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Crank-Nicolson orthogonal spline collocation method combined with WSGI difference scheme for the two-dimensional time-fractional diffusion-wave equation

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Abstract: In this paper, a discrete orthogonal spline collocation method combining with a second-order Crank-Nicolson weighted and shifted Grünwald integral (WSGI) operator is proposed for solving time-fractional wave equations based on its equivalent partial integro-differential equations. The stability and convergence of the schemes have been strictly proved. Several numerical examples in one variable and in two space variables are given to demonstrate the theoretical analysis.

Keywords: diffusion-wave equation; Crank-Nicolson method; weighted and shifted Grünwald integral operator; orthogonal spline collocation method; Caputo derivative

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1 Introduction

Recently, fractional partial differential equations (FPDEs) have attracted more and more attention, which can be used to describe some physical and chemical phenomenon more accurately than the classical integer-order differential equations. For example, when studying universal electromagnetic responses involving the unification of diffusion and wave propagation phenomena, there are processes that are modeled by equations with time fractional derivatives of order $\gamma \in (1, 2)$ [1]. Generally, the analytical solutions of fractional partial differential equations are difficult to obtain, so many authors have resorted to numerical solution techniques based on convergence and stability. Various kinds of numerical methods for solving FPDEs have been proposed by researchers, such as finite element method [2, 3], finite difference method [4–6], meshless method [7, 8], wavelets method [9], spline collocation method [10–12] and so forth.

In this study, we consider the following two-dimensional time-fractional diffusion-wave equation

$${}_0^C D_t^\gamma u(x, y, t) = \Delta u(x, y, t) - u(x, y, t) + f(x, y, t), (x, y, t) \in \Omega \times (0, T] \quad (1.1)$$

subject to the initial condition

$$u(x, y, 0) = \phi(x, y), \quad \frac{\partial u(x, y, 0)}{\partial t} = \varphi(x, y), \quad (x, y) \in \Omega, \quad (1.2)$$

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and the boundary condition

$$u(x, y, t) = 0, (x, y, t) \in \partial\Omega \times (0, T], \quad (1.3)$$

where Δ is Laplace operator, $\Omega = [0, 1] \times [0, 1]$ with boundary $\partial\Omega$, $\phi(x, y)$, $\varphi(x, y)$ and $f(x, y, t)$ are given sufficiently smooth functions in their respective domains and ${}_0^C D_t^\gamma$ denotes the Caputo derivative of order γ ($1 < \gamma < 2$), which reads as follows:

$${}_0^C D_t^\gamma u(x, y, t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{\partial^2 u(x, y, s)}{\partial s^2} (t-s)^{1-\gamma} ds,$$

in which $\Gamma(\cdot)$ is the Gamma function. Without loss of generality, we assume that $\phi(x, y) \equiv 0$ in (1.2), since we can solve the equation for $v(x, y, t) = u(x, y, t) - \phi(x, y)$ in general.

Most of the numerical algorithms in [1–8] employed the L_1 scheme to approximate fractional derivatives. Recently, Tian et al. [13] proposed second- and third-order approximations for Riemann-Liouville fractional derivative via the weighted and shifted Grünwald difference (WSGD) operators. Thereafter, some related research work covering the WSGD idea were done by many scholars. In [14], Liu et al. developed a high-order local discontinuous Galerkin method combined with WSGD approximation for a Caputo time-fractional sub-diffusion equation. In [15], Chen considered the numerical solutions of the multi-term time fractional diffusion and diffusion-wave equations with variable coefficients, which the time fractional derivative was approximated by WSGD operator. In [16], Yang proposed a new numerical approximation, using WSGD operator with second order in time direction and orthogonal spline collocation method in spatial direction, for the two-dimensional distributed-order time fractional reaction-diffusion equation. Following the idea of WSGD operator, Wang and Vong [17] used compact finite difference WSGI scheme for the temporal Caputo fractional diffusion-wave equation. However, the numerical methods with WSGI approximation have been rarely studied. Cao et al. [18] applied the idea of WSGI approximation combining with finite element method to solve the time fractional wave equation.

Orthogonal spline collocation (OSC) method has evolved as a valuable technique for solving different types of partial differential equations [19–23]. The popularity of OSC is due to its conceptual simplicity, wide applicability and easy implementation. Comparing with finite difference method and the Galerkin finite element method, OSC method has the following advantages: the calculation of the coefficients in the equation determining the approximate solution is fast since there is no need to calculate the integrals; and it provides approximations to the solution and spatial derivatives. Moreover, OSC scheme always leads to the almost block diagonal linear system, which can be solved by the software packages efficiently [24]. Another feature of OSC method lies in its super-convergence [25].

Motivated and inspired by the work mentioned above, the main goal of this paper is to propose a high-order OSC approximation method combined with second order WSGI operator for solving two-dimensional time-fractional wave equation, which is abbreviated as WSGI-OSC in forthcoming sections. The remainder of the paper is organized as follows. In Section 2, some notations and preliminaries are presented. In Section 3, the fully discrete scheme combining WSGI operator with second order and orthogonal spline collocation scheme is formulated. Stability and convergence analysis of WSGI-OSC scheme are presented in Section 4. Section 5 provides detailed description of the WSGI-OSC scheme. In Section 6, several numerical experiments are carried out to confirm the convergence analysis. Finally, the conclusion is drawn in Section 7.

2 Discrete-time OSC scheme

2.1 Preliminaries

In this section, we will introduce some notations and basic lemmas. For some positive integers N_x and N_y , δ_x and δ_y are two uniform partitions of $\bar{I} = [0, 1]$ which are defined as follows:

$$\delta_x : 0 = x_0 < x_1 < \cdots < x_{N_x} = 1, \quad \delta_y : 0 = y_0 < y_1 < \cdots < y_{N_y} = 1,$$

and $h_i^x = x_i - x_{i-1}$, $I_i^x = (x_{i-1}, x_i)$, $1 \leq i \leq N_x$, and $h_j^y = y_j - y_{j-1}$, $I_j^y = (y_{j-1}, y_j)$, $1 \leq j \leq N_y$, $h = \max(\max_{1 \leq i \leq N_x} h_i^x, \max_{1 \leq j \leq N_y} h_j^y)$. Let $M_r(\delta_x)$ and $M_r(\delta_y)$ be the space of piecewise polynomial of degree at most $r \geq 3$, defined by

$$M_r(\delta_x) = \{v \in C^1[0, 1] : v|_{I_i^x} \in P_r, 1 \leq i \leq N_x, v(0) = v(1) = 0\},$$

$$M_r(\delta_y) = \{v \in C^1[0, 1] : v|_{I_j^y} \in P_r, 1 \leq j \leq N_y, v(0) = v(1) = 0\},$$

where P_r denotes the set of polynomial of degree at most r . It is easy to know that the dimension of the spaces $M_x(\delta_x)$ and $M_y(\delta_y)$ are $(r-1)N_x := M_x$ and $(r-1)N_y := M_y$, respectively.

Let $\delta = \delta_x \otimes \delta_y$ be a quasi-uniform partition of Ω , and $M_r(\delta) = M_r(\delta_x) \otimes M_r(\delta_y)$ with the dimension of $M_x \times M_y$. Let $\{\lambda_j\}_{j=1}^{r-1}$ denotes the nodes for the $\{r-1\}$ -point Gaussian quadrature rule on the interval $\bar{I}$ with corresponding weights $\{\omega_j\}_{j=1}^{r-1}$. Denote by

$$\mathcal{G}_x = \{\xi_{i,l}^x\}_{i,l=1}^{N_x, r-1} \text{ and } \mathcal{G}_y = \{\xi_{j,m}^y\}_{j,m=1}^{N_y, r-1}$$

as the sets of Gauss points in x and y direction, respectively, where

$$\xi_{i,l}^x = x_{i-1} + h_i^x \lambda_l, \quad \xi_{j,m}^y = y_{j-1} + h_j^y \lambda_m, \quad 1 \leq l, m \leq r-1.$$

Let $\mathcal{G} = \{\xi = (\xi^x, \xi^y) : \xi^x \in \mathcal{G}_x, \xi^y \in \mathcal{G}_y\}$. For the functions u and v defined on $\mathcal{G}$, the inner product $\langle u, v \rangle$ and norm $\|v\|_{M_r}$ are respectively defined by

$$\langle u, v \rangle = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} h_i^x h_j^y \sum_{l=1}^{r-1} \sum_{m=1}^{r-1} \omega_l \omega_m (uv)(\xi_{i,l}^x, \xi_{j,m}^y), \quad \|v\|_{M_r}^2 = \langle v, v \rangle.$$

For m a nonnegative integer, let $H^m(\Omega)$ denotes the usual Sobolev space with norm

$$\|v\|_{H^m} = \left(\sum_{l=0}^m \sum_{i+j=l} \left\| \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \right\|^2 \right)^{\frac{1}{2}},$$

where the norm $\|\cdot\|$ denotes the usual L_2 norm, sometimes it is written as $\|\cdot\|_{H^0}$ for convenience. The following important lemmas are required in our forthcoming analysis. First, we introduce the differentiable (resp. twice differentiable) map $W : [0, T] \rightarrow M_r(\delta)$ by

$$\Delta(u - W) = 0 \text{ on } \mathcal{G} \times [0, T], \quad (2.1)$$

where u is the solution of the Eqs.(1.1)-(1.3). Then we have the following estimates for $u - W$ and its time derivatives.

