Research Article

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B-maximal commutators, commutators of B-singular integral operators and B-Riesz potentials on B-Morrey spaces

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Abstract: In this article, we consider the Laplace-Bessel differential operator

$$\Delta_{B_{k,n}} = \sum_{i=1}^k \left(\frac{\partial^2}{\partial x_i^2} + \frac{y_i}{x_i} \frac{\partial}{\partial x_i} \right) + \sum_{i=k+1}^n \frac{\partial^2}{\partial x_i^2}, y_1 > 0, \dots, y_k > 0.$$

Furthermore, we define B -maximal commutators, commutators of B -singular integral operators and B -Riesz potentials associated with the Laplace-Bessel differential operator. Moreover, we also obtain the boundedness of the B -maximal commutator $M_{b,y}$ and the commutator $[b, A_y]$ of the B -singular integral operator and Hardy-Littlewood-Sobolev-type theorem for the commutator $[b, I_{\alpha,y}]$ of the B -Riesz potential on B -Morrey spaces $L_{p,\lambda,y}$, when $b \in \text{BMO}_y$.

Keywords: commutator, generalized shift operator, B -Riesz potential, B -maximal function, B -Morrey space, bounded mean oscillation (BMO) space

MSC 2010: 42B20, 42B25, 42B35

1 Introduction

The Laplace-Bessel differential operator, defined by

$$\Delta_{B_{k,n}} = \sum_{i=1}^k \left(\frac{\partial^2}{\partial x_i^2} + \frac{y_i}{x_i} \frac{\partial}{\partial x_i} \right) + \sum_{i=k+1}^n \frac{\partial^2}{\partial x_i^2}, y_1 > 0, \dots, y_k > 0,$$

is an important technical tool in the Fourier-Bessel harmonic analysis and applications. It has been investigated by many researchers, such as Muckenhoupt, Stein, Kipriyanov, Trimeche, Lyakhov, Stempak, Gadjeiev, Aliev, Guliyev, Bayrakci, Hasanov, Serbetci, Keskin and Ekincioglu [1–17].

Given a linear operator T acting on functions and given a function b , the commutator $[T, b]$ is formally defined by

$$[T, b]f = T(bf) - bT(f).$$

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The first result on commutators was obtained by Coifman, Rochberg and Weiss [18]. They showed that if T is a classical Calderon-Zygmund singular integral operator and $b \in \text{BMO}$, then the commutator $[T, b]$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$. Later, Chanillo [19] proved that commutators characterize Riesz potentials on the bounded mean oscillation (BMO) space.

Classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$, $0 < \lambda < n$, $1 \leq p < \infty$ were introduced in 1938 by Morrey [20] to study the local behavior of solutions to second-order elliptic partial differential equations. Morrey spaces defined as the set of all functions $f \in L_{\text{loc}}^p(\mathbb{R}^n)$ such that the following norm is finite

$$\|f\|_{L_{p,\lambda}} \equiv \|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p},$$

where $B(x, r)$ is the ball in $\mathbb{R}^n$ centered at x and with radius $r > 0$. Moreover, for the readers, it is convenient to declare in what sense $L_{p,\lambda}(\mathbb{R}^n)$ is trivial when $\lambda < 0$, $\lambda > n$ and $L_{p,0}(\mathbb{R}^n) \cong L_p(\mathbb{R}^n)$, $L_{p,n}(\mathbb{R}^n) \cong L_\infty(\mathbb{R}^n)$.

Classical Morrey spaces are investigated by Adams [21] in a study related to Riesz potentials and recently also by Chiarenza and Frasca [22] in studying the boundedness of the Hardy-Littlewood maximal operator, Riesz potentials and singular integral operators. Moreover, Mixed Morrey Spaces, a generalization of Morrey spaces, are defined by Scapellato [23,24] and their applications to partial differential equations of parabolic type are examined. Furthermore, Morrey spaces generated by generalized translation operator and their applications have been research areas for many mathematicians such as Guliyev, Hasanov, Zeren, Ekincioglu and Uygur [5,9,10,25,26].

In this article, we consider Morrey spaces $L_{p,\lambda,\gamma}$ called B -Morrey space, B -maximal commutators, commutators of singular integral operators and Riesz potentials associated with the Laplace-Bessel differential operator. Moreover, we prove the boundedness of the B -maximal commutator $M_{b,\gamma}$ and the commutator $[b, A_\gamma]$ of the B -singular integral operator on the B -Morrey space, $1 < p < \infty$ when $b \in \text{BMO}_\gamma$. We also obtain the Hardy-Littlewood-Sobolev-type theorem for the commutator $[b, I_{\alpha,\gamma}]$ of the B -Riesz potential when $b \in \text{BMO}_\gamma$.

The article is organized as follows. Section 2 contains some basic definitions and results on the Lebesgue and Morrey spaces. Main results and their proofs are given in Section 3.

2 Preliminaries

Let $\mathbb{R}_{k,+}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n: x_1 > 0, \dots, x_k > 0, 1 \leq k \leq n\}$, $E(x, r) = \{y \in \mathbb{R}_{k,+}^n: |x - y| < r\}$, $y = (y_1, \dots, y_k)$, $y_1 > 0, \dots, y_k > 0$, $|y| = y_1 + \dots + y_k$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $(x')^\gamma dx = x_1^{\gamma_1} \dots x_k^{\gamma_k} dx_1 \dots dx_k$. For a measurable set $E \subset \mathbb{R}_{k,+}^n$, let $|E|_\gamma = \int_E (x')^\gamma dx$ and $|E(0, r)|_\gamma = \omega(n, k, \gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n, k, \gamma) = \int_{E(0,1)} (x')^\gamma dx = \frac{\pi^{n-k/2}}{2^k} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Let $L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}_{k,+}^n)$, $1 \leq p < \infty$ be the space of all measurable functions on $\mathbb{R}_{k,+}^n$ with the norm

$$\|f\|_{L_{p,\gamma}} = \left(\int_{\mathbb{R}_{k,+}^n} |f(x)|^p (x')^\gamma dx \right)^{\frac{1}{p}} < \infty.$$

In the case $p = \infty$, the space $L_{\infty,\gamma}(\mathbb{R}_{k,+}^n)$ is defined by means of the usual modification

$$\|f\|_{L_\infty} = \text{ess sup} |f(x)|, x \in \mathbb{R}_{k,+}^n.$$

The weak- $L_{p,\gamma}$ space $WL_{p,\gamma} \equiv WL_{p,\gamma}(\mathbb{R}_{k,+}^n)$ is defined by

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r |\{x \in \mathbb{R}_{k,+}^n : |f(x)| > r\}|_\gamma^{1/p}, \quad 1 \leq p < \infty.$$

The weighted space $L_{p,\omega,\gamma} \equiv L_{p,\omega,\gamma}(\mathbb{R}_{k,+}^n)$ is the space of measurable functions on $\mathbb{R}_{k,+}^n$ defined by

$$\|f\|_{L_{p,\omega,\gamma}} = \left(\int_{\mathbb{R}_{k,+}^n} |f(x)|^p \omega(x) (x')^\gamma dx \right)^{1/p}, \quad 1 \leq p < \infty$$

and $L_{\infty,\omega,\gamma}(\mathbb{R}_{k,+}^n) = L_{\infty}(\mathbb{R}_{k,+}^n)$.

Definition 2.1. For $1 < p < \infty$ and a nonnegative locally integrable function ω on $\mathbb{R}_{k,+}^n$, ω is in the Muckenhoupt class $A_{p,\gamma}$ if it satisfies the condition

$$\sup_{x \in \mathbb{R}_{k,+}^n, r > 0} \left(\frac{1}{|E(x, r)|_\gamma} \int_{E(x, r)} \omega(y) (y')^\gamma dy \right) \left(\frac{1}{|E(x, r)|_\gamma} \int_{E(x, r)} \omega^{-\frac{1}{p-1}}(y) (y')^\gamma dy \right)^{p-1} < \infty$$

and a weight function ω belongs to the class $A_{1,\gamma}$ if there exists a positive constant C such that for any $x \in \mathbb{R}_{k,+}^n$ and $r > 0$,

$$|E(x, r)|_\gamma^{-1} \int_{E(x, r)} \omega(y) (y')^\gamma dy \leq C \operatorname{ess\,sup}_{y \in E(x, r)} \frac{1}{\omega(y)}.$$

Definition 2.2. The generalized shift operator T^γ is defined by

$$T^\gamma f(x) = C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f((x', y')_\beta, x'' - y'') dv(\beta),$$

where

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)} = \frac{2^k}{\pi^k} \omega(2k, k, \gamma)$$

$x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{n-k}$, $1 \leq k \leq n$, $(x', y')_\beta = ((x_1, y_1)_{\beta_1}, \dots, (x_k, y_k)_{\beta_k})$, $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos \beta_i + y_i^2)^{1/2}$, $1 \leq i \leq k$ and $dv(\beta) = \prod_{i=1}^k \sin^{\gamma_i-1} \beta_i \, d\beta_1 \dots d\beta_k$.

