarrow \mathcal{N}(M) \supseteq \mathbf{G}$.

Lemma 2.5. [14] Let $M \in \mathbb{C}_n^{\text{CM}}$ with $\text{rk}(M) = r$. Then, there exists a unitary matrix V such that

$$M = V \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} V^*, \quad (2.2)$$

where $T \in \mathbb{C}^{r \times r}$ is nonsingular. Furthermore,

$$M^\oplus = V \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*. \quad (2.3)$$

3 Main results

3.1 Solution of (1.3)

Theorem 3.1. Let $M \in \mathbb{C}_n^{\text{CM}}$ and $b \in \mathbb{C}^n$. Then,

$$x = M^{\oplus} b \quad (3.1)$$

is the unique solution of (1.3).

Proof. From $x \in \mathcal{R}(M)$, it follows that there exists $y \in \mathbb{C}^n$ for which $x = My$. Then, x is the solution of (1.3) if and only if y is the solution of

$$\|M^2 y - b\|_F = \min. \quad (3.2)$$

Let the decomposition of M be as in (2.2). Denote

$$V^* y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad V^* b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{and} \quad M^{\oplus} b = V \begin{bmatrix} T^{-1} b_1 \\ 0 \end{bmatrix}, \quad (3.3)$$

where y_1, b_1 and $T^{-1}b_1 \in \mathbb{C}^{\text{rk}(M)}$. It follows that

$$\|Mx - b\|_F^2 = \left\| \begin{bmatrix} T^2 y_1 + TSy_2 - b_1 \\ -b_2 \end{bmatrix} \right\|_F^2 = \|T^2 y_1 + TSy_2 - b_1\|_F^2 + \|b_2\|_F^2.$$

Since T is invertible, we have $\min_{y_1, y_2} \|T^2 y_1 + TSy_2 - b_1\|_F^2 = 0$, that is, $\|M^2 y - b\|_F = \min = \|b_2\|_F$, in which $y_2 \in \mathbb{C}^{n-\text{rk}(M)}$ is arbitrary, and $y_1 = T^{-2}b_1 - T^{-1}Sy_2$. It follows that

$$x = My = V \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} V^* y = V \begin{bmatrix} Ty_1 + Sy_2 \\ 0 \end{bmatrix} = V \begin{bmatrix} T^{-1}b_1 \\ 0 \end{bmatrix} = M^{\oplus} b,$$

that is, (3.1) is the unique solution of (1.3). $\square$

3.2 Determinantal formulas

When $M \in \mathbb{C}^{n \times n}$ is nonsingular, it is well known that the solution of (1.2) is unique and $x = M^{-1}b$. Let $x = (x_1, x_2, \dots, x_n)^T$. Then,

$$x_i = \frac{\det(M(i \rightarrow b))}{\det(M)}, \quad i = 1, 2, \dots, n, \quad (3.4)$$

is called Cramer's rule for solving (1.2). In [29], Ben-Israel gets a Cramer's rule for obtaining the least-norm solution of the consistent linear system (1.2),

$$x_i = \frac{\det \begin{pmatrix} M(i \rightarrow b) & U \\ V^*(i \rightarrow 0) & 0 \end{pmatrix}}{\det \begin{pmatrix} M & U \\ V^* & 0 \end{pmatrix}}, \quad i = 1, 2, \dots, n,$$

where U and V are of full column rank, $\mathcal{R}(U) = \mathcal{N}(M^*)$ and $\mathcal{R}(V) = \mathcal{N}(M)$. In [26], Wang gives a Cramer's rule for the unique solution $x \in \mathcal{R}(M^k)$ of (1.2), where $b \in \mathcal{R}(M^k)$ and $\text{Ind}(M) = k$. In [30], Ji proposes two new condensed Cramer's rules for the unique solution $x \in \mathcal{R}(M^k)$ of (1.2), where $b \in \mathcal{R}(M^k)$ and

$\text{Ind}(M) = k$. More details of Cramer's rules for finding restricted solutions of (1.2) can be found in [1,3,31–36]. In Theorems 3.4 and 3.6, we will give two Cramer's rules for the unique solution of (1.3).

First of all, we give the following two lemmas to prepare for a Cramer's rule for core inverse in Theorem 3.4.