Lemma 2.1. [26] If $\partial^l u / \partial t^l \in H^{r+3-j}$, for all $t \in [0, T]$, $l = 0, 1, 2$, $j = 0, 1, 2$, and W is defined by (2.1), then there exists a constant C such that

$$\left\| \frac{\partial^l(u - W)}{\partial t^l} \right\|_{H^j} \leq Ch^{r+1-j} \left\| \frac{\partial^l u}{\partial t^l} \right\|_{H^{r+3-j}}. \quad (2.2)$$

Lemma 2.2. [26] If $\partial^i u / \partial t^i \in H^{r+3}$, for $t \in [0, T]$, $i = 0, 1$, then

$$\left\| \frac{\partial^{l+i}(u - W)}{\partial x^{l_1} \partial y^{l_2} \partial t^i} \right\|_{M_r} \leq Ch^{r+1-l} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{H^{r+3}}, \quad (2.3)$$

where $0 \leq l = l_1 + l_2 \leq 4$.

Lemma 2.3. [27] If $u, v \in M_r(\delta)$, then

$$\langle -\Delta u, v \rangle = \langle u, -\Delta v \rangle, \quad (2.4)$$

and there exists a positive constant C such that

$$\langle -\Delta u, u \rangle \geq C \|\nabla u\|^2 \geq 0. \quad (2.5)$$

Lemma 2.4. [28] The norms $\|\cdot\|_{M_r}$ and $\|\cdot\|$ are equivalent on $M_r(\delta)$.

Throughout the paper, we denote $C > 0$ a constant which is independent of mesh sizes h and τ . The following Young's inequality will also be used repeatedly,

$$XY \leq \varepsilon X^2 + \frac{1}{4\varepsilon} Y^2, \quad X, Y \in \mathbb{R}, \varepsilon > 0. \quad (2.6)$$

2.2 Construction of the fully discrete orthogonal spline collocation scheme

In this subsection, we consider discrete-time OSC schemes for solving the Eqs.(1.1)-(1.3). Our main idea of the proposed method is to transform the time fractional diffusion-wave equation into its equivalent partial integro-differential equation. To construct the continuous-time OSC scheme to the solution u of (1.1), we introduce the Riemann-Liouville fractional integral which is defined by

$${}_0\mathcal{J}_t^\alpha u(x, y, t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(x, y, s)}{(t-s)^{1-\alpha}} ds, \quad (2.7)$$

where $0 < \alpha = \gamma - 1 < 1$.

We integrate the equation(1.1) using Riemann-Liouville fractional integral operator ${}_0\mathcal{J}_t^\alpha$ defined in (2.7), then the problem is transformed into its equivalent partial integro-differential equation as follows

$$u_t(x, y, t) - {}_0\mathcal{J}_t^\alpha \Delta u(x, y, t) + {}_0\mathcal{J}_t^\alpha u(x, y, t) = {}_0\mathcal{J}_t^\alpha f(x, y, t) + \varphi(x, y). \quad (2.8)$$

Let $t_k = k\tau$, $k = 0, 1, \dots, N$, where $\tau = T/N$ is the time step size. For the convenience of description, we define $D_t u^{n+1} = \frac{u^{n+1} - u^n}{\tau}$, and $u^{n+\frac{1}{2}} = \frac{u^{n+1} + u^n}{2}$, where $u^n \triangleq u(x, y, t_n)$. Based on the idea of weighted and shifted Grünwald difference operator, Wang and Vong ([17]) established the second order accuracy approximation formula of the Riemann-Liouville fractional integral operator ${}_0\mathcal{J}_t^\alpha u^{n+1}$, which is called as WSGI approximation,

$${}_0\mathcal{J}_t^\alpha u^{n+1} = \tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} u^{n+1-k} + \tilde{E} \triangleq {}_0I_t^\alpha u^{n+1} + \tilde{E}, \quad (2.9)$$

where $\tilde{E} = O(\tau^2)$ and

$$\lambda_0^{(\alpha)} = (1 - \frac{\alpha}{2})\omega_0^{(\alpha)}, \quad \lambda_k^{(\alpha)} = (1 - \frac{\alpha}{2})\omega_k^{(\alpha)} + \frac{\alpha}{2}\omega_{k-1}^{(\alpha)}, \quad k \geq 1, \quad (2.10)$$

here

$$\omega_k^{(\alpha)} = (-1)^k \binom{-\alpha}{k}, \quad \omega_0^{(\alpha)} = 1, \quad \omega_k^{(\alpha)} = \left(1 + \frac{\alpha-1}{k}\right) \omega_{k-1}^{(\alpha)}, \quad k \geq 1. \quad (2.11)$$

By using the Crank-Nicolson difference scheme and WSGI approximation formula to discretize the equation (2.8), we obtain the semi-discrete scheme in time direction

$$D_t u^{n+1} - {}_0I_t^\alpha \Delta u^{n+\frac{1}{2}} + {}_0\mathcal{J}_t^\alpha u^{n+\frac{1}{2}} = g^{n+\frac{1}{2}} + E^{n+\frac{1}{2}}, \quad (2.12)$$

where $g^{n+\frac{1}{2}} = {}_0\mathcal{J}_t^\alpha f^{n+\frac{1}{2}} + \varphi(x, y)$, $E^{n+\frac{1}{2}} = \tilde{E} + E_c^{n+\frac{1}{2}} = O(\tau^2)$, $E_c^{n+\frac{1}{2}} = D_t u^{n+\frac{1}{2}} - u_t(t_{n+\frac{1}{2}}) = O(\tau^2)$. Then by using (2.9), (2.12), the fully discrete WSGI-OSC scheme for Eqs(1.1) consists in finding $\{u_h^n\}_{n=0}^{N-1} \subset M_r(\delta)$ such that

$$\frac{u_h^{n+1} - u_h^n}{\tau} - \tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} \Delta u_h^{n+\frac{1}{2}-k} + \tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} u_h^{n+\frac{1}{2}-k} = g^{n+\frac{1}{2}}. \quad (2.13)$$

For the needs of analysis, we give the following equivalent Galerkin weak formulation of the equation (2.12) by multiplying the equation with $v \in H_0^1$ and integrating with respect to spatial domain Ω

$$(D_t u^{n+1}, v) + ({}_0 I_t^\alpha \nabla u^{n+\frac{1}{2}}, \nabla v) + ({}_0 I_t^\alpha u^{n+\frac{1}{2}}, v) = (g^{n+\frac{1}{2}}, v) + (E^{n+\frac{1}{2}}, v). \quad (2.14)$$

We take the space $M_r(\delta) \subset H_0^1$ and obtain the fully discrete scheme as follows:

$$\left(\frac{u_h^{n+1} - u_h^n}{\tau}, v_h \right) + \tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} (\nabla u_h^{n+\frac{1}{2}-k}, \nabla v_h) + \tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} (u_h^{n+\frac{1}{2}-k}, v_h) = (g^{n+\frac{1}{2}}, v_h), \forall v_h \in M_r(\delta) \quad (2.15)$$

3 Stability and convergence analysis of WSGI-OSC scheme

In this section, we will give the stability and convergence analysis for fully-discrete WSGI-OSC scheme (2.13). To this end, we further need the following lemmas.

