Research Article

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A pair of equations in unlike powers of primes and powers of 2

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Abstract: In this article, we show that every pair of large even integers satisfying some necessary conditions can be represented in the form of a pair of one prime, one prime squares, two prime cubes, and 187 powers of 2.

Keywords: circle method, Linnik problem, powers of 2

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1 Introduction

As an approximation to Goldbach's problem, Linnik proved in 1951 [1] under the assumption of the Generalized Riemann Hypothesis (GRH), and later in 1953 [2] unconditionally, that each large even integer N is a sum of two primes p_1, p_2 and a bounded number of powers of 2, namely

$$N = p_1 + p_2 + 2^{v_1} + \cdots + 2^{v_k}. \quad (1.1)$$

In 2002, Heath-Brown and Puchta [3] applied a rather different approach to this problem and showed that $k = 13$ and, on the GRH, $k = 7$. In 2003, Pintz and Ruzsa [4] established this latter result and announced that $k = 8$ is acceptable unconditionally. Elsholtz, in an unpublished manuscript, which is yet to appear in print, showed that $k = 12$; this was proved independently by Liu and Lü [5].

In 1999, Liu et al. [6] proved that every large even integer N can be written as a sum of four squares of primes and a bounded number of powers of 2, namely

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}. \quad (1.2)$$

And Platt and Trudgian [7] got that $k = 45$ suffices.

In 2001, Liu and Liu [8] proved that every large even integer N can be written as a sum of eight cubes of primes and a bounded number of powers of 2, namely

$$N = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{v_1} + \cdots + 2^{v_k}. \quad (1.3)$$

The acceptable value $k = 330$ was determined by Platt and Trudgian [7].

In 2011, Liu and Lü [9] considered a hybrid problem of (1.1)–(1.3),

$$N = p_1 + p_2^2 + p_3^3 + p_4^3 + 2^{v_1} + \cdots + 2^{v_k}. \quad (1.4)$$

They showed that $k = 161$ is acceptable and Platt and Trudgian [7] revised it to 156.

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As a generalization, recently, Hu and Yang [10] first considered the simultaneous representation of pairs of positive even integers $N_2 \gg N_1 > N_2$, in the form

$$\begin{cases} N_1 = p_1 + p_2^2 + p_3^3 + p_4^3 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_5 + p_6^2 + p_7^3 + p_8^3 + 2^{v_1} + \cdots + 2^{v_k}, \end{cases} \quad (1.5)$$

where k is a positive integer. They proved that the simultaneous Eq. (1.5) is solvable for $k = 455$.

The primary purpose of this article is to sharpen this result considerably by establishing the following theorem.

Theorem 1.1. *For $k = 187$, Eq. (1.5) is solvable for every pair of sufficiently large positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$.*

Our proof of Theorem 1.1 uses the Hardy-Littlewood circle method. We make a new estimate of minor arcs and draw on some strategies adopted in the works of Hu and Yang [10] and Kong and Liu [11].

Throughout this article, the letter ϵ denotes a positive constant, which is arbitrarily small but may not be the same at different occurrences.

2 The proof of Theorem 1.1

Throughout this article, we assume that N_i , $i = 1, 2$ are sufficiently large even integers satisfying $N_2 \gg N_1 > N_2$. Then, we set

$$P_i = N_i^{1/9-2\epsilon}, \quad Q_i = N_i^{8/9+\epsilon}, \quad L = \log_2 N_1$$

for $i = 1, 2$.

We define the major arcs $\mathcal{M}_1, \mathcal{M}_2$ and minor arcs $C(\mathcal{M}_1), C(\mathcal{M}_2)$ as usual, namely

$$\mathcal{M}_i = \bigcup_{q \leq P_i} \bigcup_{\substack{1 \leq a \leq q \\ (a, q) = 1}} \mathcal{M}_i(a, q), \quad C(\mathcal{M}_i) = \left[\frac{1}{Q_i}, 1 + \frac{1}{Q_i} \right] \setminus \mathcal{M}_i,$$

where $i = 1, 2$ and

$$\mathcal{M}_i(a, q) = \left\{ \alpha_i : \left| \alpha_i - \frac{a}{q} \right| \leq \frac{1}{qQ_i} \right\}.$$