Lemma 3.2. *Let $M \in \mathbb{C}_n^{\text{CM}}$ with $\text{rk}(M) = r$, and let $L \in \mathbb{C}^{n \times (n-r)}$ with $\text{rk}(L) = n - r$ and $\mathcal{R}(L) = \mathcal{N}(M^*)$. Then,*

$$M^{\oplus}M + (I_n - M^{\oplus}M)L(L^*L)^{-1}L^* = I_n. \quad (3.5)$$

Proof. Let M be as in (2.2), applying Lemma 2.2, we see that

$$\mathcal{R}(M) \oplus \mathcal{N}(M) = \mathbb{C}^n. \quad (3.6)$$

Denote $M_1 = I_n - M^{\oplus}M$ and $M_2 = L(L^*L)^{-1}L^*$.

Applying Lemmas 2.1, 2.3 and $M^{\oplus}M = M^{\#}M$, we have

$$M^{\oplus}M = P_{\mathcal{R}(M), \mathcal{N}(M)}, \quad (3.7)$$

$$M_1 = I - M^{\oplus}M = P_{\mathcal{N}(M), \mathcal{R}(M)}. \quad (3.8)$$

Since $(L(L^*L)^{-1}L^*)(L(L^*L)^{-1}) = L(L^*L)^{-1}$ and $L^*(L(L^*L)^{-1})L^* = L^*$, applying Lemma 2.3, we obtain

$$M_2 = P_{\mathcal{R}(L), \mathcal{N}(L^*)} = P_{\mathcal{N}(M^*), \mathcal{R}(M)}. \quad (3.9)$$

Since $\mathcal{R}(L) = \mathcal{N}(M^*)$, we obtain $M_2M_1 = M_2$ and

$$M_1M_2 = P_{\mathcal{N}(M), \mathcal{R}(M)}. \quad (3.10)$$

Therefore, applying Lemma 2.1, (3.7) and (3.10), we gain

$$M^{\oplus}M + M_1M_2 = P_{\mathcal{R}(M), \mathcal{N}(M)} + P_{\mathcal{N}(M), \mathcal{R}(M)} = I_n,$$

i.e., (3.5). □

In [28, Theorems 3.2 and 3.3], let $M \in \mathbb{C}_n^{\text{CM}}$, $b \in \mathbb{C}^n$ and $b \in \mathcal{R}(M)$, and let M_b and M_c be of the full column ranks with $\mathcal{N}(M^*) = \mathcal{R}(M_b)$ and $\mathcal{N}(M_c^*) = \mathcal{R}(M)$. Then,

$$\begin{bmatrix} M & M_b \\ M_c^* & 0 \end{bmatrix}$$

is invertible and the unique solution $x = M^{\oplus}b$ of (1.2) satisfying

$$x_i = \det \left(\begin{bmatrix} M(i \rightarrow b) & M_b \\ M_c^*(i \rightarrow 0) & 0 \end{bmatrix} \right) / \det \left(\begin{bmatrix} M & M_b \\ M_c^* & 0 \end{bmatrix} \right),$$

where $i = 1, 2, \dots, n$. In Lemma 3.3 and Theorem 3.4, we give the unique least-squares solution of (1.3) in a similar way.

Lemma 3.3. *Let M and L be as in Lemma 3.2. Then,*

$$G = \begin{bmatrix} M & L \\ L^* & 0 \end{bmatrix} \quad (3.11)$$

is invertible and

$$G^{-1} = \begin{bmatrix} M^{\oplus} & (I_n - M^{\oplus}M)L(L^*L)^{-1} \\ (L^*L)^{-1}L^* & 0 \end{bmatrix}. \quad (3.12)$$

Proof. Since $\mathcal{R}(L) = \mathcal{N}(M^*)$ and $M^\oplus = M^\#MM^\dagger$, we have $M^\oplus L = M^\#MM^\dagger L = 0$ and $(L^*L)^{-1}L^*M = 0$. Furthermore, applying (3.5), we have

$$\begin{bmatrix} M^\oplus & (I_n - M^\oplus M)L(L^*L)^{-1} \\ (L^*L)^{-1}L^* & 0 \end{bmatrix} \begin{bmatrix} M & L \\ L^* & 0 \end{bmatrix} = \begin{bmatrix} M^\oplus M + (I_n - M^\oplus M)L(L^*L)^{-1}L^* & M^\oplus L \\ (L^*L)^{-1}L^*M & (L^*L)^{-1}L^*L \end{bmatrix} = I_{2n-r},$$

that is, G is invertible and G^{-1} is of the form (3.12). $\square$

Based on Lemmas 3.2 and 3.3, we get a Cramer's rule for the unique solution of (1.3).