It is well known that T^γ is closely related to the Laplace-Bessel differential operator $\Delta_{B_{k,n}}$, see [13,27] for details.

Lemma 2.3. For all $x \in \mathbb{R}_{k,+}^n$

$$\int_{E(x,t)} g(y) (y')^\gamma dy = C_{\gamma,k}^{-1} \int_{B((x,0),t)} g(\sqrt{z_1^2 + \bar{z}_1^2}, \dots, \sqrt{z_k^2 + \bar{z}_k^2}, z'') dv(z, \bar{z}'),$$

where $B((x, 0), t) = \{(z, \bar{z}') \in \mathbb{R}^n \times (0, \infty)^k : |(x_1 - \sqrt{z_1^2 + \bar{z}_1^2}, \dots, x_k - \sqrt{z_k^2 + \bar{z}_k^2}, x'' - z'')| < t\}$, $dv(z, \bar{z}') = (\bar{z}')^{\gamma-1} dz d\bar{z}'$, $d\bar{z}' = d\bar{z}'_1 \dots d\bar{z}'_k$ and $(\bar{z}')^{\gamma-1} = (\bar{z}'_1)^{\gamma_1-1} \dots (\bar{z}'_k)^{\gamma_k-1}$.

Lemma 2.4. For all $x \in \mathbb{R}_{k,+}^n$

$$\int_{E(0,r)} T^\gamma g(x) (y')^\gamma dy = \int_{E((x,0),r)} g(\sqrt{z_1^2 + \bar{z}_1^2}, \dots, \sqrt{z_k^2 + \bar{z}_k^2}, z'') dv(z, \bar{z}'),$$

where $E((x, 0), r) = \{(z, \bar{z}') \in \mathbb{R}^n \times (0, \infty)^k : |x - z, \bar{z}'| < r\}$.

Lemma 2.5. For all $x \in \mathbb{R}_{k,+}^n$

$$\int_{\mathbb{R}_{k,+}^n} T^\gamma g(x) \varphi(y) (y')^\gamma dy = \int_{\mathbb{R}^n \times (0, \infty)^k} g(\sqrt{z_1^2 + \bar{z}_1^2}, \dots, \sqrt{z_k^2 + \bar{z}_k^2}, z'') \varphi(z, \bar{z}') dv(z, \bar{z}').$$

The proofs of Lemmas 2.3–2.5 are immediately obtained by changing the variables as follows: $z'' = y''$, $z_i = y_i \cos \alpha_i$, $\bar{z}_i = y_i \sin \alpha_i$, $0 \leq \alpha_i < \pi$, $i = 1, \dots, k$, $y \in \mathbb{R}_{k,+}^n$, $\bar{z}' = (\bar{z}_1, \dots, \bar{z}_k)$ and $(z, \bar{z}') \in \mathbb{R}^n \times (0, \infty)^k$, $1 \leq k \leq n$.

Lemma 2.6. [28] Let $0 < \alpha < Q$. Then, $|T^\gamma x|^{\alpha-Q} - |y|^{\alpha-Q} \leq 2^{Q-\alpha+1} |y|^{\alpha-Q-1} |x|$, $2|x| \leq |y|$.

Definition 2.7. [7] Let $1 \leq p < \infty$, $0 \leq \lambda \leq Q$. B -Morrey space $L_{p,\lambda,\gamma} = L_{p,\lambda,\gamma}(\mathbb{R}_{k,+}^n)$, associated with the Laplace-Bessel differential operator, is defined as the space of locally integrable functions f with the finite norm

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{t>0, x \in \mathbb{R}_{k,+}^n} \left(t^{-\lambda} \int_{E(0,t)} T^\gamma (|f|)^p(x) (y')^\gamma dy \right)^{1/p}.$$

Note that $L_{p,0,\gamma} = L_{p,\gamma}$, $L_{p,Q,\gamma} = L_{\infty}$. If $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda,\gamma} = \emptyset$.

Definition 2.8. [9] Let $1 \leq p < \infty$, $0 \leq \lambda \leq Q$. The weak B -Morrey space $WL_{p,\lambda,\gamma} = WL_{p,\lambda,\gamma}(\mathbb{R}_{k,+}^n)$ is defined by

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0, x \in \mathbb{R}_{k,+}^n} \left(t^{-\lambda} \int_{\{y \in E(0,t): T^\gamma [|f|](x) > r\}} (y')^\gamma dy \right)^{1/p}.$$

Note that $L_{p,\lambda,\gamma} \subset WL_{p,\lambda,\gamma}$ and $\|f\|_{WL_{p,\lambda,\gamma}} \leq \|f\|_{L_{p,\lambda,\gamma}}$.

Definition 2.9. The B -BMO space generated by the generalized shift operator and denoted by $BMO_\gamma = BMO_\gamma(\mathbb{R}_{k,+}^n)$ is defined as the space of locally integrable functions f with the finite norm

$$\|f\|_{BMO_\gamma} = \sup_{t>0, x \in \mathbb{R}_{k,+}^n} |E(0,t)|_\gamma^{-1} \int_{E(0,t)} T^\gamma f(x) - f_{E(0,t)}(x) | (y')^\gamma dy|$$

or

$$\|f\|_{BMO_\gamma} = \inf_C \sup_{t>0, x \in \mathbb{R}_{k,+}^n} |E(0,t)|_\gamma^{-1} \int_{E(0,t)} |T^\gamma f(x) - C| (y')^\gamma dy,$$

where $f_{E(0,t)}(x) = |E(0,t)|_\gamma^{-1} \int_{E(0,t)} T^\gamma f(x) (y')^\gamma dy$. Moreover, the $BMO(X, \nu)$ space is defined as the space of all functions f such that the following norm is finite

$$\|f\|_{BMO(X,\nu)} = \sup_{t>0, x \in X} \nu(B(x,t))^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}| d\nu(y) < \infty,$$

where $f_{B(x,t)} = \nu(B(x,t))^{-1} \int_{B(x,t)} f(y) d\nu(y)$.

We also define the sharp maximal operator $M_\gamma^\#$ generated by the generalized shift operator. Besides its close relation to B -BMO, it is also a very convenient tool for a pointwise control of many operators arising in the Fourier-Bessel harmonic analysis.

The sharp maximal operator $M_\gamma^\#$ is defined by

$$M_\gamma^\# f(x) = \sup_{t>0} |E(0,t)|_\gamma^{-1} \int_{E(0,t)} |T^\gamma f(x) - f_{E(0,t)}(x)| (y')^\gamma dy.$$

If ν is a doubling measure (i.e., there exists a constant C such that $\nu(2Q) \leq C\nu(Q)$ for all cubes Q), then it satisfies the following John-Nirenberg inequality.

Theorem 2.10. [29] *Let $f \in \text{BMO}(X, \nu)$ and ν be a doubling measure. Then, for every $r > 0$ there exists two constants c_1 and c_2 , independent of f, r such that*

$$\nu(\{y \in B(x, t) : |f(x) - f_{B(x,t)}| > r\}) \leq c_1 \nu(B(x, t)) e^{-c_2 r / \|f\|_{\text{BMO}(X, \nu)}}.$$

It is clear that $\text{BMO}(X, \nu) = \text{BMO}_p(X, \nu)$ provided that the John-Nirenberg inequality holds.

Theorem 2.11. [30]

(1) *Let $f \in L_{1,y}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. If*

$$\|f\|_{\text{BMO}_{p,y}} = \sup_{t>0, x \in \mathbb{R}_{k,+}^n} \left(|E(0, t)|_y^{-1} \int_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|^p (y')^y dy \right)^{1/p} < \infty,$$

then for any $1 < p < \infty$

$$\|f\|_{\text{BMO}_y} \leq \|f\|_{\text{BMO}_{p,y}} \leq A_p \|f\|_{\text{BMO}_y},$$

where the constant A_p depends on p .