Lemma 3.1. [17] Let $\{\lambda_k^{(\alpha)}\}_{k=0}^\infty$ defined in (2.10), then for any positive integer k and real vector $(v_1, v_2, \dots, v_k)^T \in \mathbb{R}^k$, it holds that

$$\sum_{n=0}^{k-1} \left(\sum_{p=0}^n \lambda_p^{(\alpha)} v_{n+1-p} \right) v_{n+1} \geq 0.$$

Lemma 3.2. (Gronwall's inequality) [29] Assume that k_n and p_n are nonnegative sequence, and the sequence ϕ_n satisfies

$$\phi_0 \leq g_0, \quad \phi_n \leq \phi_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} k_l p_l, \quad n \geq 1,$$

where, $g_0 \geq 0$. Then the sequence ϕ_n satisfies

$$\phi_n \leq \left(g_0 + \sum_{l=0}^{n-1} p_l \right) \exp \left(\sum_{l=0}^{n-1} k_l \right), \quad n \geq 1.$$

Theorem 3.1. The fully-discrete WSGI-OSC scheme (2.15) is unconditionally stable for sufficiently small $\tau > 0$, it holds

$$\|u_h^{L+1}\|^2 \leq C \left(\|u_h^0\|^2 + \max_{0 \leq n \leq N-1} \|g^{n+\frac{1}{2}}\|^2 \right), \quad 1 \leq L \leq N-1. \quad (3.1)$$

Proof. Taking $v_h = u_h^{n+\frac{1}{2}} = \frac{u_h^{n+1} + u_h^n}{2}$ in (2.15) and applying the Cauchy-Schwarz inequality and Young inequality, it gives that

$$\frac{1}{2\tau} (\|u_h^{n+1}\|^2 - \|u_h^n\|^2) + \tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} \left[(\nabla u_h^{n+\frac{1}{2}-k}, \nabla v_h) + (u_h^{n+\frac{1}{2}-k}, v_h) \right] \leq \frac{1}{2} (\|g^{n+\frac{1}{2}}\|^2 + \|u_h^{n+\frac{1}{2}}\|^2). \quad (3.2)$$

Summing (3.2) for n from 0 to L ($0 \leq n \leq N-1$), we obtain

$$\frac{1}{2\tau} \sum_{n=0}^L (\|u_h^{n+1}\|^2 - \|u_h^n\|^2) + \tau^\alpha \sum_{n=0}^L \sum_{k=0}^n \lambda_k^{(\alpha)} \left[(\nabla u_h^{n+\frac{1}{2}-k}, \nabla v_h) + (u_h^{n+\frac{1}{2}-k}, v_h) \right] \leq \frac{1}{2} \sum_{n=0}^L (\|g^{n+\frac{1}{2}}\|^2 + \|u_h^{n+\frac{1}{2}}\|^2). \quad (3.3)$$

Multiplying the above equation by 2τ , also using Lemma 1, then dropping the nonnegative terms

$$2\tau^{\alpha+1} \sum_{n=0}^L \sum_{k=0}^n \lambda_k^{(\alpha)} \left[(\nabla u_h^{n+\frac{1}{2}-k}, \nabla v_h) + (u_h^{n+\frac{1}{2}-k}, v_h) \right],$$

we have

$$\begin{aligned}
 \|u_h^{L+1}\|^2 &\leq \|u_h^0\|^2 + \tau \sum_{n=0}^L (\|g^{n+\frac{1}{2}}\|^2 + \|u_h^{n+\frac{1}{2}}\|^2) \\
 &\leq \|u_h^0\|^2 + T \max_{0 \leq n \leq N-1} \|g^{n+\frac{1}{2}}\|^2 + \tau \sum_{n=0}^L \|u_h^{n+\frac{1}{2}}\|^2 \\
 &\leq \|u_h^0\|^2 + T \max_{0 \leq n \leq N-1} \|g^{n+\frac{1}{2}}\|^2 + \tau \sum_{n=0}^L \frac{1}{2} (\|u_h^{n+1}\|^2 + \|u_h^n\|^2).
 \end{aligned} \tag{3.4}$$

Then, it gives that,

$$(1 - \frac{1}{2}\tau) \|u_h^{L+1}\|^2 \leq (1 + \frac{1}{2}\tau) \|u_h^0\|^2 + T \max_{0 \leq n \leq N-1} \|g^{n+\frac{1}{2}}\|^2 + \tau \sum_{n=1}^L \|u_h^n\|^2. \tag{3.5}$$

Provided the time step τ is sufficiently small, there exists a positive constant C such that

$$\|u_h^{L+1}\|^2 \leq C(\|u_h^0\|^2 + T \max_{0 \leq n \leq N-1} \|g^{n+\frac{1}{2}}\|^2 + \tau \sum_{n=1}^L \|u_h^n\|^2). \tag{3.6}$$

Using Gronwall's Lemma 3.2, we get

$$\|u_h^{L+1}\|^2 \leq C(\|u_h^0\|^2 + \max_{0 \leq n \leq N-1} \|g^{n+\frac{1}{2}}\|^2). \tag{3.7}$$

The proof is complete.

Theorem 3.2. Suppose u is the exact solution of (1.1)-(1.3), and $u_h^n (0 \leq n \leq N-1)$ is the solution of the problem (2.13) with $u_h^0 = W^0$, then there exists a positive constant C , independent of h and τ such that

$$\|u(t_n) - u_h^n\|^2 \leq C(\tau^2 + h^{r+1}). \tag{3.8}$$

Proof. With W defined in (2.1), we set

$$\eta^n = W^n - u^n, \quad \zeta^n = u_h^n - W^n, \quad 0 \leq n \leq N, \tag{3.9}$$

thus we have

$$u^n - u_h^n = \eta^n + \zeta^n. \tag{3.10}$$

Because the estimate of η^n are provided by Lemma 2.2, it is sufficient to bound ζ^n , then use the triangle inequality to bound $u^n - u_h^n$. Firstly, from (1.1), (2.1), (2.13), and (2.15), then for $v_h \in M_r(\delta)$, we obtain

$$\begin{aligned}
 (\frac{\eta^{n+1} - \eta^n}{\tau}, v_h) + \tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} (\nabla \eta^{n+\frac{1}{2}-k}, \nabla v_h) + \tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} (\eta^{n+\frac{1}{2}-k}, v_h) \\
 = -\tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} (\zeta^{n+\frac{1}{2}-k}, v_h) - (\frac{\zeta^{n+1} - \zeta^n}{\tau}, v_h) + (E^{n+\frac{1}{2}}, v_h),
 \end{aligned} \tag{3.11}$$

where $E^{n+\frac{1}{2}}$ is defined in (2.12). Taking $v_h = \eta^{n+\frac{1}{2}}$ in (3.11), we have

$$\begin{aligned}
 (\frac{\eta^{n+1} - \eta^n}{\tau}, \eta^{n+\frac{1}{2}}) + \tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} (\nabla \eta^{n+\frac{1}{2}-k}, \nabla \eta^{n+\frac{1}{2}}) + \tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} (\eta^{n+\frac{1}{2}-k}, \eta^{n+\frac{1}{2}}) \\
 = -\tau^\alpha \sum_{k=0}^n \lambda_k^{(\alpha)} (\zeta^{n+\frac{1}{2}-k}, \eta^{n+\frac{1}{2}}) - (\frac{\zeta^{n+1} - \zeta^n}{\tau}, \eta^{n+\frac{1}{2}}) + (E^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}).
 \end{aligned} \tag{3.12}$$

Multiplying (3.12) by 2τ , and summing from $n = 0$ to $n = L - 1$ ($1 \leq n \leq N + 1$), it follows that

$$\begin{aligned} & \sum_{n=0}^{L-1} (\|\eta^{n+1}\|^2 - \|\eta^n\|^2) + 2\tau^{\alpha+1} \sum_{n=0}^{L-1} \sum_{k=0}^n \lambda_k^{(\alpha)} \left[(\nabla \eta^{n+\frac{1}{2}-k}, \nabla \eta^{n+\frac{1}{2}}) + (\eta^{n+\frac{1}{2}-k}, \eta^{n+\frac{1}{2}}) \right] \\ &= -2\tau^{\alpha+1} \sum_{n=0}^{L-1} \sum_{k=0}^n \lambda_k^{(\alpha)} (\zeta^{n+\frac{1}{2}-k}, \eta^{n+\frac{1}{2}}) - 2\tau \sum_{n=0}^{L-1} \left(\frac{\zeta^{n+1} - \zeta^n}{\tau}, \eta^{n+\frac{1}{2}} \right) + 2\tau \sum_{n=0}^{L-1} (E^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}) \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (3.13)$$

Next, we will give the estimate of I_1 , I_2 and I_3 , respectively.