Theorem 3.4. Let M and b be as in Lemma 3.2, and let L be as in Lemma 3.2. Then, (1.3) has the unique solution $x = (x_1, x_2, \dots, x_n)^T$ satisfying

$$x_i = \det \left(\begin{bmatrix} M(i \rightarrow b) & L \\ L^*(i \rightarrow 0) & 0 \end{bmatrix} \right) / \det \left(\begin{bmatrix} M & L \\ L^* & 0 \end{bmatrix} \right), \quad (3.13)$$

where $i = 1, 2, \dots, n$.

Proof. Since G is invertible, applying Lemma 3.3, we get the unique solution $\hat{x} = G^{-1}\hat{b}$ of $G\hat{x} = \hat{b}$, in which $\hat{x}^* = [x^* \ y^*]^*$ and $\hat{b}^* = [b^* \ 0]^*$. It follows from (3.12) that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} M^\oplus & (I_n - M^\oplus M)L(L^*L)^{-1} \\ (L^*L)^{-1}L^* & 0 \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} M^\oplus b \\ (L^*L)^{-1}L^*b \end{bmatrix}.$$

Applying (3.4) we obtain (3.13). $\square$

In the following theorem, we give a characterization of the core inverse and prepare for a Cramer's rule for the core inverse in Theorem 3.6.

Theorem 3.5. Let M and L be as in Lemma 3.2. Then,

$$M^\oplus = (MM^*M + LL^*)^{-1}MM^*. \quad (3.14)$$

Proof. Since $\mathcal{R}(L) = \mathcal{N}(M^*)$, $M \in \mathbb{C}_n^{\text{CM}}$ and $\mathcal{R}(M)^\perp = \mathcal{N}(M^*)$, we obtain

$$\begin{aligned} (LL^*)(LL^*)^\dagger &= P_{\mathcal{N}(M^*), \mathcal{R}(M)}, \\ (MM^*M)(MM^*M)^\oplus &= P_{\mathcal{R}(M), \mathcal{N}(M^*)} \end{aligned}$$

and

$$\begin{aligned} (MM^*M + LL^*)((MM^*M)^\oplus + (LL^*)^\dagger - (MM^*M)^\oplus MM^*M(LL^*)^\dagger) \\ = (MM^*M)(MM^*M)^\oplus + (LL^*)(LL^*)^\dagger = P_{\mathcal{R}(M), \mathcal{N}(M^*)} + P_{\mathcal{N}(M^*), \mathcal{R}(M)} = I_n. \end{aligned}$$

Therefore, $MM^*M + LL^*$ is invertible.

Since $(LL^*)^\dagger MM^* = 0$ and $(MM^*M)^\oplus MM^* = M^\oplus$, we have

$$(MM^*M + LL^*)^{-1}MM^* = (MM^*M)^\oplus MM^* + (LL^*)^\dagger MM^* = M^\oplus.$$

It follows that we get (3.14). $\square$

Theorem 3.6. Let M and L be as in Lemma 3.2. Then, (1.3) has the unique solution $x = (x_1, x_2, \dots, x_n)^T$ satisfying

$$x_j = \frac{\det(MM^*M + LL^*)(j \rightarrow MM^*b)}{\det(MM^*M + LL^*)}, \quad (3.15)$$

where $j = 1, 2, \dots, n$.

Proof. Applying Theorems 3.5 to 3.1, we have

$$x = (MM^*M + LL^*)^{-1}MM^*b,$$

that is,

$$(MM^*M + LL^*)x = MM^*b.$$

It follows from (3.4) that we get (3.15). $\square$

In [30], Ji obtains the condensed determinantal expressions of $M^\dagger$ and M^D . By using Theorem 3.5, we get a condensed determinantal expression of $M^\oplus$.

Theorem 3.7. Let M and L be defined as in (3.11). Then, the core inverse $M^\oplus$ is given by:

$$M_{i,j}^\oplus = \frac{\det(MM^*M + LL^*)(i \rightarrow (MM^*)e_j)}{\det(MM^*M + LL^*)}, \quad (3.16)$$

where $1 \leq i, j \leq n$.

Proof. Since $MM^*M + LL^*$ is invertible, we consider

$$(MM^*M + LL^*)x = (MM^*)e_j$$

and get the solution

$$e_i^T x = \frac{\det(MM^*M + LL^*)(i \rightarrow (MM^*)e_j)}{\det(MM^*M + LL^*)},$$

in which $i, j = 1, \dots, n$.