(2) *Let $f \in \text{BMO}_y$. Then, there exists a constant $c > 0$ such that*

$$|f_{E(0,r)} - f_{E(0,t)}| \leq c \|f\|_{\text{BMO}_y} \ln \frac{t}{r}, \quad 0 < 2r < t,$$

where c is independent of f, x, r and t .

Now let us define the B -maximal operator M_y and the B -Riesz potential $I_{\alpha,y}$ generated by the generalized shift operator as follows:

$$M_y f(x) = \sup_{r>0} \frac{1}{|E(0, r)|_y} \int_{E(0,r)} T^y(|f|)(x) (y')^y dy,$$

$$I_{\alpha,y} f(x) = \int_{\mathbb{R}_{k,+}^n} f(y) T^y |x|^{\alpha-Q} (y')^y dy, \quad 0 < \alpha < Q.$$

The following theorem deals with the boundedness of the B -maximal operator in the weighted $L_{p,\omega,y}$ space.

Theorem 2.12. [31]

(1) *If $f \in L_{1,\omega,y}$, $\omega \in A_{1,y}$, then $M_y f \in WL_{1,\omega,y}$ and*

$$\|M_y f\|_{WL_{1,\omega,y}} \leq c_1 \|f\|_{L_{1,\omega,y}},$$

where c_1 depends on ω, y and n .

(2) *If $f \in L_{p,\omega,y}$, $\omega \in A_{p,y}$, $1 < p < \infty$, then $M_y f \in L_{p,\omega,y}$ and*

$$\|M_y f\|_{L_{p,\omega,y}} \leq c_2 \|f\|_{L_{p,\omega,y}},$$

where c_2 depends on p, ω, y and n .

The following theorems deal with the boundedness of B -Riesz potentials $I_{\alpha,y}$ and the B -maximal operator M_y in B -Morrey spaces and the first one known as Hardy-Littlewood-Sobolev-type inequality for B -Riesz potentials.

Theorem 2.13. [25] *Let $0 < \alpha < Q$, $0 \leq \lambda < Q - \alpha$ and $1 \leq p \leq \frac{Q-\lambda}{\alpha}$.*

(1) *If $1 < p < \frac{Q-\lambda}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$ is necessary and sufficient condition for the boundedness of $I_{\alpha,y}$ from $L_{p,\lambda,y}$ to $L_{q,\lambda,y}$.*

- (2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\alpha}{Q - \lambda}$ is necessary and sufficient condition for the boundedness of $I_{\alpha, \gamma}$ from $L_{1, \lambda, \gamma}$ to $WL_{q, \lambda, \gamma}$.

Theorem 2.14. [9, 25]

- (1) If $f \in L_{1, \lambda, \gamma}$, $0 \leq \lambda < Q$, then $M_\gamma f \in WL_{1, \lambda, \gamma}$ and

$$\|M_\gamma f\|_{WL_{1, \lambda, \gamma}} \leq c_1 \|f\|_{L_{1, \lambda, \gamma}},$$

where c_1 depends only on λ, γ, k and n .

- (2) If $f \in L_{p, \lambda, \gamma}$, $1 < p < \infty$, $0 \leq \lambda < Q$, then $M_\gamma f \in L_{p, \lambda, \gamma}$ and

$$\|M_\gamma f\|_{L_{p, \lambda, \gamma}} \leq c_2 \|f\|_{L_{p, \lambda, \gamma}},$$

where c_2 depends on p, λ, γ, k and n .

3 Main results and proofs

3.1 Maximal commutators in $L_{p, \lambda, \gamma}$

Given a measurable function b , the commutator of the B -maximal operator M_γ is formally defined by

$$[M_\gamma, b]f = M_\gamma(bf) - bM_\gamma(f)$$

and the B -maximal commutator is defined by

$$M_{b, \gamma}(f)(x) = \sup_{r > 0} |E(0, r)|_\gamma^{-1} \int_{E(0, r)} T^\gamma |(b(x) - b(y))f(x)| (y')^\gamma dy, \quad \forall x \in \mathbb{R}_{k, +}^n.$$

Lemma 3.1.1. Let $1 < s < \infty$, $b \in \text{BMO}_\gamma$. Then, there exists $c_1 > 0$ such that

$$M_\gamma^\#(M_{b, \gamma}f)(x) \leq c_1 \|b\|_{\text{BMO}_\gamma} ((M_\gamma(M_\gamma f)^s)^{1/s}(x) + M_\gamma(M_\gamma |f|^s)^{1/s}(x)), \quad x \in \mathbb{R}_{k, +}^n.$$

Proof. By the boundedness of M_γ and the pointwise inequality, we have

$$M_\gamma^\#(M_{b, \gamma}f)(x) \leq 2M_\gamma(M_{b, \gamma}f)(x), \quad x \in \mathbb{R}_{k, +}^n.$$

Using Hölder's inequality, we have

$$\begin{aligned} & |E(0, t)|_\gamma^{-1} \int_{E(0, t)} T^z (|b(x) - b(z)| |f(x)|) (z')^\gamma dz \\ & \leq |E(0, t)|_\gamma^{-1} \int_{E(0, t)} T^z (|b(x) - b_{E(0, t)}| |f(x)|) (z')^\gamma dz + |E(0, t)|_\gamma^{-1} \int_{E(0, t)} |b(z) - b_{E(0, t)}| T^z (|f|)(x) (z')^\gamma dz \\ & \leq |E(0, t)|_\gamma^{-1} \left(\int_{E(0, t)} T^z |b(x) - b_{E(0, t)}|^{s'} (z')^\gamma dz \right)^{1/s'} \left(\int_{E(0, t)} T^z (|f|)^s (x) (z')^\gamma dz \right)^{1/s} \\ & \quad + |E(0, t)|_\gamma^{-1} \int_{E(0, t)} |b(z) - b_{E(0, t)}| T^z (|f|)(x) (z')^\gamma dz \\ & \leq C \|b\|_{\text{BMO}_\gamma} (M_\gamma |f|^s)^{1/s}(x) + |E(0, t)|_\gamma^{-1} \int_{E(0, t)} |b(z) - b_{E(0, t)}| T^z (|f|)(x) (z')^\gamma dz \end{aligned}$$

and

$$\begin{aligned}
& |E(0, r)|_y^{-1} \int_{E(0, r)} T^y \left[|E(0, t)|_y^{-1} \left(\int_{E(0, t)} |b(z) - b_{E(0, t)}| T^z(|f|)(x)(z')^y dz \right) \right] (y')^y dy \\
& \leq |E(0, r)|_y^{-1} \int_{E(0, r)} T^y [|b(z) - b_{E(0, t)}| M_y f(x)] (y')^y dy \\
& \leq |E(0, r)|_y^{-1} \left(\int_{E(0, r)} T^y |b(z) - b_{E(0, r)}|^{s'} (y')^y dy \right)^{1/s'} \left(\int_{E(0, r)} T^y (M_y f)^s(x) (y')^y dy \right)^{1/s} \\
& \quad + |E(0, r)|_y^{-1} \int_{E(0, r)} T^y [|b_{E(0, t)} - b_{E(0, r)}| M_y f(x)] (y')^y dy \\
& \leq C \|b\|_{\text{BMO}_y} (M_y (M_y f)^s)^{1/s}(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_y(M_{b,y}f)(x) &= \sup_{r>0} |E(0, r)|_y^{-1} \int_{E(0, r)} T^y [M_{b,y}f](x) (y')^y dy \\
&\leq C \|b\|_{\text{BMO}_y} \left((M_y (M_y f)^s)^{1/s}(x) + \sup_{r>0} |E(0, r)|_y^{-1} \int_{E(0, r)} T^y (M_y |f|^s)^{1/s}(x) (y')^y dy \right) \quad (3.1) \\
&\leq C \|b\|_{\text{BMO}_y} ((M_y (M_y f)^s)^{1/s}(x) + M_y (M_y |f|^s)^{1/s}(x)). \quad \square
\end{aligned}$$

Proposition 3.1.2. [32] Let (X, d, ν) be a space of homogeneous type in the sense of d be a pseudometric and ν be a positive Borel regular measure satisfying the following doubling condition: there exists a constant c such that

$$\nu(B(x, 2r)) \leq c \nu(B(x, r)) < \infty \quad \text{for all } x \in X \text{ and } r > 0.$$

There exists a positive constant C such that for any weight ω and any nonnegative function f with $\nu(\{x \in X: f(x) > \beta\}) < \infty$ for any $\beta > 0$,

- (i) if $\nu(X) = \infty$, then $\int_X f(x) \omega(x) d\nu(x) \leq C \int_X M^\# f(x) M\omega(x) d\nu(x)$,
- (ii) if $\nu(X) < \infty$, then $\int_X f(x) \omega(x) d\nu(x) \leq C \int_X M^\# f(x) M\omega(x) d\nu(x) + C\omega(X) \nu_X(f)$,

where $\omega(X) = \int_X \omega(x) d\nu(x)$, $\nu_X(f) = \frac{1}{\nu(X)} \int_X f(x) d\nu(x)$. The Fefferman-Stein sharp maximal operator is defined as $M^\# f(x) = \sup_{x \in Q} \frac{1}{\nu(Q)} \int_Q |f(y) - f_Q| d\nu(y)$, where $f_Q = \frac{1}{\nu(Q)} \int_Q f$ and the supremum is taken over all cubes Q containing the point x .