$$\begin{aligned} I_1 &= -2\tau^{\alpha+1} \sum_{n=0}^{L-1} \sum_{k=0}^n \lambda_k^{(\alpha)} (\zeta^{n+\frac{1}{2}-k}, \eta^{n+\frac{1}{2}}) \\ &= -2\tau^{\alpha+1} \sum_{n=0}^{L-1} \left(\frac{{}_0J_{t_{n+1}}^\alpha \zeta + {}_0J_{t_n}^\alpha \zeta}{2} - \tilde{E}, \eta^{n+\frac{1}{2}} \right) \\ &= -2\tau^{\alpha+1} \sum_{n=0}^{L-1} \left(\frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} \frac{\zeta(x, y, s)}{(t_{n+1} - s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_n} \frac{\zeta(x, y, s)}{(t_n - s)^{1-\alpha}} ds - 2\tilde{E}, \eta^{n+\frac{1}{2}} \right) \\ &\leq \tau \sum_{n=0}^{L-1} \left(\frac{-1}{\Gamma(\alpha)\alpha} [(t_{n+1} - s)^\alpha|_0^{t_{n+1}} + (t_n - s)^\alpha|_0^{t_n}] \max_{0 \leq s \leq t_{n+1}} \|\zeta(x, y, s)\| + \|2\tilde{E}\| \right) \|\eta^{n+\frac{1}{2}}\| \\ &\leq \frac{\tau}{\Gamma(\alpha+1)} \sum_{n=0}^{L-1} \left(2T^\alpha \max_{0 \leq t \leq T} \|\zeta(x, y, t)\| + \|\tilde{E}\| \right) \|\eta^{n+\frac{1}{2}}\| \\ &\leq C\tau \sum_{n=0}^{L-1} \left(2T^\alpha \max_{0 \leq t \leq T} \|\zeta(x, y, t)\| + \|\tilde{E}\| \right) \|\eta^{n+\frac{1}{2}}\| \\ &\leq C\tau \sum_{n=0}^{L-1} \left(\tau^4 + \max_{0 \leq t \leq T} \|\zeta(x, y, t)\|^2 + \|\eta^{n+\frac{1}{2}}\|^2 \right). \end{aligned} \quad (3.14)$$

Taking advantages of mean value theorem and Cauchy-Schwarz inequality as well as Young inequality, we have $t_n \leq t_{n+\theta} \leq t_{n+1}$

$$\begin{aligned} I_2 + I_3 &= -2\tau \sum_{n=0}^{L-1} \left(\frac{\zeta^{n+1} - \zeta^n}{\tau}, \eta^{n+\frac{1}{2}} \right) + 2\tau \sum_{n=0}^{L-1} (E^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}) \\ &= \tau \sum_{n=0}^{L-1} (\|\zeta_t(x, y, t_{n+\theta})\|^2 + \|E^{n+\frac{1}{2}}\|^2 + 2\|\eta^{n+\frac{1}{2}}\|^2). \end{aligned} \quad (3.15)$$

Using Lemma 1, we obtain

$$2\tau^{\alpha+1} \sum_{n=0}^L \sum_{k=0}^n \lambda_k^{(\alpha)} \left[(\nabla \eta^{n+\frac{1}{2}-k}, \nabla \eta) + (\eta^{n+\frac{1}{2}-k}, \eta) \right] \geq 0. \quad (3.16)$$

Substituting (3.14), (3.15), (3.16) in (3.13) and removing the nonnegative terms, we attain

$$\|\eta^L\|^2 \leq \|\eta^0\|^2 + C\tau \sum_{n=0}^{L-1} \left(\tau^4 + \max_{0 \leq t \leq T} \|\zeta(x, y, t)\|^2 + \|\eta^{n+\frac{1}{2}}\|^2 \right) + \tau \sum_{n=0}^{L-1} \left(\|\zeta_t(x, y, t_{n+\theta})\|^2 + \|E^{n+\frac{1}{2}}\|^2 + 2\|\eta^{n+\frac{1}{2}}\|^2 \right), \quad (3.17)$$

that is

$$(1 - C\tau) \|\eta^L\|^2 \leq C\tau \sum_{n=0}^{L-1} \|\eta^n\|^2 + C\tau \sum_{n=0}^{L-1} \left(\tau^4 + \max_{0 \leq t \leq T} \|\zeta(x, y, t)\|^2 + \|\zeta_t(x, y, t_{n+\theta})\|^2 \right). \quad (3.18)$$

Using the Gronwall's inequality, Lemma 2.2 and triangle inequality, in the case that the time step τ is sufficiently small, there exists a positive constant C such that

$$\|\eta^L\|^2 \leq \exp(CT) \cdot C\tau \sum_{n=0}^{L-1} \left(\tau^4 + Ch^{2r+2} \|u\|_{H^{r+3}}^2 + Ch^{2r+2} \|u_t\|_{H^{r+3}}^2 \right) \leq C(\tau^4 + h^{2r+2}) \quad (3.19)$$

and

$$\|u(t_L) - u_h^L\|^2 \leq (\|\eta^L\| + \|\zeta^L\|)^2 \leq C(\tau^4 + h^{2r+2}) \quad (3.20)$$

which completes the proof.

4 Description of the WSGI-OSC scheme

It can be observed from the fully discrete scheme (2.13) that we need to handle a two-dimensional partial differential equation for each time level, that is

$$\begin{aligned} (1 + \frac{1}{2}\tau^{\alpha+1}\lambda_0^{(\alpha)})u_h^{n+1} - \frac{1}{2}\tau^{\alpha+1}\lambda_0^{(\alpha)}\Delta u_h^{n+1} = & -\frac{1}{2}\tau^{\alpha+1} \sum_{k=1}^{n+1} \lambda_k^{(\alpha)} (-\Delta u_h^{n+1-k} + u_h^{n+1-k}) \\ & - \frac{1}{2}\tau^{\alpha+1} \sum_{k=0}^n \lambda_k^{(\alpha)} (-\Delta u_h^{n-k} + u_h^{n-k}) + \tau \frac{g^{n+1} + g^n}{2} + u_h^n \end{aligned} \quad (4.1)$$

We denote $\alpha_0 = \frac{1}{2}\tau^{\alpha+1}\lambda_0^{(\alpha)}$, $\beta_0 = \frac{1}{2}\tau^{\alpha+1}$, then the above equation can be rewritten as

$$\begin{aligned} (1 + \alpha_0)u_h^{n+1} - \alpha_0\Delta u_h^{n+1} = & \beta_0 \sum_{k=1}^{n+1} \lambda_k^{(\alpha)} (\Delta u_h^{n+1-k} - u_h^{n+1-k}) + \beta_0 \sum_{k=0}^n \lambda_k^{(\alpha)} (\Delta u_h^{n-k} - u_h^{n-k}) + \tau \frac{g^{n+1} + g^n}{2} + u_h^n, \\ & n = 0, \dots, N-1. \end{aligned} \quad (4.2)$$

For applying the numerical schemes, firstly, we usually represent u_h^n by the base functions of $M_r(\delta)$, then solve the coefficients of the representation formula. Letting

$$M_r(\delta_x) = \text{span}\{\Phi_1, \Phi_2, \dots, \Phi_{M_x-1}, \Phi_{M_x}\}, M_r(\delta_y) = \text{span}\{\Psi_1, \Psi_2, \dots, \Psi_{M_y-1}, \Psi_{M_y}\},$$

then

$$u_h^n(x, y) = \sum_{j=1}^{M_y} \sum_{i=1}^{M_x} \hat{u}_{i,j}^n \Phi_i(x) \Psi_j(y),$$

where $\{\hat{u}_{i,j}^n\}_{i,j=1}^{M_x, M_y}$ are unknown coefficients to be determined. Setting

$$\hat{\mathbf{u}} = [\hat{u}_{1,1}^n, \hat{u}_{1,2}^n, \dots, \hat{u}_{1,M_y}^n, \hat{u}_{2,1}^n, \hat{u}_{2,2}^n, \dots, \hat{u}_{M_x, M_y}^n]^T,$$

then the equation (4.2) can be written in the following form by Kronecker product

$$\begin{aligned} \left\{ (1 + \alpha_0)(B^x \otimes B^y) + \alpha_0(A^x \otimes B^y + B^x \otimes A^y) \right\} \hat{\mathbf{u}}^{n+1} = & -\beta_0 \left\{ A^x \otimes B^y + B^x \otimes A^y + B^x \otimes B^y \right\} \left(\sum_{k=1}^{n+1} \lambda_k^{(\alpha)} \hat{\mathbf{u}}^{n+1-k} \right. \\ & \left. + \sum_{k=0}^n \lambda_k^{(\alpha)} \hat{\mathbf{u}}^{n-k} \right) + (B^x \otimes B^y) \hat{\mathbf{u}}^n + \frac{1}{2}\tau(\mathbf{G}_1^{n+1} + \mathbf{G}_2^n), \quad (4.3) \\ & n = 0, \dots, N-1, \end{aligned}$$