It follows from (3.14) and $M_{i,j}^\oplus = e_i^T M^\oplus e_j$ that we get (3.16). $\square$

3.3 Examples

In the following examples, we show that our results are effective.

Example 3.1.

Let $M = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. It is easy to check that $\mathcal{R}(L) = \mathcal{N}(M^*)$. By applying Lemma 3.3,

we have $M^\oplus = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then,

$$(I_n - M^\oplus M)L(L^*L)^{-1} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, (L^*L)^{-1}L^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \det(G) = 1 \text{ and } G^{-1} = \begin{bmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

By applying Theorem 3.1, we get the solution of (1.3) is $x = M^\oplus b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\text{For } \det \begin{pmatrix} 1 & 0 & 0 & -2 & -2 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = 1 \text{ and } \det \begin{pmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 0, \text{ by applying}$$

Theorem 3.4, we get $x_1 = \frac{1}{1}$, $x_2 = \frac{0}{1}$ and $x_3 = \frac{0}{1}$. Therefore, the solution of (1.3) is $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\text{For } \det(MM^*M + LL^*) = \det \begin{pmatrix} 9 & 18 & 18 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 9, \quad MM^*b = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix},$$

$$\det(MM^*M + LL^*)(1 \rightarrow MM^*b) = \det \begin{pmatrix} 9 & 18 & 18 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 9,$$

$\det(MM^*M + LL^*)(2 \rightarrow MM^*b) = 0$, $\det(MM^*M + LL^*)(3 \rightarrow MM^*b) = 0$, and by applying Theorem 3.6, we get

$$x_1 = \frac{9}{9}, \quad x_2 = \frac{0}{9} \text{ and } x_3 = \frac{0}{9}. \text{ Therefore, the solution of (1.3) is } x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Example 3.2.

Let

$$M = \begin{bmatrix} 1/4 & 1/8 & 1/8 \\ 1/4 & 1/8 & 1/8 \\ 1/4 & 0 & 1/4 \end{bmatrix}.$$

Then,

$$M^\# = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 3 & -2 \\ 1 & -5 & 6 \end{bmatrix}, \quad M^\# = \begin{bmatrix} 4/3 & 4/3 & 0 \\ 8/3 & 8/3 & -4 \\ -4/3 & -4/3 & 4 \end{bmatrix}, \quad M^\oplus = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

with $\text{rk}(L) = n - r$ and $\mathcal{R}(L) = \mathcal{N}(M^*)$. It is easy to check that

$$MM^*M + LL^* = \begin{bmatrix} 137/128 & -125/128 & 3/64 \\ -119/128 & 131/128 & 3/64 \\ 5/64 & 3/128 & 7/128 \end{bmatrix}.$$

By applying Theorem 3.5, we get

$$(MM^*M + LL^*)^{-1}MM^* = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix} = M^\oplus.$$

For

$$\begin{aligned} \det(MM^*M + LL^*) &= 3/4096, \\ \det(MM^*M + LL^*)(1 \rightarrow (MM^*)e_1) &= 3/2048, \\ \det(MM^*M + LL^*)(1 \rightarrow (MM^*)e_2) &= 3/2048, \\ \det(MM^*M + LL^*)(1 \rightarrow (MM^*)e_3) &= -3/2048, \\ \det(MM^*M + LL^*)(2 \rightarrow (MM^*)e_1) &= 3/2048, \end{aligned}$$

$$\begin{aligned}
\det(MM^*M + LL^*)(2 \rightarrow (MM^*)e_1) &= 3/2048, \\
\det(MM^*M + LL^*)(2 \rightarrow (MM^*)e_2) &= 3/2048, \\
\det(MM^*M + LL^*)(2 \rightarrow (MM^*)e_3) &= -3/2048, \\
\det(MM^*M + LL^*)(3 \rightarrow (MM^*)e_1) &= -3/2048, \\
\det(MM^*M + LL^*)(3 \rightarrow (MM^*)e_2) &= -3/2048, \\
\det(MM^*M + LL^*)(3 \rightarrow (MM^*)e_3) &= 9/2048,
\end{aligned}$$

by applying Theorem 3.7, we get

$$\begin{aligned}
M_{11} &= 2, & M_{12} &= 2, & M_{13} &= -2, \\
M_{21} &= 2, & M_{22} &= 2, & M_{23} &= -2, \\
M_{31} &= -2, & M_{32} &= -2, & M_{33} &= 6,
\end{aligned}$$

that is,

$$M^{\oplus} = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}.$$

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