Lemma 3.1.3. Let $f \in L_{p,y}$, $\omega \in A_{p,y}$, $1 < p < \infty$. Then, $\|f\omega^{1/p}\|_{L_{p,y}} \leq C \|\omega^{1/p} M_y^\# f\|_{L_{p,y}}$, where a constant $c > 0$ is independent of f .

Proof. Let $f \in L_{p,y}(\mathbb{R}_{k,+}^n)$. We need to introduce another maximal operator defined on a space of homogeneous type (X, d, ν) . Here, $X = \mathbb{R}^n \times (0, \infty)^k = \{(x, \bar{x}'): x \in \mathbb{R}^n, \bar{x}' \in (0, \infty)^k\}$ equipped with the continuous pseudometric d and the positive measure ν satisfying the doubling condition:

$$\nu(E((x, \bar{x}'), 2r)) \leq c \nu(E((x, \bar{x}'), r)),$$

where a constant c is independent of $(x, \bar{x}')$, $r > 0$ and $E((x, \bar{x}'), r) = \{(y, \bar{y}') \in X: d((x, \bar{x}'), (y, \bar{y}')) < r\}$, $d\nu(x, \bar{x}') = (\bar{x}')^{y-1} dx d\bar{x}'$, $(\bar{x}')^{y-1} = (\bar{x}_1)^{y_1-1} \dots (\bar{x}_k)^{y_k-1}$ and $d((x, \bar{x}'), (y, \bar{y}')) = |(x, \bar{x}') - (y, \bar{y}')| \equiv (|x - y|^2 + (\bar{x}' - \bar{y}')^2)^{1/2}$. We define the maximal operator M_ν by

$$M_\nu \bar{f}(x, \bar{x}') = \sup_{r>0} \frac{1}{\nu(E((x, \bar{x}'), r))} \int_{E((x, \bar{x}'), r)} |\bar{f}(y, \bar{y}')| d\nu(y, \bar{y}'),$$

where $\bar{f}(x, \bar{x}') = f(\sqrt{x_1^2 + \bar{x}_1^2}, \dots, \sqrt{x_k^2 + \bar{x}_k^2}, x'')$, $x'' \in \mathbb{R}^{n-k}$. Coifman and Weiss [33] showed that the maximal operator M_ν is of weak type (1, 1) and is bounded on $L_p = L_p(X, d, \nu)$ for $1 < p < \infty$. The sharp maximal operator is defined by

$$M_\nu^\# \bar{f}(x, \bar{x}') = \sup_{r>0} \frac{1}{\nu(E((x, 0), r))} \int_{E((x, 0), r)} |T^y \bar{f}(x, \bar{x}') - \bar{f}_{E((x, 0), r)}(x, \bar{x}')| d\nu(y, \bar{y}'),$$

where $\bar{f}_{E((x, 0), r)}(x, \bar{x}') = \frac{1}{\nu(E((x, 0), r))} \int_{E((x, 0), r)} T^y \bar{f}(x, \bar{x}') d\nu(y, \bar{y}')$. Moreover, Guliyev and Hasanov [9, 25] proved that

$$M_\nu f(\sqrt{x_1^2 + \bar{x}_1^2}, \dots, \sqrt{x_k^2 + \bar{x}_k^2}, x'') = M_\nu \bar{f}(\sqrt{x_1^2 + \bar{x}_1^2}, \dots, \sqrt{x_k^2 + \bar{x}_k^2}, x'', 0)$$

and

$$M_\nu f(x) = M_\nu \bar{f}(x, 0).$$

So according to Proposition 3.1.2, we have

$$\begin{aligned} \|f\omega^{\frac{1}{p}}\|_{L_{p,y}} &= \|\bar{f}\omega^{\frac{1}{p}}\|_{L_p} = \sup_{\|\bar{g}\|_{L_{p',y}} \leq 1} \left| \int_X \bar{f}(y, \bar{y}') \bar{g}(y, \bar{y}') \omega^{\frac{1}{p}}(y, \bar{y}') d\nu(y, \bar{y}') \right|, \quad 1/p + 1/p' = 1 \\ &\leq C \sup_{\|\bar{g}\|_{L_{p',y}} \leq 1} \left| \int_X M_\nu^\# \bar{f}(y, \bar{y}') M_\nu(\bar{g}\omega^{\frac{1}{p}})(y, \bar{y}') d\nu(y, \bar{y}') \right| \\ &= C \sup_{\|g\|_{L_{p',y}} \leq 1} \left| \int_{\mathbb{R}_{k,+}^n} M_\nu^\# f(y) M_\nu(g\omega^{\frac{1}{p}})(y) (y')^y dy \right|. \end{aligned}$$

Finally, using Hölder's inequality with the conjugate exponents p , p' and Theorem 2.12, we get

$$\begin{aligned} \|f\omega^{\frac{1}{p}}\|_{L_{p,y}} &\leq C \sup_{\|g\|_{L_{p',y}} \leq 1} \|\omega^{\frac{1}{p}} M_\nu^\# f\|_{L_{p,y}} \|\omega^{\frac{1}{p}} M_\nu(g\omega^{\frac{1}{p}})\|_{L_{p',y}} \\ &\leq C \sup_{\|g\|_{L_{p',y}} \leq 1} \|\omega^{\frac{1}{p}} M_\nu^\# f\|_{L_{p,y}} \|g\|_{L_{p',y}} \\ &\leq C \|\omega^{\frac{1}{p}} M_\nu^\# f\|_{L_{p,y}}. \end{aligned}$$

□

Theorem 3.1.4. Let $b \in \text{BMO}_Y$, $1 < p < \infty$ and $\omega \in A_{p,Y}$. Then, the B -maximal commutator $M_{b,Y}$ is bounded on the weighted space $L_{p,\omega,Y}$.

Proof. By using Lemmas 3.1.1, 3.1.3 and Theorem 2.12, we have $M_{b,Y}$ is bounded on the space $L_{p,\omega,Y}$. □

The commutator $M_{b,Y}$ of the B -maximal operator M_Y and the B -maximal commutator $[M_Y, b]$ are essentially different from each other. Indeed, the commutator $M_{b,Y}$ is a positive and sublinear operator, but $[M_Y, b]$ is neither positive nor sublinear. However, if b satisfies some conditions, then the operator $M_{b,Y}$ controls $[M_Y, b]$.

A necessary and sufficient condition for the boundedness of the commutator $M_{b,Y}$ on the B -Morrey space $L_{p,\lambda,Y}$ is given by the following theorem.

Theorem 3.1.5. Let $1 < p < \infty$, $0 \leq \lambda < Q$. Then, the commutator $M_{b,Y}$ is bounded on the B -Morrey space $L_{p,\lambda,Y}$ if and only if $b \in \text{BMO}_Y$.