where

$$A^x = (a_{i,j}^x)_{i,j=1}^{M_x}, a_{i,j}^x = -\Phi_j''(\xi_i^x), B^x = (b_{i,j}^x)_{i,j=1}^{M_x}, b_{i,j}^x = \Phi_j(\xi_i^x),$$

$$A^y = (a_{i,j}^y)_{i,j=1}^{M_y}, a_{i,j}^y = -\Psi_j''(\xi_i^y), B^y = (b_{i,j}^y)_{i,j=1}^{M_y}, b_{i,j}^y = \Psi_j(\xi_i^y), \quad (4.4)$$

and

$$\mathbf{G}_1^{n+1} = [g^{n+1}(\xi_1^x, \xi_1^y), g^{n+1}(\xi_1^x, \xi_2^y), \dots, g^{n+1}(\xi_1^x, \xi_{M_y}^y), g^{n+1}(\xi_2^x, \xi_1^y), \dots, g^{n+1}(\xi_{M_x}^x, \xi_{M_y}^y)]^T, \quad (4.5)$$

$$\mathbf{G}_2^n = [g^n(\xi_1^x, \xi_1^y), g^n(\xi_1^x, \xi_2^y), \dots, g^n(\xi_1^x, \xi_{M_y}^y), g^n(\xi_2^x, \xi_1^y), g^n(\xi_2^x, \xi_2^y), \dots, g^n(\xi_{M_x}^x, \xi_{M_y}^y)]^T. \quad (4.6)$$

The matrices A^x , B^x , A^y and B^y are $M_x \times M_y$ having the following structure,

$$\begin{bmatrix} \times & \times & \times & & & & & & & \\ & \times & \times & \times & & & & & & \\ & & \times & \times & \times & \times & & & & \\ & & \times & \times & \times & \times & & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & \times & \times & \times & & \\ & & & & & \times & \times & \times & & \end{bmatrix}. \quad (4.7)$$

We carry out the WSGI-OSC scheme in piecewise Hermite cubic spline space $M_3(\delta)$, which satisfies zero boundary condition. Detailedly, we choose the basis of cubic Hermite polynomials [30], namely, for $1 \leq i \leq K-1$, it follows that

$$\phi_i(x) = \begin{cases} \frac{-2(x-x_{i-1})^3}{h^3} + \frac{3(x-x_{i-1})^2}{h^2}, & x_{i-1} \leq x \leq x_i, \\ \frac{2(x-x_{i-1})^3}{h^3} + \frac{3(x-x_{i+1})^2}{h^2}, & x_i \leq x \leq x_{i+1}, \\ 0, & x < x_{i-1} \text{ or } x > x_{i+1}, \end{cases} \quad (4.8)$$

and

$$\psi_i(x) = \begin{cases} \frac{(x-x_{i-1})^2(x-x_i)}{h^2}, & x_{i-1} \leq x \leq x_i, \\ \frac{(x-x_i)(x-x_{i+1})^2}{h^2}, & x_i \leq x \leq x_{i+1}, \\ 0, & x < x_{i-1} \text{ or } x > x_{i+1}. \end{cases} \quad (4.9)$$

Note that functions $\phi_i(x), \psi_i(x)$ satisfy zero boundary conditions $\phi_i(0) = \phi_i(1) = \psi_i(0) = \psi_i(1) = 0$. Renumber the basis functions and let

$$\{\psi_0, \phi_1, \psi_1, \phi_2, \dots, \phi_{K-1}, \psi_{K-1}, \psi_K\} = \{\Phi_1, \Phi_2, \Phi_3, \dots, \Phi_{2K}\},$$

then

$$M_3(\delta_x) = \text{span}\{\Phi_1, \Phi_2, \Phi_3, \dots, \Phi_{2K}\}, M_3(\delta_y) = \text{span}\{\Phi_1, \Phi_2, \Phi_3, \dots, \Phi_{2K}\}.$$

In order to recover the coefficient matrix of the equations (4.3), we need to calculate the values of the basis functions at the Gauss point and their second-order derivatives. They are defined as follows:

$$\begin{aligned} H_1(u_j) &= (1+2u_j)(1-u_j)^2, H_2(u_j) = u_j(1-u_j)^2 h_k, H_3(u_j) = u_j^2(3-2u_j), H_4(u_j) = u_j^2(u_j-1)h_k, \\ I_1(u_j) &= (12u_j-6)/h_k^2, I_2(u_j) = (6u_j-4)/h_k, I_3(u_j) = (6-12u_j)/h_k^2, I_4(u_j) = (6u_j-2)/h_k, \end{aligned} \quad (4.10)$$

where $u_1 = (3-\sqrt{3})/6$, $u_2 = (3+\sqrt{3})/6$, H_i and I_i denotes the formulas of Hermite polynomials and their second-order derivatives at Gauss points, respectively. Based on the above descriptions of basis functions, we give an example of matrix A^x and B^x in the case of $N_x = N_y = 5$ and $h_k = 1/N_x$. We have

$$A^x = \begin{bmatrix} I_2(u_1) & I_3(u_1) & I_4(u_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_2(u_2) & I_3(u_2) & I_4(u_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_1(u_1) & I_2(u_1) & I_3(u_1) & I_4(u_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & I_1(u_2) & I_2(u_2) & I_3(u_2) & I_4(u_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_1(u_1) & I_2(u_1) & I_3(u_1) & I_4(u_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & I_1(u_2) & I_2(u_2) & I_3(u_2) & I_4(u_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_1(u_1) & I_2(u_1) & I_3(u_1) & I_4(u_1) & 0 \\ 0 & 0 & 0 & 0 & 0 & I_1(u_2) & I_2(u_2) & I_3(u_2) & I_4(u_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_1(u_1) & I_2(u_1) & I_4(u_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_1(u_2) & I_2(u_2) & I_4(u_2) \end{bmatrix}. \quad (4.11)$$

$$B^x = \begin{bmatrix} H_2(u_1) & H_3(u_1) & H_4(u_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ H_2(u_2) & H_3(u_2) & H_4(u_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & H_1(u_1) & H_2(u_1) & H_3(u_1) & H_4(u_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & H_1(u_2) & H_2(u_2) & H_3(u_2) & H_4(u_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_1(u_1) & H_2(u_1) & H_3(u_1) & H_4(u_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & H_1(u_2) & H_2(u_2) & H_3(u_2) & H_4(u_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & H_1(u_1) & H_2(u_1) & H_3(u_1) & H_4(u_1) & 0 \\ 0 & 0 & 0 & 0 & 0 & H_1(u_2) & H_2(u_2) & H_3(u_2) & H_4(u_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_1(u_1) & H_2(u_1) & H_4(u_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_1(u_2) & H_2(u_2) & H_4(u_2) \end{bmatrix}. \quad (4.12)$$

It can be seen from the tensor product calculation that the WSGI-OSC scheme requires the solution of an almost block diagonal linear system at each time level, which can be solved efficiently by the software package COLROW [24].

5 Numerical experiments

In this section, four examples are given to demonstrate our theoretical analysis. In our implementations, we adopt the space of piecewise Hermite bicubics ($r = 3$) on uniform partitions of $\bar{I}$ in both x and y directions with $N_x = N_y = K$. The forcing term $f(x, y, t)$ is approximated by the piecewise Hermite interpolant projection in the Gauss points. To check the accuracy of WSGI-OSC scheme, we present L_∞ and L_2 errors at $T = 1$ and the corresponding convergence order defined by

$$\text{Convergence order} \approx \frac{\log(e_m/e_{m+1})}{\log(h_m/h_{m+1})},$$

where $h_m = 1/K$ is the time step size and e_m is the norm of the corresponding error.

Example 1 We consider the following one-dimensional time-fractional diffusion-wave equation

$$\begin{aligned} {}^C_0 D_t^\gamma u(x, t) &= \frac{\partial^2 u(x, t)}{\partial x^2} - u(x, t) + f(x, t), \quad 0 < x < 1, \quad 0 < t \leq 1, \\ u(x, 0) &= 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad 0 \leq x \leq 1, \\ u(0, t) &= u(1, t) = 0, \quad 0 \leq t \leq 1, \end{aligned} \quad (5.1)$$

where $f(x, t) = \frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\gamma} x^2 (1-x)^2 e^x - 2t^3 e^x (1-4x+4x^3)$. The analytical solution of this equation is $u(x, t) = t^3 x^2 (1-x)^2 e^x$.