It follows from the definition of P_i and Q_i that the arcs $\mathcal{M}_1(a, q)$ and $\mathcal{M}_2(a, q)$ are mutually disjoint, respectively. We further define

$$\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 = \{(\alpha_1, \alpha_2) : \alpha_1 \in \mathcal{M}_1, \alpha_2 \in \mathcal{M}_2\}, \quad C(\mathcal{M}) = \left[\frac{1}{Q_i}, 1 + \frac{1}{Q_i} \right]^2 \setminus \mathcal{M}.$$

As in [10], let $\delta = 10^{-4}$ and

$$U_i = \left(\frac{N_i}{16(1 + \delta)} \right)^{1/3}, \quad V_i = U_i^{5/6} \quad (2.1)$$

for $i = 1, 2$. We set

$$f(\alpha_i, N_i) = \sum_{p \leq N_i} (\log p) e(p\alpha_i), \quad g(\alpha_i, N_i) = \sum_{p \leq N_i^{1/2}} (\log p) e(p^2\alpha_i), \quad (2.2)$$

$$S(\alpha_i, U_i) = \sum_{p \sim U_i} (\log p) e(p^3 \alpha_i), \quad T(\alpha_i, V_i) = \sum_{p \sim V_i} (\log p) e(p^3 \alpha_i), \quad (2.3)$$

and set

$$G(\alpha_i) = \sum_{v \leq L} e(2^v \alpha_i), \quad \mathcal{E}_\lambda := \{\alpha_i \in [0, 1] : |G(\alpha_i)| \geq \lambda L\}, \quad (2.4)$$

for $i = 1, 2$.

Let

$$R(N_1, N_2) = \sum \log p_1 \log p_2 \dots \log p_8$$

be the weighted number of solutions of (1.5) in $(p_1, \dots, p_8, v_1, \dots, v_k)$ with

$$p_1 \leq N_1, \quad p_2 \leq N_1^{1/2}, \quad p_3 \sim U_1, \quad p_4 \sim V_1, \quad p_5 \leq N_2, \quad p_6 \leq N_2^{1/2}, \quad p_7 \sim U_2, \quad p_8 \sim V_2, \quad v_j \leq L,$$

for $j = 1, 2, \dots, k$. Then, $R(N_1, N_2)$ can be written as follows:

$$\left(\iint_{\mathcal{M}} + \iint_{C(\mathcal{M}) \cap \mathcal{E}_\lambda} + \iint_{C(\mathcal{M}) \setminus \mathcal{E}_\lambda} \right) f(\alpha_1, N_1) g(\alpha_1, N_1) S(\alpha_1, U_1) T(\alpha_1, V_1) f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) \\ \times T(\alpha_2, V_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 := R_1(N_1, N_2) + R_2(N_1, N_2) + R_3(N_1, N_2).$$

We will establish Theorem 1.1 by estimating the term $R_1(N_1, N_2)$, $R_2(N_1, N_2)$, and $R_3(N_1, N_2)$. We need to show that $R(N_1, N_2) > 0$ for every pair of sufficiently large positive even integers $N_2 \gg N_1 > N_2$.

Let

$$C_i(q, a) = \sum_{h=1}^q e\left(\frac{ah^i}{q}\right)$$

for $i = 1, 2, 3, 4$. Then, we write

$$B(n, q) = \sum_{\substack{a=1 \\ (a, q)=1}}^q C_1(q, a) C_2(q, a) C_3(q, a) C_4(q, a) e\left(-\frac{an}{q}\right), \quad A(n, q) = \frac{B(n, q)}{\varphi^4(q)}, \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q).$$

Now, we need to quote two lemmas from Hu and Yang [10, Lemmas 2.3 and 2.4] as follows:

Lemma 2.1. *For all integers $n \equiv 0 \pmod{2}$, we have $\mathfrak{S}(n) \geq 0.2448$.*