Proof. Sufficiency: Let the commutator $M_{b,Y}$ be bounded on $L_{p,\lambda,Y}$, $1 < p < \infty$. Since

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{t>0, x \in \mathbb{R}_{k,+}^n} \left(t^{-\lambda} \int_{E(0,t)} T^\gamma [|f|]^p(x) (y')^\gamma dy \right)^{1/p},$$

let us consider $f = \chi_{E(0,r)}$. We have

$$\begin{aligned} \|\chi_{E(0,r)}\|_{L_{p,\lambda,\gamma}} &= \sup_{t>0, x \in \mathbb{R}_{k,+}^n} \left(t^{-\lambda} \int_{E(0,t)} T^\gamma [\chi_{E(0,r)}]^p(x) (y')^\gamma dy \right)^{1/p} \\ &= \sup_{t>0, x \in \mathbb{R}_{k,+}^n} \left(t^{-\lambda} \int_{E(x,t)} \chi_{E(0,r)}(y) (y')^\gamma dy \right)^{1/p} \\ &= \sup_{E(x,t) \subset E(0,r)} (t^{-\lambda} |E(x,t) \cap E(0,r)|_\gamma)^{1/p} \leq r^{\frac{Q-\lambda}{p}}. \end{aligned}$$

Then,

$$\begin{aligned} &\frac{1}{|E(0,t)|_\gamma} \int_{E(0,t)} |T^z b(x) - b_{E(0,t)}| (z')^\gamma dz \\ &= \frac{1}{|E(0,t)|_\gamma} \int_{E(0,t)} \left| T^z b(x) - \frac{1}{|E(0,t)|_\gamma} \int_{E(0,t)} T^z b(y) (y')^\gamma dy \right| (z')^\gamma dz \\ &\leq \frac{1}{|E(0,t)|_\gamma} \int_{E(0,t)} \frac{1}{|E(0,t)|_\gamma} \int_{E(0,t)} |T^z b(x) - T^z b(y)| (y')^\gamma dy (z')^\gamma dz \\ &\leq \frac{1}{|E(0,t)|_\gamma} \int_{E(0,t)} M_{b,\gamma} \chi_{E(0,t)}(z) (z')^\gamma dz \leq C t^{-Q+\lambda} \|M_{b,\gamma} \chi_{E(0,t)}\|_{L_{p,\lambda,\gamma}} \|\chi_{E(0,t)}\|_{L_{p',\lambda,\gamma}} \\ &\leq C t^{\frac{Q-\lambda}{p'} + \frac{Q-\lambda}{p} - Q + \lambda} \leq C. \end{aligned}$$

This shows that $b \in \text{BMO}_\gamma$.

Necessity: Let $b \in \text{BMO}_\gamma$ and $f \in L_{p,\lambda,\gamma}$, $1 < p < \infty$, $0 \leq \lambda < Q$. Taking into account the properties of $A_{p,\gamma}$ we easily obtain $(M_\gamma \chi_{E(0,t)})^\delta \in A_{p,\gamma}$, for any $0 < \delta < 1$. Then, by using Lemma 3.1.3 and Theorem 3.1.4, we obtain

$$\begin{aligned} &\int_{E(0,t)} T^\gamma (|M_{b,\gamma} f|)^p(x) (y')^\gamma dy \\ &= \int_{\mathbb{R}_{k,+}^n} T^\gamma (|M_{b,\gamma} f|)^p(x) \chi_{E(0,t)}(y) (y')^\gamma dy \\ &\leq \int_{\mathbb{R}_{k,+}^n} T^\gamma (|M_{b,\gamma} f|)^p(x) (M_\gamma \chi_{E(0,t)}(y))^\delta (y')^\gamma dy \leq C \|b\|_{\text{BMO}_\gamma}^p \int_{\mathbb{R}_{k,+}^n} T^\gamma (|f|)^p(x) (M_\gamma \chi_{E(0,t)}(y))^\delta (y')^\gamma dy \\ &\leq C \|b\|_{\text{BMO}_\gamma}^p \left(\int_{E(0,t)} T^\gamma (|f|)^p(x) (y')^\gamma dy + \sum_{j=1}^{\infty} \int_{E(0,2^{j+1}t) \setminus E(0,2^j t)} T^\gamma (|f|)^p(x) (M_\gamma \chi_{E(0,t)}(y))^\delta (y')^\gamma dy \right) \\ &\leq C \|b\|_{\text{BMO}_\gamma}^p \left(\int_{E(0,t)} T^\gamma (|f|)^p(x) (y')^\gamma dy + \sum_{j=1}^{\infty} \int_{E(0,2^{j+1}t) \setminus E(0,2^j t)} T^\gamma (|f|)^p(x) \frac{t^{Q\delta}}{(|y| + t)^{Q\delta}} (y')^\gamma dy \right) \\ &\leq C \|b\|_{\text{BMO}_\gamma}^p \|f\|_{L_{p,\lambda,\gamma}}^p \left(t^\lambda + \sum_{j=1}^{\infty} \frac{1}{(2^j + 1)^{Q\delta}} (2^{j+1}t)^\lambda \right) \\ &\leq C t^\lambda \|b\|_{\text{BMO}_\gamma}^p \|f\|_{L_{p,\lambda,\gamma}}^p. \end{aligned}$$

Finally, we have

$$\|M_{b,y}f\|_{p,\lambda,y}^p = \sup_{t>0, x \in \mathbb{R}_{k,+}^n} t^{-\lambda} \int_{E(0,t)} T^y (|M_{b,y}f|)^p(x) (y')^y dy \leq \|b\|_{\text{BMO}_Y}^p \|f\|_{L_{p,\lambda,y}}^p. \quad \square$$

Theorem 3.1.6. Let $0 \leq \lambda < Q$, $b \in \text{BMO}_Y$. Then, the commutator $M_{b,y}$ is bounded from $L_{1,\lambda,y}$ to $WL_{1,\lambda,y}$.

Proof. Let $0 \leq \lambda < Q$, $f \in L_{1,\lambda,y}$. The result can be obtained from $f(x) \leq M_Y f(x)$. Hence, by using (3.1) and Theorem 2.14, we have

$$\|M_{b,y}f\|_{WL_{1,\lambda,y}} \leq \|M_Y(M_{b,y}f)\|_{WL_{1,\lambda,y}} \leq \|b\|_{\text{BMO}_Y} \|(M_Y(M_Y f)^s)^{1/s} + M_Y(M_Y |f|^s)^{1/s}\|_{WL_{1,\lambda,y}} \leq \|b\|_{\text{BMO}_Y} \|f\|_{L_{1,\lambda,y}}. \quad \square$$

3.2 Commutators of the B -Riesz potential in B -Morrey spaces

In this section, we consider the commutator of the B -Riesz potential defined by

$$[b, I_{\alpha,y}]f(x) = \int_{\mathbb{R}_{k,+}^n} (b(x) - b(y)) |y|^{\alpha-Q} T^y f(x) (y')^y dy, \quad 0 < \alpha < Q$$

and the operator $|b, I_{\alpha,y}|$ given by

$$|b, I_{\alpha,y}|f(x) = \int_{\mathbb{R}_{k,+}^n} |b(x) - b(y)| |y|^{\alpha-Q} T^y [|f|](x) (y')^y dy, \quad 0 < \alpha < Q.$$

Theorem 3.2.1. Let $0 < \alpha < Q$, $0 \leq \lambda < Q - \alpha$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$ and $1 < p < \frac{Q-\lambda}{\alpha}$. Then, the commutator $|b, I_{\alpha,y}|$ is bounded from $L_{p,\lambda,y}$ to $L_{q,\lambda,y}$ if and only if $b \in \text{BMO}_Y$.

Proof. Sufficiency: Let $|b, I_{\alpha,y}|$ be bounded from $L_{p,\lambda,y}$ to $L_{q,\lambda,y}$, $1 < p < \frac{Q-\lambda}{\alpha}$. We have

$$\begin{aligned} & \frac{1}{|E(0,t)|_Y} \int_{E(0,t)} |T^z b(x) - f_{E(0,t)}| (z')^y dz \\ &= \frac{1}{|E(0,t)|_Y} \int_{E(0,t)} \left| T^z b(x) - \frac{1}{|E(0,t)|_Y} \int_{E(0,t)} T^z b(y) (y')^y dy \right| (z')^y dz \\ &\leq \frac{1}{|E(0,t)|_Y^{1+\frac{\alpha}{Q}}} \int_{E(0,t)} \frac{1}{|E(0,t)|_Y^{1-\frac{\alpha}{Q}}} \int_{E(0,t)} |T^z b(x) - T^z b(y)| (y')^y dy (z')^y dz \\ &\leq \frac{1}{|E(0,t)|_Y^{1+\frac{\alpha}{Q}}} \int_{E(0,t)} \int_{E(0,t)} \frac{|T^z(b(x) - b(y))|}{|y|^{Q-\alpha}} (y')^y dy (z')^y dz \\ &\leq \frac{1}{|E(0,t)|_Y^{1+\frac{\alpha}{Q}}} \int_{E(0,t)} |b, I_{\alpha,y}| \chi_{E(0,t)}(z) (z')^y dz \\ &\leq Ct^{-Q-\alpha+\lambda} \| |b, I_{\alpha,y}| \chi_{E(0,t)} \|_{L_{q,\lambda,y}} \| \chi_{E(0,t)} \|_{L_{q',\lambda,y}} \leq Ct^{\frac{Q-\lambda}{q'} + \frac{Q-\lambda}{p} - Q - \alpha + \lambda} \leq C. \end{aligned}$$

This shows that $b \in \text{BMO}_Y$.