From the theoretical analysis, the numerical convergence order of WSGI-OSC (4.2) is expected to be $O(\tau^2 + h^4)$ when $r = 3$. In order to check the second order accuracy in time direction, we select $\tau = h$ so that the error caused by the spatial approximation can be negligible. Table 1 lists L_∞ and L_2 errors and the corresponding convergence orders of WSGI-OSC scheme for $\gamma \in (1, 2)$. We observe that our scheme generates the temporal accuracy with the order 2. To test the spatial approximation accuracy, Table 2 shows that our scheme has the accuracy of 4 in spatial direction, where the temporal step size $\tau = h^2$ is fixed. Numerical solution and global error for $\gamma = 1.3$, $h = 1/32$, $\tau = 1/32$ are shown in Figure 1.

Example 2 Consider the following one-dimensional fractional diffusion-wave equation

$$\begin{aligned} {}^C_0 D_t^\gamma u(x, t) &= \frac{\partial^2 u(x, t)}{\partial x^2} - u(x, t) + f(x, t), \quad 0 < x < 1, \quad 0 < t \leq 1, \\ u(x, 0) &= 0, \quad \frac{\partial u(x, 0)}{\partial t} = -\sin \pi x, \quad 0 \leq x \leq 1, \\ u(0, t) &= u(1, t) = 0, \quad 0 \leq t \leq 1, \end{aligned} \quad (5.2)$$

Table 1: The L_∞ , L_2 errors and temporal convergence orders with $\tau = h$ for Example 1.

γ	τ	L_∞ error	Convergence order	L_2 error	Convergence order
1.1	$\frac{1}{10}$	7.0727×10^{-5}		4.4681×10^{-5}	
	$\frac{1}{20}$	1.7932×10^{-5}	1.9798	1.1012×10^{-5}	2.0206
	$\frac{1}{40}$	4.5623×10^{-6}	1.9747	2.7487×10^{-6}	2.0022
	$\frac{1}{80}$	1.1483×10^{-6}	1.9903	6.8758×10^{-7}	1.9992
1.3	$\frac{1}{10}$	2.6081×10^{-4}		1.7238×10^{-4}	
	$\frac{1}{20}$	6.6648×10^{-5}	1.9684	4.2518×10^{-5}	2.0194
	$\frac{1}{40}$	1.6825×10^{-6}	1.9860	1.0577×10^{-5}	2.0072
	$\frac{1}{80}$	4.2263×10^{-7}	1.9931	2.6387×10^{-6}	2.0030
1.5	$\frac{1}{10}$	4.1657×10^{-4}		2.7911×10^{-4}	
	$\frac{1}{20}$	1.0633×10^{-4}	1.9701	6.8593×10^{-5}	2.0247
	$\frac{1}{40}$	2.6736×10^{-5}	1.9916	1.7020×10^{-5}	2.0108
	$\frac{1}{80}$	6.7115×10^{-6}	1.9941	4.2405×10^{-6}	2.0050
1.7	$\frac{1}{10}$	5.3422×10^{-4}		3.6265×10^{-4}	
	$\frac{1}{20}$	1.3701×10^{-4}	1.9632	8.9343×10^{-5}	2.0212
	$\frac{1}{40}$	3.4419×10^{-5}	1.9930	2.2160×10^{-5}	2.0114
	$\frac{1}{80}$	8.6292×10^{-6}	1.9959	5.5175×10^{-6}	2.0059
1.9	$\frac{1}{10}$	5.7600×10^{-4}		3.9339×10^{-4}	
	$\frac{1}{20}$	1.4884×10^{-4}	1.9523	9.7244×10^{-5}	2.0163
	$\frac{1}{40}$	3.7391×10^{-5}	1.9930	2.4112×10^{-5}	2.0119
	$\frac{1}{80}$	9.3633×10^{-6}	1.9976	5.9996×10^{-6}	2.0068
1.95	$\frac{1}{10}$	5.6941×10^{-4}		3.8862×10^{-4}	
	$\frac{1}{20}$	1.4696×10^{-4}	1.9540	9.6061×10^{-5}	2.0163
	$\frac{1}{40}$	3.6917×10^{-5}	1.9931	2.3812×10^{-5}	2.0123
	$\frac{1}{80}$	9.2425×10^{-6}	1.9979	5.9232×10^{-6}	2.0072

(a)

(b)

Figure 1: Numerical solution (a) and global error (b) for Example 1 with $\gamma = 1.3$, $h = 1/32$, $\tau = 1/32$.

Table 2: The L_∞ , L_2 errors and spatial convergence orders with $\tau = h^2$ for Example 1.

γ	h	L_∞ error	Convergence order	L_2 error	Convergence order
1.1	$\frac{1}{10}$	2.4371×10^{-6}		1.7740×10^{-6}	
	$\frac{1}{20}$	1.5377×10^{-7}	3.9863	1.0837×10^{-7}	4.0329
	$\frac{1}{40}$	9.6290×10^{-9}	3.9972	6.6928×10^{-9}	4.0172
	$\frac{1}{80}$	6.0225×10^{-10}	3.9989	4.1576×10^{-10}	4.0088
1.3	$\frac{1}{10}$	3.8377×10^{-6}		2.6750×10^{-6}	
	$\frac{1}{20}$	2.4364×10^{-7}	3.9774	1.6332×10^{-7}	4.0338
	$\frac{1}{40}$	1.5241×10^{-8}	3.9987	1.0085×10^{-8}	4.0174
	$\frac{1}{80}$	9.5308×10^{-10}	3.9992	6.2644×10^{-10}	4.0089
1.5	$\frac{1}{10}$	4.7527×10^{-6}		3.2535×10^{-6}	
	$\frac{1}{20}$	3.0159×10^{-7}	3.9781	1.9851×10^{-7}	4.0347
	$\frac{1}{40}$	1.8863×10^{-8}	3.9990	1.2256×10^{-8}	4.0177
	$\frac{1}{80}$	1.1798×10^{-9}	3.9990	7.6129×10^{-10}	4.0089
1.7	$\frac{1}{10}$	5.1530×10^{-6}		3.4857×10^{-6}	
	$\frac{1}{20}$	3.2579×10^{-7}	3.9834	2.1258×10^{-7}	4.0354
	$\frac{1}{40}$	2.0382×10^{-8}	3.9986	1.3123×10^{-8}	4.0178
	$\frac{1}{80}$	1.2754×10^{-9}	3.9982	8.1509×10^{-10}	4.0090
1.9	$\frac{1}{10}$	4.6730×10^{-6}		3.0735×10^{-6}	
	$\frac{1}{20}$	2.9311×10^{-7}	3.9948	1.8735×10^{-7}	4.0361
	$\frac{1}{40}$	1.8412×10^{-8}	3.9927	1.1563×10^{-8}	4.0181
	$\frac{1}{80}$	1.1509×10^{-9}	3.9999	7.1819×10^{-10}	4.0090
1.95	$\frac{1}{10}$	4.3316×10^{-6}		2.8280×10^{-6}	
	$\frac{1}{20}$	2.7151×10^{-7}	3.9958	1.7235×10^{-7}	4.0364
	$\frac{1}{40}$	1.7062×10^{-8}	3.9922	1.0637×10^{-8}	4.0182
	$\frac{1}{80}$	1.0665×10^{-9}	3.9999	6.6066×10^{-10}	4.0091

where $f(x, t) = \left[\frac{2}{\Gamma(3-\gamma)} t^{2-\gamma} + (t^2 - t)\pi^2 + (t^2 - t) \right] \sin \pi x$. The analytical solution of this equation is $u(x, t) = (t^2 - t) \sin \pi x$.

In order to test the temporal accuracy of the proposed method, we choose $\tau = h$ to avoid contamination of the spatial error. The maximum L_∞ , L_2 errors and temporal convergence orders are shown in Table 3. To check the convergence order in space, the time step τ and space step h are chosen such that $\tau = h^2$, and $\gamma = 1.1, 1.3, 1.5, 1.7, 1.9, 1.95$. Table 4 presents the maximum L_∞ , L_2 errors and spatial convergence orders. The results in Tables 3 and 4 demonstrate the expected convergence rates of 2 order in time and 4 order in space simultaneously. Numerical solution and global error at $T = 1$ with $\gamma = 1.5$, $h = 1/32$, $\tau = 1/32$ are shown in Figure 2.