Lemma 2.2. *Let $\mathcal{A}(N_i, k) = \{n_i \geq 2 : n_i = N_i - 2^{v_1} - \dots - 2^{v_k}\}$ with $k \geq 2$. Then, for $N_1 \equiv N_2 \equiv 0 \pmod{2}$, we have*

$$\sum_{\substack{n_1 \in \mathcal{A}(N_1, k) \\ n_2 \in \mathcal{A}(N_2, k) \\ n_1 \equiv n_2 \equiv 0 \pmod{2}}} J(n_1) J(n_2) \geq 5.4671 N_1^{10/9} N_2^{10/9} L^k,$$

where $J(n)$ is defined as

$$J(n) := \sum_{\substack{m_1 + m_2 + m_3 + m_4 = n \\ m_1 \leq N_1, m_2 \leq N_1, U_1^2 < m_3 \leq 8U_1^2, V_1^2 < m_4 \leq 8V_1^2}} m_2^{-1/2} (m_3 m_4)^{-2/3}.$$

Lemma 2.3. *For every pair of sufficiently large positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$,*

$$R_1(N_1, N_2) \geq 0.00101 N_1^{10/9} N_2^{10/9} L^k.$$

Proof. By Hu and Yang [10, Lemma 2.2] we note that

$$\begin{aligned} R_1(N_1, N_2) &= \iint_{\mathcal{M}} f(\alpha_1, N_1) g(\alpha_1, N_1) S(\alpha_1, U_1) T(\alpha_1, V_1) f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2) \\ &\quad \times G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &\geq \left(\frac{1}{2 \cdot 3^2}\right)^2 \sum_{\substack{n_1 \in \mathcal{A}(N_1, k) \\ n_2 \in \mathcal{A}(N_2, k)}} \mathfrak{S}(n_1) \mathfrak{S}(n_2) J(n_1) J(n_2). \end{aligned}$$

Then, using Lemmas 2.1 and 2.2 the lemma now follows. $\square$

Lemma 2.4. *We have*

$$\text{meas}(\mathcal{E}_\lambda) \ll N_1^{-E(\lambda)},$$

with $E(0.9322) > 845/1008 + 10^{-10}$.

Proof. According to Hu and Yang [12, Lemma 5.1], the following lemma can be calculated by computer. $\square$

Lemma 2.5. *For every pair of sufficiently large positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$,*

$$R_2(N_1, N_2) \ll N_1^{10/9} N_2^{10/9} L^{k-1}.$$

Proof. By the definition of $C(\mathcal{M})$, we have

$$C(\mathcal{M}) \subset \{(\alpha_1, \alpha_2) : \alpha_1 \in C(\mathcal{M}_1), \alpha_2 \in [0, 1]\} \cup \{(\alpha_1, \alpha_2) : \alpha_1 \in [0, 1], \alpha_2 \in C(\mathcal{M}_2)\}.$$

Then, $R_2(N_1, N_2)$ is

$$\begin{aligned} &\iint_{C(\mathcal{M}) \cap \mathcal{E}_\lambda} f(\alpha_1, N_1) g(\alpha_1, N_1) S(\alpha_1, U_1) T(\alpha_1, V_1) f(\alpha_2, N_2) g(\alpha_2, N_2) \\ &\quad \times S(\alpha_2, U_2) T(\alpha_2, V_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\ &\ll L^k \left(\iint_{\substack{(\alpha_1, \alpha_2) \in C(\mathcal{M}_1) \times [0, 1] \\ |G(\alpha_1 + \alpha_2)| \geq \lambda L}} + \iint_{\substack{(\alpha_1, \alpha_2) \in [0, 1] \times C(\mathcal{M}_2) \\ |G(\alpha_1 + \alpha_2)| \geq \lambda L}} \right) f(\alpha_1, N_1) g(\alpha_1, N_1) \\ &\quad \times S(\alpha_1, U_1) T(\alpha_1, V_1) f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2) d\alpha_1 d\alpha_2 \\ &:= L^k \left(\iint_1 + \iint_2 \right), \end{aligned}$$

where we use the trivial bound of $G(\alpha_1 + \alpha_2)$. We note that

$$\begin{aligned} \iint_1 &\ll \iint_{\substack{(\alpha_1, \alpha_2) \in C(\mathcal{M}_1) \times [0, 1] \\ |G(\alpha_1 + \alpha_2)| \geq \lambda L}} |f(\alpha_1, N_1) g(\alpha_1, N_1) S(\alpha_1, U_1) T(\alpha_1, V_1) f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2)| d\alpha_1 d\alpha_2 \\ &\ll N_1^{1965/1008 + \epsilon} \iint_{\substack{(\alpha_1, \alpha_2) \in [0, 1]^2 \\ |G(\alpha_1 + \alpha_2)| \geq \lambda L}} |f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2)| d\alpha_1 d\alpha_2, \end{aligned}$$