Necessity: Let $f \in L_{p,\lambda,y}$. Then,

$$|b, I_{\alpha,y}|f(x) = \left(\int_{E(0,t)} + \int_{\mathbb{R}_{k,+}^n \setminus E(0,t)} \right) T^y [|b - b(x)]f(x) |y|^{\alpha-Q} (y')^y dy \equiv F_1(x, t) + F_2(x, t). \quad (3.2)$$

First, let us estimate $F_1(x, t)$. By using Hölder's inequality, we have

$$\begin{aligned}
 F_1(x, t) &= \int_{E(0, t)} T^\gamma [b - b(x)] f(x) |y|^{\alpha-Q} (y')^\gamma dy \\
 &\leq \sum_{j=-\infty}^{-1} (2^j t)^{\alpha-Q} \int_{E(0, 2^{j+1}t) \setminus E(0, 2^j t)} T^\gamma [b - b(x)] f(x) (y')^\gamma dy \\
 &\leq \sum_{j=-\infty}^{-1} (2^j t)^{\alpha-Q} \left(\int_{E(0, 2^{j+1}t) \setminus E(0, 2^j t)} |T^\gamma b(x) - b|^{p'} (y')^\gamma dy \right)^{1/r'} \\
 &\quad \times \left(\int_{E(0, 2^{j+1}t) \setminus E(0, 2^j t)} T^\gamma [|f|]^r (x) (y')^\gamma dy \right)^{1/r} \\
 &\leq \|b\|_{\text{BMO}_\gamma} (M_\gamma(|f|^r)(x))^{1/r} \sum_{j=-\infty}^{-1} (2^j t)^\alpha.
 \end{aligned}$$

Hence,

$$F_1(x, t) \leq Ct^\alpha \|b\|_{\text{BMO}_\gamma} (M_\gamma(|f|^r)(x))^{1/r}. \quad (3.3)$$

Now by using Hölder's inequality, let us estimate $F_2(x, t)$. We get

$$\begin{aligned}
 F_2(x, t) &\leq \int_{\mathbb{R}_{k,+}^n \setminus E(0, t)} T^\gamma (|b - b(x)| |f(x)|) |y|^{\alpha-Q} (y')^\gamma dy \\
 &\leq \sum_{j=0}^{\infty} (2^j t)^{\alpha-Q} \int_{E(0, 2^{j+1}t) \setminus E(0, 2^j t)} T^\gamma (|b - b(x)| |f(x)|) (y')^\gamma dy \\
 &\leq \sum_{j=0}^{\infty} (2^j t)^{\alpha-Q} \left(\int_{E(0, 2^{j+1}t) \setminus E(0, 2^j t)} |T^\gamma b(x) - b|^{p'} (y')^\gamma dy \right)^{1/p'} \\
 &\quad \times \left(\int_{E(0, 2^{j+1}t) \setminus E(0, 2^j t)} T^\gamma [|f|]^p (x) (y')^\gamma dy \right)^{1/p}, \quad 1/p + 1/p' = 1 \\
 &\leq 2^{\frac{Q-\lambda}{p'}} t^{\alpha - \frac{Q-\lambda}{p}} \|b\|_{\text{BMO}_\gamma} \|f\|_{L_{p,\lambda,\gamma}} \sum_{j=0}^{\infty} 2^{j\left(\alpha - \frac{Q-\lambda}{p}\right)}.
 \end{aligned}$$

Thus,

$$F_2(x, t) \leq Ct^{\alpha - \frac{Q-\lambda}{p}} \|b\|_{\text{BMO}_\gamma} \|f\|_{L_{p,\lambda,\gamma}}. \quad (3.4)$$

So, from (3.3) and (3.4) we have

$$|b, I_{\alpha,\gamma} f(x)| \leq Ct^\alpha \|b\|_{\text{BMO}_\gamma} (M_\gamma(|f|^r)(x))^{1/r} + Ct^{\alpha - \frac{Q-\lambda}{p}} \|b\|_{\text{BMO}_\gamma} \|f\|_{L_{p,\lambda,\gamma}}.$$

Minimizing with respect to $t = [(M_\gamma(|f|^r)(x))^{-1/r} \|f\|_{L_{p,\lambda,\gamma}}]^{p/(Q-\lambda)}$ we obtain

$$|b, I_{\alpha,\gamma} f(x)| \leq C \|b\|_{\text{BMO}_\gamma} ((M_\gamma(|f|^r)(x))^{1/r})^{p/q} \|f\|_{L_{p,\lambda,\gamma}}^{1-p/q}.$$

Hence, by Theorem 2.14, we get

$$\begin{aligned} \int_{E(0,t)} T^\gamma [||b, I_{\alpha,\gamma}|f||^q](x)(y')^\gamma dy &\leq C \|b\|_{\text{BMO}_\gamma}^q \|f\|_{L_{p,\lambda,\gamma}}^{q-p} \int_{E(0,t)} T^\gamma (M_\gamma(|f|^r)(x))^{p/r} (y')^\gamma dy \\ &\leq C t^\lambda \|b\|_{\text{BMO}_\gamma}^q \|f\|_{L_{p,\lambda,\gamma}}^{q-p} \|f\|_{L_{p,\lambda,\gamma}}^p = C t^\lambda \|b\|_{\text{BMO}_\gamma}^q \|f\|_{L_{p,\lambda,\gamma}}^q. \end{aligned}$$

So,

$$\| |b, I_{\alpha,\gamma}|f \|_{L_{p,\lambda,\gamma}} \leq C \|b\|_{\text{BMO}_\gamma} \|f\|_{L_{p,\lambda,\gamma}}. \quad \square$$

Theorem 3.2.2. Let $0 < \alpha < Q$, $0 \leq \lambda < Q - \alpha$, $1 - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$ and $b \in \text{BMO}_\gamma$. Then, the commutator $|b, I_{\alpha,\gamma}|$ is bounded from $L_{1,\lambda,\gamma}$ to $WL_{q,\lambda,\gamma}$.

Proof. Let $f \in L_{1,\lambda,\gamma}$. We have

$$|\{x \in E(0, t) : ||b, I_{\alpha,\gamma}|f|(x)| > 2\beta\}|_\gamma \leq |\{x \in E(0, t) : F_1(x, t) > \beta\}|_\gamma + |\{x \in E(0, t) : F_2(x, t) > \beta\}|_\gamma,$$

where $F_1(x, t)$ and $F_2(x, t)$ are defined in (3.2). Now taking into account (3.3) and Theorem 2.14, we have

$$|\{x \in E(0, t) : |F_1(x, t)| > \beta\}|_\gamma \leq \left| \left\{ x \in E(0, t) : (M_\gamma(|f|^r)(x))^{1/r} > \frac{\beta}{C t^\alpha \|b\|_{\text{BMO}_\gamma}} \right\} \right|_\gamma \leq \frac{C t^\alpha}{\beta} \|b\|_{\text{BMO}_\gamma} \|f\|_{L_{1,\lambda,\gamma}}.$$

Also taking into account (3.4), for $\beta = C t^{\alpha - \frac{Q-\lambda}{p}} \|b\|_{\text{BMO}_\gamma} \|f\|_{L_{1,\lambda,\gamma}}$ we have

$$F_2(x, t) \leq \beta, \quad \text{that is, } |\{x \in E(0, t) : F_2(x, t) > \beta\}|_\gamma = 0.$$

Finally,

$$|\{x \in E(0, t) : ||b, I_{\alpha,\gamma}|f|(x)| > 2\beta\}|_\gamma \leq C \frac{1}{\beta^q} \|b\|_{\text{BMO}_\gamma}^q \frac{1}{\beta^q} \|f\|_{L_{1,\lambda,\gamma}}^q. \quad \square$$

The following theorem is known as the Hardy-Littlewood-Sobolev-type theorem for the commutator of the B -Riesz potentials and it gives a necessary and sufficient condition for the boundedness of the operator $|b, I_{\alpha,\gamma}|$ from $L_{p,\lambda,\gamma}$ to $L_{q,\lambda,\gamma}$

Theorem 3.2.3. Let $0 < \alpha < Q$, $0 \leq \lambda < Q - \alpha$, $b \in \text{BMO}_\gamma$ and $1 \leq p < \frac{Q-\lambda}{\alpha}$.