Example 3 Consider the following two-dimensional fractional diffusion-wave equation

$$\begin{aligned}
 {}^C_0 D_t^\gamma u(x, y, t) - \Delta u(x, y, t) + u(x, y, t) &= f(x, y, t), \\
 u(x, y, 0) &= 0, \quad \frac{\partial u(x, y, 0)}{\partial t} = 0, \quad (x, y) \in \Omega, \\
 u(x, y, t) &= 0, \quad (x, y, t) \in \partial\Omega \times (0, T],
 \end{aligned} \tag{5.3}$$

Table 3: The L_∞ , L_2 errors and temporal convergence orders with $\tau = h$ for Example 2.

γ	τ	L_∞ error	Convergence order	L_2 error	Convergence order
1.1	$\frac{1}{10}$	2.7779×10^{-5}		1.8686×10^{-5}	
	$\frac{1}{20}$	6.9405×10^{-6}	2.0009	4.5452×10^{-6}	2.0395
	$\frac{1}{40}$	1.7225×10^{-6}	2.0105	1.1135×10^{-6}	2.0292
	$\frac{1}{80}$	4.2704×10^{-7}	2.0121	2.7427×10^{-7}	2.0215
1.3	$\frac{1}{10}$	6.8399×10^{-5}		4.6042×10^{-5}	
	$\frac{1}{20}$	1.6912×10^{-5}	2.0159	1.1079×10^{-5}	2.0551
	$\frac{1}{40}$	4.1818×10^{-6}	2.0158	2.7032×10^{-6}	2.0352
	$\frac{1}{80}$	1.0358×10^{-6}	2.0134	6.6503×10^{-7}	2.0232
1.5	$\frac{1}{10}$	1.0251×10^{-4}		6.9519×10^{-5}	
	$\frac{1}{20}$	2.5114×10^{-5}	2.0292	1.6555×10^{-5}	2.0701
	$\frac{1}{40}$	6.2025×10^{-6}	2.0176	4.0327×10^{-6}	2.0375
	$\frac{1}{80}$	1.5384×10^{-6}	2.0114	9.9335×10^{-7}	2.0214
1.7	$\frac{1}{10}$	1.4424×10^{-4}		9.9816×10^{-5}	
	$\frac{1}{20}$	3.5642×10^{-5}	2.0168	2.4001×10^{-5}	2.0562
	$\frac{1}{40}$	8.8717×10^{-6}	2.0063	5.8868×10^{-6}	2.0275
	$\frac{1}{80}$	2.2116×10^{-6}	2.0041	1.4572×10^{-6}	2.0143
1.9	$\frac{1}{10}$	1.9061×10^{-4}		1.2932×10^{-4}	
	$\frac{1}{20}$	4.7290×10^{-5}	2.0110	3.1561×10^{-5}	2.0347
	$\frac{1}{40}$	1.1810×10^{-5}	2.0015	7.7937×10^{-6}	2.0178
	$\frac{1}{80}$	2.9519×10^{-6}	2.0003	1.9367×10^{-6}	2.0087
1.95	$\frac{1}{10}$	2.0110×10^{-4}		1.3455×10^{-4}	
	$\frac{1}{20}$	4.9930×10^{-5}	2.0099	3.2930×10^{-5}	2.0306
	$\frac{1}{40}$	1.2482×10^{-5}	2.0001	8.1369×10^{-6}	2.0169
	$\frac{1}{80}$	3.1218×10^{-6}	1.9994	2.0222×10^{-6}	2.0085

(a)

(b)

Figure 2: Numerical solution (a) and global error (b) for Example 2 with $\gamma = 1.5$ at $T = 1$ ($h = 1/32$, $\tau = 1/32$).

Table 4: The L_∞, L_2 errors and spatial convergence orders with $\tau = h^2$ for Example 2.

γ	h	L_∞ error	Convergence order	L_2 error	Convergence order
1.1	$\frac{1}{10}$	9.8873×10^{-8}		7.9806×10^{-8}	
	$\frac{1}{20}$	6.3013×10^{-9}	3.9719	5.0124×10^{-9}	3.9929
	$\frac{1}{40}$	4.0052×10^{-10}	3.9757	3.1726×10^{-10}	3.9818
	$\frac{1}{80}$	2.5425×10^{-11}	3.9775	2.0139×10^{-11}	3.9776
1.3	$\frac{1}{10}$	2.0590×10^{-7}		1.1973×10^{-7}	
	$\frac{1}{20}$	1.2348×10^{-8}	4.0596	7.0248×10^{-9}	4.0912
	$\frac{1}{40}$	7.5090×10^{-10}	4.0395	4.2273×10^{-10}	4.0547
	$\frac{1}{80}$	4.6084×10^{-11}	4.0263	2.5826×10^{-11}	4.0328
1.5	$\frac{1}{10}$	3.1378×10^{-7}		1.8224×10^{-7}	
	$\frac{1}{20}$	1.9140×10^{-8}	4.0351	1.0838×10^{-8}	4.0716
	$\frac{1}{40}$	1.1827×10^{-9}	4.0165	6.6114×10^{-10}	4.0350
	$\frac{1}{80}$	7.3513×10^{-11}	4.0079	4.0836×10^{-11}	4.0170
1.7	$\frac{1}{10}$	4.2637×10^{-7}		2.5157×10^{-7}	
	$\frac{1}{20}$	2.6414×10^{-8}	4.0127	1.5170×10^{-8}	4.0516
	$\frac{1}{40}$	1.6453×10^{-9}	4.0049	9.3246×10^{-10}	4.0241
	$\frac{1}{80}$	1.0270×10^{-10}	4.0019	5.7825×10^{-11}	4.0113
1.9	$\frac{1}{10}$	6.2873×10^{-7}		3.5996×10^{-7}	
	$\frac{1}{20}$	3.9276×10^{-8}	4.0007	2.1898×10^{-8}	4.0389
	$\frac{1}{40}$	2.4548×10^{-9}	4.0000	1.3510×10^{-9}	4.0188
	$\frac{1}{80}$	1.5342×10^{-10}	4.0000	8.3898×10^{-11}	4.0092
1.95	$\frac{1}{10}$	6.7414×10^{-7}		3.9216×10^{-7}	
	$\frac{1}{20}$	4.2262×10^{-8}	3.9956	2.3894×10^{-8}	4.0367
	$\frac{1}{40}$	2.6423×10^{-9}	3.9995	1.4745×10^{-9}	4.0184
	$\frac{1}{80}$	1.6515×10^{-10}	3.9999	9.1572×10^{-11}	4.0091

where $\Omega = [0, 1] \times [0, 1], T = 1, f(x, y, t) = \left[\frac{\Gamma(4)}{\Gamma(4-\gamma)} t^{3-\gamma} xy(1-x)(1-y) + t^3 xy(7-3y-3x-xy) \right] e^{x+y}$. The exact solution of the equation is $u(x, y, t) = t^3 xy(1-x)(1-y)e^{x+y}$.

Similar to the selection of parameters in Examples 1 and 2, Tables 5 and 6 list the maximum L_∞, L_2 errors and convergence orders, respectively. The similar convergence rates in time and space are also obtained. As we hope, the convergence order of all numerical results match that of the theoretical analysis. Figure 3 plots the numerical solution and global error at $T = 1$ with $\gamma = 1.7, h = 1/32, \tau = 1/32$.

Example 4 Consider the following two-dimensional fractional diffusion-wave equation

$$\begin{aligned}
 {}^C_0 D_t^\gamma u(x, y, t) - \Delta u(x, y, t) + u(x, y, t) &= f(x, y, t), \\
 u(x, y, 0) &= 0, \frac{\partial u(x, y, 0)}{\partial t} = 0, (x, y) \in \Omega, \\
 u(x, y, t) &= 0, (x, y, t) \in \partial\Omega \times (0, T]
 \end{aligned} \tag{5.4}$$

where $\Omega = [0, 1] \times [0, 1], T = 1, f(x, y, t) = \left[\frac{\Gamma(3+\gamma)}{2} + (2\pi^2 + 1)t^\gamma \right] t^2 \sin \pi x \sin \pi y$. The exact solution of the equation is $u(x, y, t) = t^{2+\gamma} \sin \pi x \sin \pi y$.

Tables 7 and 8 display L_∞ and L_2 errors and the corresponding convergence orders in time and space for some $\gamma \in (1, 2)$. Once again, the expected convergence rates with second-order accuracy in time direction

Table 5: The L_∞ , L_2 errors and temporal convergence orders for Example 3.