where we use the bound from [10, Lemma 2.5] and [13, Lemma 2.5] as follows:

$$\begin{aligned} \sup_{\alpha_1 \in C(\mathcal{M}_1)} |f(\alpha_1, N_1)| &\ll N_1^{17/18+\epsilon}, \quad \sup_{\alpha_1 \in C(\mathcal{M}_1)} |g(\alpha_1, N_1)| \ll N_1^{7/16+\epsilon}, \\ \sup_{\alpha_1 \in C(\mathcal{M}_1)} |S(\alpha_1, U_1)| &\ll N_1^{13/42+\epsilon}, \quad \sup_{\alpha_1 \in C(\mathcal{M}_1)} |T(\alpha_1, V_1)| \ll N_1^{65/252+\epsilon}. \end{aligned}$$

By using the integral transformation of $\beta = \alpha_1 + \alpha_2$ and the periodicity of $G(\beta)$, we have

$$\begin{aligned} &\iint_{\substack{(\alpha_1, \alpha_2) \in [0, 1]^2 \\ |G(\alpha_1 + \alpha_2)| \geq \lambda L}} |f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2)| d\alpha_1 d\alpha_2 \\ &= \int_0^1 |f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2)| \left(\int_{\substack{\beta \in [\alpha_2, 1+\alpha_2] \\ |G(\beta)| \geq \lambda L}} d\beta \right) d\alpha_2. \end{aligned}$$

By the simple orthogonality and [14, Lemma 4.12], we have

$$\int_0^1 |f^2(\alpha_2, N_2)| d\alpha_2 \ll N_2^{1+\epsilon}$$

and

$$\int_0^1 |g^2(\alpha_2, N_2) S^2(\alpha_2, U_2) T^2(\alpha_2, V_2)| d\alpha_2 \ll N_2^{11/9+\epsilon}.$$

From Lemma 2.4 and Cauchy's inequality, we have

$$\iint_{\substack{(\alpha_1, \alpha_2) \in [0, 1]^2 \\ |G(\alpha_1 + \alpha_2)| \geq \lambda L}} |f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2)| d\alpha_1 d\alpha_2 \ll N_2^{10/9+\epsilon} N_1^{-E(\lambda)}.$$

We choose $\lambda = 0.9322$ and get

$$\iint_1 \ll N_1^{1965/1008-845/1008-\epsilon} N_2 \ll N_1^{10/9-\epsilon} N_2^{10/9},$$

since $N_2 \gg N_1 > N_2$. Similarly,

$$\iint_2 \ll N_2^{1965/1008-845/1008-\epsilon} N_1 \ll N_2^{10/9-\epsilon} N_1^{10/9}.$$

Then,

$$R_2(N_1, N_2) \ll N_1^{10/9-\epsilon} N_2^{10/9} L^k + N_1^{10/9} N_2^{10/9-\epsilon} L^k \ll N_1^{10/9} N_2^{10/9} L^{k-1}. \quad \square$$

Lemma 2.6. For every pair of sufficiently large even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$,

$$R_3(N_1, N_2) \leq 260.757 \lambda^{k-9} N_1^{10/9} N_2^{10/9} L^k.$$

Proof. First, we need to estimate

$$I = \iint_{(\alpha_1, \alpha_2) \in [0, 1]^2} |g^4(\alpha_1, N_1) g^4(\alpha_2, N_2) G^{28}(\alpha_1 + \alpha_2)| d\alpha_1 d\alpha_2.$$