(1) If $1 < p < \frac{Q-\lambda}{\alpha}$, then $|b, I_{\alpha,\gamma}|$ is bounded from $L_{p,\lambda,\gamma}$ to $L_{q,\lambda,\gamma}$ if and only if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$.

(2) If $p = 1$, then $|b, I_{\alpha,\gamma}|$ is bounded from $L_{1,\lambda,\gamma}$ to $WL_{q,\lambda,\gamma}$ if and only if $1 - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$.

Proof.

(1) Sufficiency: Let $1 < p < \frac{Q-\lambda}{\alpha}$. Let us define $f_t(x) = f(tx)$, $t > 0$. Then,

$$\|f_t\|_{L_{p,\lambda,\gamma}} = t^{-\frac{Q}{p}} \sup_{r>0, x \in \mathbb{R}_{k,+}^n} \left(r^{-\lambda} \int_{E(0,tr)} T^\gamma [|f|]^p(tx)(y')^\gamma dy \right)^{1/p} = t^{-\frac{Q-\lambda}{p}} \|f\|_{L_{p,\lambda,\gamma}}$$

and we have

$$\begin{aligned} \| |b, I_{\alpha,\gamma}|f_t \|_{L_{q,\lambda,\gamma}} &= t^{-\alpha} \sup_{r>0, x \in \mathbb{R}_{k,+}^n} \left(r^{-\lambda} \int_{E(0,r)} T^{\gamma} [||b, I_{\alpha,\gamma}|f||^q](tx)(y')^\gamma dy \right)^{1/q} \\ &= t^{-\alpha - \frac{Q}{q}} \sup_{r>0, x \in \mathbb{R}_{k,+}^n} \left(r^{-\lambda} \int_{E(0,tr)} T^\gamma [||b, I_{\alpha,\gamma}|f||^q](x)(y')^\gamma dy \right)^{1/q} \\ &= t^{-\alpha - \frac{Q-\lambda}{q}} \| |b, I_{\alpha,\gamma}|f \|_{L_{q,\lambda,\gamma}}. \end{aligned}$$

Since the operator $|b, I_{\alpha, \gamma}|$ is bounded from $L_{p, \lambda, \gamma}$ to $L_{q, \lambda, \gamma}$, we get

$$\| |b, I_{\alpha, \gamma}| f \|_{L_{q, \lambda, \gamma}} \leq c_1 t^{\alpha + \frac{Q-\lambda}{q} - \frac{Q-\lambda}{p}} \|b\|_{\text{BMO}_\gamma} \|f\|_{L_{p, \lambda, \gamma}},$$

where c_1 depends on $p, q, \lambda, \gamma, k, n$. If $\frac{Q-\lambda}{p} < \frac{Q-\lambda}{q} + \alpha$, then $\| |b, I_{\alpha, \gamma}| f \|_{L_{q, \lambda, \gamma}} = 0$ for all $f \in L_{p, \lambda, \gamma}$ as $t \rightarrow 0$. Similarly, if $\frac{Q-\lambda}{p} > \frac{Q-\lambda}{q} + \alpha$, then $\| |b, I_{\alpha, \gamma}| f \|_{L_{q, \lambda, \gamma}} = 0$ for all $f \in L_{p, \lambda, \gamma}$ as $t \rightarrow \infty$. Therefore, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$. The necessity follows from Theorem 3.2.1.

(2) Sufficiency: Let $|b, I_{\alpha, \gamma}|$ bounded from $L_{1, \lambda, \gamma}$ to $WL_{q, \lambda, \gamma}$. We have

$$\begin{aligned} \| |b, I_{\alpha, \gamma}| f_t \|_{WL_{q, \lambda, \gamma}} &= \sup_{r>0} r \sup_{\tau>0, x \in \mathbb{R}_{k,+}^n} \left(\tau^{-\lambda} \int_{\{y \in E(0, \tau) : T^y |b, I_{\alpha, \gamma}| f_t (x) > r\}} (y')^\gamma dy \right)^{1/q} \\ &= t^{-\alpha} \sup_{r>0} r t^\alpha \sup_{\tau>0, x \in \mathbb{R}_{k,+}^n} \left(\tau^{-\lambda} \int_{\{y \in E(0, \tau) : T^y |b, I_{\alpha, \gamma}| f (tx) > r t^\alpha\}} (y')^\gamma dy \right)^{1/q} \\ &= t^{-\alpha - \frac{Q}{q}} \sup_{r>0} r t^\alpha \sup_{\tau>0, x \in \mathbb{R}_{k,+}^n} \left(t^\lambda (t\tau)^{-\lambda} \int_{\{y \in E(0, t\tau) : T^y |b, I_{\alpha, \gamma}| f (x) > r t^\alpha\}} (y')^\gamma dy \right)^{1/q} \\ &= t^{-\alpha - \frac{Q-\lambda}{q}} \| |b, I_{\alpha, \gamma}| f \|_{WL_{q, \lambda, \gamma}}. \end{aligned}$$

Since the operator $|b, I_{\alpha, \gamma}|$ bounded from $L_{1, \lambda, \gamma}$ to $WL_{q, \lambda, \gamma}$, we get

$$\| |b, I_{\alpha, \gamma}| f \|_{WL_{q, \lambda, \gamma}} \leq c_2 t^{\alpha + \frac{Q-\lambda}{q} - (Q-\lambda)} \|b\|_{\text{BMO}_\gamma} \|f\|_{L_{1, \lambda, \gamma}},$$

where c_2 depends on q, λ, γ, k, n .

If $1 < \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then $\| |b, I_{\alpha, \gamma}| f \|_{WL_{q, \lambda, \gamma}} = 0$ for all $f \in L_{1, \lambda, \gamma}$ as $t \rightarrow 0$. Similarly, if $1 > \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then we have $\| |b, I_{\alpha, \gamma}| f \|_{WL_{q, \lambda, \gamma}} = 0$ for all $f \in L_{1, \lambda, \gamma}$ as $t \rightarrow \infty$. Therefore, we have $1 = \frac{1}{q} + \frac{\alpha}{Q-\lambda}$. The necessity of the theorem follows from Theorem 3.2.2. $\square$

3.3 Commutators of B -singular integrals in B -Morrey spaces

Let us consider the Calderón-Zygmund-type B -singular integral operator generated by the generalized shift operator, defined by

$$A_\gamma f(x) = \int_{\mathbb{R}_{k,+}^n} T^\gamma f(x) K(y) (y')^\gamma dy,$$

where the kernel $K(x)$ is a singular kernel and satisfying the following conditions:

$$\begin{aligned} \left| \int_{\{x \in \mathbb{R}_{k,+}^n : \varepsilon < |x| < r\}} K(x) (x')^\gamma dx \right| &\leq C, \quad 0 < \varepsilon < r < \infty, \\ \int_{\{x \in \mathbb{R}_{k,+}^n : r < |x| < 4r\}} |K(x)| (x')^\gamma dx &\leq C, \quad 0 < r < \infty, \\ \int_{\{x \in \mathbb{R}_{k,+}^n : |x| \geq 4|y|\}} |T^\gamma K(x) - K(x)| (x')^\gamma dx &\leq C, \quad |y| < \frac{1}{4}. \end{aligned}$$

We know that

$$A_\gamma f(x) = \lim_{\varepsilon \rightarrow 0+} A_{\varepsilon, \gamma} f(x), \quad (3.5)$$

where $A_{\varepsilon, \gamma} f(x) = \int_{\{y \in \mathbb{R}_{k,+}^n : |y| > \varepsilon\}} T^\gamma f(x) K(y) (y')^\gamma dy$, $\varepsilon > 0$ (see [27, 34]).