γ	N	L_∞ error	Convergence order	L_2 error	Convergence order
1.1	10	1.6611×10^{-4}		8.3486×10^{-5}	
	15	7.5461×10^{-5}	1.9461	3.7876×10^{-5}	1.9493
	20	4.2909×10^{-5}	1.9624	2.1509×10^{-5}	1.9669
	25	2.7589×10^{-5}	1.9792	1.3841×10^{-5}	1.9754
1.3	10	5.1729×10^{-4}		2.6164×10^{-4}	
	15	2.3249×10^{-4}	1.9724	1.1769×10^{-4}	1.9704
	20	1.3148×10^{-4}	1.9813	6.6585×10^{-5}	1.9799
	25	8.4565×10^{-5}	1.9779	4.2760×10^{-5}	1.9847
1.5	10	7.7899×10^{-4}		3.9475×10^{-4}	
	15	3.4829×10^{-4}	1.9853	1.7651×10^{-4}	1.9850
	20	1.9648×10^{-4}	1.9899	9.9627×10^{-5}	1.9882
	25	1.2607×10^{-4}	1.9886	6.3896×10^{-5}	1.9905
1.7	10	9.8958×10^{-4}		5.0659×10^{-4}	
	15	4.3990×10^{-4}	1.9995	2.2433×10^{-4}	2.0090
	20	2.4748×10^{-4}	1.9995	1.2609×10^{-4}	2.0028
	25	1.5841×10^{-4}	1.9994	8.0685×10^{-5}	2.0006
1.9	10	1.1985×10^{-3}		6.2808×10^{-4}	
	15	5.3173×10^{-4}	2.0044	2.7856×10^{-4}	2.0052
	20	2.9891×10^{-4}	2.0022	1.5657×10^{-4}	2.0026
	25	1.9123×10^{-4}	2.0015	1.0018×10^{-4}	2.0014

(a)

(b)

Figure 3: Numerical solution (a) and global error (b) for Example 3 with $\gamma = 1.7$ at $T = 1$ ($h = 1/32$, $\tau = 1/32$).

and fourth-order accuracy in spatial direction can be observed from two tables. Numerical solution and global error at $T = 1$ with $\gamma = 1.9$, $h = 1/32$, $\tau = 1/32$ are displayed in Figure 4.

Table 6: The L_∞ , L_2 errors and spatial convergence orders for Example 3.

γ	N	L_∞ error	Convergence order	L_2 error	Convergence order
1.1	10	1.7277×10^{-6}		5.8164×10^{-7}	
	15	3.7129×10^{-7}	3.7921	1.1583×10^{-7}	3.9799
	20	1.2237×10^{-7}	3.8582	3.6753×10^{-8}	3.9902
	25	5.1343×10^{-8}	3.8922	1.5074×10^{-8}	3.9942
1.3	10	4.5383×10^{-6}		2.0806×10^{-6}	
	15	8.9315×10^{-7}	4.0091	4.1188×10^{-7}	3.9946
	20	2.8185×10^{-7}	4.0092	1.3042×10^{-7}	3.9974
	25	1.1523×10^{-7}	4.0083	5.3439×10^{-8}	3.9984
1.5	10	7.1532×10^{-6}		3.3974×10^{-6}	
	15	1.4118×10^{-6}	4.0021	6.7176×10^{-7}	3.9976
	20	4.4624×10^{-7}	4.0036	2.1262×10^{-7}	3.9988
	25	1.8263×10^{-7}	4.0037	8.7104×10^{-8}	3.9993
1.7	10	9.2188×10^{-6}		4.4527×10^{-6}	
	15	1.8187×10^{-6}	4.0031	8.7956×10^{-7}	4.0000
	20	5.7483×10^{-7}	4.0038	2.7830×10^{-7}	3.9999
	25	2.3526×10^{-7}	4.0036	1.1399×10^{-7}	3.9999
1.9	10	1.1444×10^{-5}		5.7230×10^{-6}	
	15	2.2505×10^{-6}	4.0110	1.1299×10^{-6}	4.0011
	20	7.1020×10^{-7}	4.0091	3.5746×10^{-7}	4.0005
	25	2.9046×10^{-7}	4.0068	1.4641×10^{-7}	4.0003

(a)

(b)

Figure 4: Numerical solution (a) and global error (b) for Example 4 with $\gamma = 1.9$ at $T = 1$ ($h = 1/32$, $\tau = 1/32$).

Table 7: The L_∞ , L_2 errors and temporal convergence orders for Example 4.

γ	N	L_∞ error	Convergence order	L_2 error	Convergence order
1.1	10	8.8381×10^{-4}		4.4449×10^{-4}	
	15	3.9978×10^{-4}	1.9566	2.0073×10^{-4}	1.9607
	20	2.2738×10^{-4}	1.9615	1.1385×10^{-4}	1.9711
	25	1.4625×10^{-4}	1.9775	7.3238×10^{-5}	1.9771
1.3	10	3.1514×10^{-3}		1.5847×10^{-3}	
	15	1.4225×10^{-3}	1.9617	7.1426×10^{-4}	1.9654
	20	8.0814×10^{-4}	1.9656	4.0464×10^{-4}	1.9752
	25	5.1939×10^{-4}	1.9811	2.6009×10^{-4}	1.9807
1.5	10	5.3058×10^{-3}		2.6680×10^{-3}	
	15	2.3861×10^{-3}	1.9709	1.1981×10^{-3}	1.9745
	20	1.3534×10^{-3}	1.9711	6.7766×10^{-4}	1.9808
	25	8.6906×10^{-4}	1.9851	4.3519×10^{-4}	1.9847
1.7	10	7.2062×10^{-3}		3.6236×10^{-3}	
	15	3.2347×10^{-3}	1.9755	1.6242×10^{-3}	1.9792
	20	1.8321×10^{-3}	1.9760	9.1737×10^{-4}	1.9857
	25	1.1754×10^{-3}	1.9893	5.8858×10^{-4}	1.9889
1.9	10	8.0346×10^{-3}		4.0402×10^{-3}	
	15	3.6198×10^{-3}	1.9665	1.8175×10^{-3}	1.9701
	20	2.0516×10^{-3}	1.9736	1.0273×10^{-3}	1.9833
	25	1.3162×10^{-3}	1.9893	6.5910×10^{-4}	1.9888

Table 8: The L_∞ , L_2 errors and spatial convergence orders for Example 4.

γ	N	L_∞ error	Convergence order	L_2 error	Convergence order
1.1	10	1.2725×10^{-5}		6.5169×10^{-5}	
	15	2.5700×10^{-6}	3.9453	1.2858×10^{-5}	4.0029
	20	8.0847×10^{-7}	4.0202	4.0665×10^{-5}	4.0014
	25	3.3300×10^{-7}	3.9751	1.6653×10^{-5}	4.0009
1.3	10	3.6079×10^{-5}		1.8230×10^{-5}	
	15	7.1968×10^{-6}	3.9758	3.6064×10^{-6}	3.9964
	20	2.2773×10^{-6}	3.9997	1.1417×10^{-6}	3.9982
	25	9.3472×10^{-7}	3.9907	4.6773×10^{-7}	3.9989
1.5	10	5.7773×10^{-5}		2.9133×10^{-5}	
	15	1.1491×10^{-5}	3.9829	5.7620×10^{-6}	3.9968
	20	3.6402×10^{-6}	3.9959	1.8240×10^{-6}	3.9984
	25	1.4930×10^{-6}	3.9942	7.4725×10^{-7}	3.9991
1.7	10	7.6488×10^{-5}		3.8541×10^{-5}	
	15	1.5190×10^{-5}	3.9868	7.6189×10^{-6}	3.9981
	20	4.8133×10^{-6}	3.9948	2.4113×10^{-6}	3.9991
	25	1.9734×10^{-6}	3.9959	9.8780×10^{-7}	3.9994
1.9	10	8.4694×10^{-5}		4.2666×10^{-5}	
	15	1.6803×10^{-5}	3.9892	8.4290×10^{-6}	3.9997
	20	5.3240×10^{-6}	3.9951	2.6670×10^{-6}	3.9999
	25	2.1823×10^{-6}	3.9968	1.0924×10^{-6}	4.0000

6 Conclusion

In this paper, we have constructed a Crank-Nicolson WSGI-OSC method for the two-dimensional time-fractional diffusion-wave equation. The original fractional diffusion-wave equation is transformed into its equivalent partial integro-differential equations, then Crank-Nicolson orthogonal spline collocation method with WSGI approximation is developed. The proposed method holds a higher convergence order than the convergence order $O(\tau^{3-\gamma})$ of general L_1 approximation. The stability and convergence analysis are derived. Some numerical examples are also given to confirm our theoretical analysis.

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