Following the lines in [14], we note that

$$I = \sum_{h \in \mathbb{Z}} r_{14}(h) \sum_{\substack{M_1 \leq p_1^2, p_2^2, p_3^2, p_4^2 \leq N_1 \\ M_2 \leq p_5^2, p_6^2, p_7^2, p_8^2 \leq N_2 \\ p_2^2 + p_3^2 - p_3^2 - p_4^2 = h \\ p_5^2 + p_6^2 - p_7^2 - p_8^2 = h}} \prod_{i=1}^8 \log p_i,$$

where

$$r_{14}(h) = \sum_{\substack{4 \leq v_j, \mu_j \leq L \\ \sum_{j=1}^{14} (2^{v_j} - 2^{\mu_j}) = h}} 1.$$

Let

$$\mathbf{S}(h) = \prod_{p > 2} \left(1 + \frac{\mathbf{B}(p, h)}{(p-1)^4} \right),$$

where

$$\mathbf{B}(p, h) = \sum_{\substack{a=1 \\ (a, q)=1}}^q |C_2(p, a)|^4 e(ah/p).$$

As in the proof of [14, Lemma 3.2], we treat the case $h \neq 0$ and $h = 0$ separately and obtain

$$I \leq 128^2 N_1 N_2 \sum_{h \neq 0} r_{14}(h) \mathbf{S}^2(h) + O(N_1 N_2 L^{26}).$$

Next, we estimate $\sum_{h \neq 0} r_{14}(h) \mathbf{S}^2(h)$. Note that

$$\mathbf{B}(p, h) = \begin{cases} -(p+1)^2, & \text{if } p \equiv 3 \pmod{4} \text{ and } p \nmid h, \\ -(p^2 + 6p + 1) - 4p(p+1) \left(\frac{h}{p} \right), & \text{if } p \equiv 1 \pmod{4} \text{ and } p \nmid h, \\ (p-1)(p+1)^2, & \text{if } p \equiv 3 \pmod{4} \text{ and } p|h, \\ (p-1)(p^2 + 6p + 1), & \text{if } p \equiv 1 \pmod{4} \text{ and } p|h. \end{cases}$$

Then, from the proof of [14, Lemma 4.3] we have

$$\mathbf{S}^2(h) \leq 8.54 \kappa^2(h) \prod_{\substack{p > 5 \\ p|h}} \left(\frac{1 + \frac{b(p)}{(p-1)^4}}{1 + \frac{a(p)}{(p-1)^4}} \right)^2,$$

where

$$\kappa(h) = \begin{cases} \frac{25 + 15 \left(\frac{h}{5} \right)}{32}, & \text{if } 3|h, 5 \nmid h, \\ \frac{3}{2}, & \text{if } 15|h, \\ 0, & \text{if } 3 \nmid h, \end{cases}$$

$$a(p) = \begin{cases} -(p+1)^2, & \text{if } p \equiv 3 \pmod{4}, \\ 3p^2 - 2p - 1, & \text{if } p \equiv 1 \pmod{4}, \end{cases}$$

$$b(p) = \begin{cases} (p-1)(p+1)^2, & \text{if } p \equiv 3 \pmod{4}, \\ (p-1)(p^2 + 6p + 1), & \text{if } p \equiv 1 \pmod{4}. \end{cases}$$

Define the multiplicative function $c(d)$ by

$$1 + \frac{1}{c(p)} = \left(\frac{1 + \frac{b(p)}{(p-1)^4}}{1 + \frac{a(p)}{(p-1)^4}} \right)^2,$$