Proposition 3.3.1. [35] *Let (X, d, ν) be a space of homogeneous type, $1 \leq p < \infty$ and $b \in \text{BMO}(X, d, \nu)$. Then, the commutator of the classical Calderón-Zygmund singular integral operator is bounded on $L_p(X, d, \nu)$.*

The commutator of the B -singular integral operator A_γ with a function b is defined by

$$[b, A_\gamma]f = bA_\gamma(f) - A_\gamma(bf).$$

In the next theorem, we prove the boundedness of the commutator $[b, A_\gamma]$ on B -Morrey spaces.

Theorem 3.3.2. *Let $1 < p < \infty$, $b \in \text{BMO}_\gamma$. Then, the commutator $[b, A_\gamma]$ is bounded on the B -Morrey space $L_{p, \lambda, \gamma}$.*

Proof. We need to introduce a specific maximal operator which is defined on a homogeneous-type space (X, d, ν) . That is a topological space $X = \mathbb{R}^n \times (0, \infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu(E((x, \bar{x}'), 2r)) \leq C_1 \nu(E((x, \bar{x}'), r)),$$

where the constant C_1 is independent of $(x, \bar{x}')$ and $r > 0$.

Furthermore, we set $E((x, \bar{x}'), r) = \{(y, \bar{y}') \in X : d((x, \bar{x}'), (y, \bar{y}')) < r\}$, $d\nu(y, \bar{y}') = (\bar{y}')^{\gamma-1} dy \, d\bar{y}'$ and $(\bar{y}')^{\gamma-1} = (\bar{y}'_1)^{\gamma_1-1} \dots (\bar{y}'_k)^{\gamma_k-1}$, $d((x, \bar{x}'), (y, \bar{y}')) = |(x, \bar{x}') - (y, \bar{y}')| = (|x - y|^2 + (\bar{x}' - \bar{y}')^2)^{\frac{1}{2}}$.

Let us define

$$[b, A_\gamma \bar{f}](x, \bar{x}') = \int_{E^*((x, \bar{x}'), r)} (b(y, \bar{y}') - b(x, \bar{x}')) \bar{f}(x - y, \bar{x}' - \bar{y}') K(y, \bar{y}') d\nu(y),$$

where $\bar{f}(x, \bar{x}') = f(\sqrt{x_1^2 + \bar{x}_1^2}, \dots, \sqrt{x_k^2 + \bar{x}_k^2}, x'')$.

With easy calculations, it is possible to prove the following inequality:

$$[b, A_\gamma f](\sqrt{z_1^2 + \bar{z}_1^2}, \dots, \sqrt{z_k^2 + \bar{z}_k^2}, z'') = [b, A_\gamma \bar{f}](\sqrt{z_1^2 + \bar{z}_1^2}, \dots, \sqrt{z_k^2 + \bar{z}_k^2}, z'', 0). \quad (3.6)$$

So,

$$[b, A_\gamma f](x) = [b, A_\gamma \bar{f}](x, 0). \quad (3.7)$$

Indeed, taking into account Lemma 2.5 and $(M\chi_{E((x, 0), r)}(y))^\theta \in A_{p, \gamma}$, $0 < \theta < 1$, we have

$$\int_{\mathbb{R}_{k,+}^n} T^\gamma [|f|]^p(x) (M_\gamma \chi_{E((0, r)}(y))^\theta (y')^\gamma dy = \int_X |\bar{f}(\sqrt{y_1^2 + \bar{y}_1^2}, \dots, \sqrt{y_k^2 + \bar{y}_k^2}, y'')|^p (M_\gamma \chi_{E((x, 0), r)}(y, \bar{y}'))^\theta d\nu(y, \bar{y}')$$

and since

$$|E(0, r)|_\gamma = \nu E((\sqrt{z_1^2 + \bar{z}_1^2}, \dots, \sqrt{z_k^2 + \bar{z}_k^2}, z''), 0, r),$$

then we have (3.6). Furthermore, taking $\bar{z}_k = 0$ in (3.6) we get (3.7). By using Lemma 2.5 and (3.6) we have

$$\begin{aligned}
& \int_{E(0,r)} T^y |[b, A_y f]|^p(x) (y')^y dy \\
& \leq \int_{\mathbb{R}_{k,+}^n} T^y |[b, A_y f]|^p(x) (M_y \chi_{E(0,r)}(y))^\theta (y')^y dy \\
& = \int_{\mathbb{R}^{n \times (0, \infty)^k}} |[b, A_y f](\sqrt{z_1^2 + \bar{z}_1^2}, \dots, \sqrt{z_k^2 + \bar{z}_k^2}, z'')|^p (M_y \chi_{E((x,0),r)}(z, \bar{z}'))^\theta dz, \bar{z}') \\
& = \int_X |[b, A_y \bar{f}](\sqrt{z_1^2 + \bar{z}_1^2}, \dots, \sqrt{z_k^2 + \bar{z}_k^2}, z'', 0)|^p (M_y \chi_{E((x,0),r)}(z, \bar{z}'))^\theta dv(z, \bar{z}').
\end{aligned}$$

Then, taking $f(y, \bar{y}') = \bar{f}(\sqrt{y_1^2 + \bar{y}_1'^2}, \dots, \sqrt{y_k^2 + \bar{y}_k'^2}, y'', 0)$ and $\varphi(y, \bar{y}') \equiv (M_y \chi_{E((x, \bar{x}'), r)}(y, \bar{y}'))^\theta$ we obtain from Proposition 3.3.1 and Lemma 2.5 that

$$\begin{aligned}
& \int_{E(0,r)} T^y |[b, A_y f](x)|^p (y')^y dy \\
& \leq \int_X |[b, A_y \bar{f}](\sqrt{y_1^2 + \bar{y}_1'^2}, \dots, \sqrt{y_k^2 + \bar{y}_k'^2}, y'', 0)|^p (M_y \chi_{E((x,0),r)}(y, \bar{y}'))^\theta dv(y, \bar{y}') \\
& \leq C \|b\|_{\text{BMO}_y}^p \int_X |\bar{f}(\sqrt{y_1^2 + \bar{y}_1'^2}, \dots, \sqrt{y_k^2 + \bar{y}_k'^2}, y'', 0)|^p (M_y \chi_{E((x,0),r)}(y, \bar{y}'))^\theta dv(y, \bar{y}') \\
& = C \|b\|_{\text{BMO}_y}^p \int_X |f(\sqrt{y_1^2 + \bar{y}_1'^2}, \dots, \sqrt{y_k^2 + \bar{y}_k'^2}, y'')|^p (M_y \chi_{E((x,0),r)}(y, \bar{y}'))^\theta dv(y, \bar{y}') \\
& = C \|b\|_{\text{BMO}_y}^p \int_{\mathbb{R}_{k,+}^n} T^y [|f|]^p(x) (M_y \chi_{E(0,r)}(y))^\theta (y')^y dy \\
& \leq C \|b\|_{\text{BMO}_y}^p \int_{E(0,r)} T^y [|f|]^p(x) (y')^y dy + C \|b\|_{\text{BMO}_y}^p \sum_{j=1}^{\infty} \int_{E(0, 2^{j+1}r) \setminus E(0, 2^j r)} T^y [|f|]^p(x) (M_y \chi_{E(0,r)}(y))^\theta (y')^y dy \\
& \leq C \|b\|_{\text{BMO}_y}^p \int_{E(0,r)} T^y [|f|]^p(x) (y')^y dy + C \|b\|_{\text{BMO}_y}^p \sum_{j=1}^{\infty} \int_{E(0, 2^{j+1}r) \setminus E(0, 2^j r)} T^y [|f|]^p(x) \frac{r^{Q\theta}}{(|y| + r)^{Q\theta}} (y')^y dy \\
& \leq C \|b\|_{\text{BMO}_y}^p \|f\|_{L_{p,\lambda,y}}^p \left(r^\lambda + \sum_{j=1}^{\infty} \frac{1}{(2^j + 1)^{Q\theta}} (2^{j+1}r)^\lambda \right) \leq C r^\lambda \|b\|_{\text{BMO}_y}^p \|f\|_{L_{p,\lambda,y}}^p.
\end{aligned}$$

Finally,

$$\|[b, A_y f]\|_{L_{p,\lambda,y}} \leq C \|b\|_{\text{BMO}_y} \|f\|_{L_{p,\lambda,y}}.$$

□

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