where d is square-free and $(30, d) = 1$. Then,

$$\begin{aligned} \sum_{h \neq 0} r_{14}(h) S^2(h) &\leq 8.54 \sum_{\substack{(h) \\ h \equiv 0 \pmod{3}}} \kappa^2(h) \prod_{\substack{p>5 \\ p|h}} \left(1 + \frac{1}{c(p)} \right) \\ &\leq 8.54 \sum_{\substack{(h) \\ h \equiv 0 \pmod{3} \\ 5 \nmid h}} \left(\frac{25 + 15 \left(\frac{h}{5} \right)^2}{32} \right)^2 \prod_{\substack{p>5 \\ p|h}} \left(1 + \frac{1}{c(p)} \right) + 8.54 \sum_{\substack{(h) \\ h \equiv 0 \pmod{15}}} \left(\frac{3}{2} \right)^2 \prod_{\substack{p>5 \\ p|h}} \left(1 + \frac{1}{c(p)} \right) \\ &= 8.54 \sum_{\substack{(h) \\ h \equiv 0 \pmod{3}}} \left(\frac{25^2 + 15^2}{32^2} \right) \sum_{\substack{p>5 \\ p|h}} \left(1 + \frac{1}{c(p)} \right) + 8.54 \sum_{\substack{(h) \\ h \equiv 0 \pmod{15}}} \left(\left(\frac{3}{2} \right)^2 - \left(\frac{25}{32} \right)^2 \right) \prod_{\substack{p>5 \\ p|h}} \left(1 + \frac{1}{c(p)} \right) \\ &= 7.09 \sum_{\substack{(h) \\ h \equiv 0 \pmod{3}}} \prod_{\substack{p>5 \\ p|h}} \left(1 + \frac{1}{c(p)} \right) + 14 \sum_{\substack{(h) \\ h \equiv 0 \pmod{15}}} \prod_{\substack{p>5 \\ p|h}} \left(1 + \frac{1}{c(p)} \right) \\ &:= 7.09 \Sigma_1 + 14 \Sigma_2, \end{aligned}$$

where the condition (h) in $\sum_{(h)}$ denotes that the summation is taken over all $v_1, \dots, v_{14}, \mu_1, \dots, \mu_{14}$ satisfying $4 \leq v_j, \mu_j \leq L$ and $h = \sum_{j=1}^{14} (2^{v_j - \mu_j}) \neq 0$.

Let us consider Σ_1 . We have

$$\begin{aligned} \Sigma_1 &\leq \sum_{\substack{d < N^\epsilon \\ p|d \Rightarrow p > 5}} \frac{\mu^2(d)}{c(d)} \sum_{\substack{4 \leq v_j, \mu_j \leq L \\ 1 \leq j \leq 14 \\ 3d|h}} 1 + O(N^{-\epsilon}) \\ &\leq L^{28} \sum_{\substack{d < N^\epsilon \\ p|d \Rightarrow p > 5}} \frac{\mu^2(d)}{c(d)} \frac{1}{\varrho(3d)^{28}} \sum_{\substack{4 \leq v_j, \mu_j \leq \varrho(3d) \\ 1 \leq j \leq 14 \\ 3d|h}} 1 + O(\epsilon) L^{28} + O(N^{-\epsilon}) \\ &\leq (c_1 + \epsilon) L^{28}, \end{aligned}$$

where $\varrho(q)$ denotes the smallest positive integer ϱ , such that $2^{\varrho(q)} \equiv 1 \pmod{q}$ and c_1 is a constant, which we will deal with later. Similarly, $\Sigma_2 \leq (c_2 + \epsilon) L^{28}$, where c_2 is a constant. Set

$$\beta(d) = \left(\frac{1}{\varrho(3d)^{28}} \sum_{\substack{4 \leq v_j, \mu_j \leq \varrho(3d) \\ 1 \leq j \leq 14 \\ 3d|h}} 1 \right)^{-1}$$

and $m(x) = \prod_{e \leq x} 2^e - 1$. Then, we have

$$\sum_{\substack{p|d \Rightarrow p > 5 \\ \beta(d) \leq x}} \frac{\mu^2(d)}{c(d)} \leq \prod_{\substack{p > 5 \\ p|m(x)}} \left(1 + \frac{1}{c(p)} \right) \leq \prod_{p > 5} \left(\frac{1 + \frac{1}{c(p)}}{\left(1 + \frac{1}{p-1} \right)^2} \right) \prod_{\substack{p > 5 \\ p|m(x)}} \left(1 + \frac{1}{p-1} \right)^2 \leq \left(\frac{8}{15} \right)^2 c_3 e^{2\gamma} \log^2 x,$$

where we use the fact $m(x)/\phi(m(x)) \leq e^\gamma \log x$ for $x \geq 9$. Here, $c_3 = \prod_{p>5} \frac{1 + \frac{1}{c(p)}}{\left(1 + \frac{1}{p-1}\right)^2} \leq 1.3904^2$ can be found in the proof of [14, Lemma 4.1]. With $M = 35$, we have

$$c_1 = \left(\int_2^M + \int_M^\infty \right) \sum_{\substack{p|d \Rightarrow p>5 \\ \beta(d) \leq x}} \frac{\mu^2(d)}{c(d)} \frac{dx}{x^2} \leq \sum_{\substack{p|d \Rightarrow p>5 \\ \beta(d) \leq M}} \frac{\mu^2(d)}{c(d)} \left(\frac{1}{\beta(d)} - \frac{1}{M} \right) + \left(\frac{8}{15} \right)^2 c_3 e^{2\gamma} \frac{(1 + \log M)^2}{M} \leq 1.12031.$$

The constant $c_2 \leq 1.10302$ can be handled in the similar way. Then, the numerical computations provide

$$\sum_{h \neq 0} r_4(h) \mathbf{S}^2(h) \leq 23.39 L^{28}.$$

So we get

$$I \leq 3.83 \times 10^5 N_1 N_2 L^{28} + O(N_1 N_2 L^{26}).$$

Next, by [11, Lemma 2.3] and [9, Lemma 2.5] we have

$$\iint_{(\alpha_1, \alpha_2) \in [0,1]^2} |f^2(\alpha_1, N_1) f^2(\alpha_2, N_2) G^4(\alpha_1 + \alpha_2)| d\alpha_1 d\alpha_2 \leq 305.8869 N_1 N_2 L^4$$

and

$$\begin{aligned} & \iint_{(\alpha_1, \alpha_2) \in [0,1]^2} |S^4(\alpha_1, U_1) T^4(\alpha_1, V_1) S^4(\alpha_2, U_2) T^4(\alpha_2, V_2)| d\alpha_1 d\alpha_2 \\ & \leq \int_0^1 |S(\alpha_1, U_1) T(\alpha_1, V_1)|^4 d\alpha_1 \int_0^1 |S(\alpha_2, U_2) T(\alpha_2, V_2)|^4 d\alpha_2 \leq 0.1289 N_1^{13/9} N_2^{13/9} \end{aligned}$$

Thus,

$$\begin{aligned} R_3(N_1, N_2) & \leq (\lambda L)^{k-9} \iint_{(\alpha_1, \alpha_2) \in [0,1]^2} |f(\alpha_1, N_1) g(\alpha_1, N_1) S(\alpha_1, U_1) T(\alpha_1, V_1) f(\alpha_2, N_2) g(\alpha_2, N_2) \\ & \quad \times S(\alpha_2, U_2) T(\alpha_2, V_2) G^9(\alpha_1 + \alpha_2)| d\alpha_1 d\alpha_2 \\ & \leq (\lambda L)^{k-9} \left(\iint_{(\alpha_1, \alpha_2) \in [0,1]^2} |f^2(\alpha_1, N_1) f^2(\alpha_2, N_2) G^4(\alpha_1 + \alpha_2)| d\alpha_1 d\alpha_2 \right)^{\frac{1}{2}} \\ & \quad \times \left(\iint_{(\alpha_1, \alpha_2) \in [0,1]^2} |g^4(\alpha_1, N_1) g^4(\alpha_2, N_2) G^{28}(\alpha_1 + \alpha_2)| d\alpha_1 d\alpha_2 \right)^{\frac{1}{4}} \\ & \quad \times \left(\iint_{(\alpha_1, \alpha_2) \in [0,1]^2} |S^4(\alpha_1, U_1) T^4(\alpha_1, V_1) S^4(\alpha_2, U_2) T^4(\alpha_2, V_2)| d\alpha_1 d\alpha_2 \right)^{\frac{1}{4}} \\ & \leq 260.757 \lambda^{k-9} N_1^{10/9} N_2^{10/9} L^k. \end{aligned}$$

□

Combining Lemmas 2.3, 2.5, and 2.6, we obtain

$$R(N_1, N_2) > 0.00101 N_1^{10/9} N_2^{10/9} L^k - 260.757 \lambda^{k-9} N_1^{10/9} N_2^{10/9} L^k.$$

We therefore solve the inequality

$$R(N_1, N_2) > 0$$

and get $k \geq 187$. Consequently, we deduce that every pair of large even integers N_1, N_2 satisfying $N_2 \gg N_1 > N_2$ can be written in the form of (1.5) for $k \geq 187$. Thus, Theorem 1.1 follows.

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