Research Article

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The H-force sets of the graphs satisfying the condition of Ore's theorem

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Abstract: Let G be a Hamiltonian graph. A nonempty vertex set $X \subseteq V(G)$ is called a Hamiltonian cycle enforcing set (in short, an H-force set) of G if every X -cycle of G (i.e., a cycle of G containing all vertices of X) is a Hamiltonian cycle. For the graph G , $h(G)$ (called the H-force number of G) is the smallest cardinality of an H-force set of G . Ore's theorem states that an n -vertex graph G is Hamiltonian if $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices u, v of G . In this article, we study the H-force sets of the graphs satisfying the condition of Ore's theorem, show that the H-force number of these graphs is possibly n , or $n - 2$, or $\frac{n}{2}$ and give a classification of these graphs due to the H-force number.

Keywords: H-force set, H-force number, Ore's theorem, weak closure

MSC 2010: 05C07, 05C45

1 Terminology and introduction

In this article, we study simple graphs without loops or parallel edges. For terminology and notations not defined here we refer the reader to [1]. Let G be a graph with n vertices. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. For a subset $X \subseteq V(G)$, the cardinality of X is denoted by $|X|$. A subgraph of G induced by a subset X is denoted by $G[X]$. In addition, $G - X = G[V(G) - X]$. Let S be a subset of $V(G)$ or a subgraph of G and v is not in S . The set $N_S(v) = \{x | vx \in E(G), x \in S\}$ and $d_S(v) = |N_S(v)|$. Let $\delta(G)$ denote the minimum degree of G .

A graph G is said to be k -connected, if $V(G) \geq k + 1$ and $G - S$ is connected for each $X \subseteq V(G)$ with $|S| \leq k - 1$. The subset $S \subseteq V(G)$ is called an *independent set* of G if any two vertices in S are nonadjacent. In addition, the subset S is called the *largest independent set* of G if there is no independent set S' such that $|S'| > |S|$.

For two disjoint graphs G and H , $G \vee H$ denotes the graph such that every vertex of G is adjacent to every vertex of H . A complete graph with n vertices is denoted by K_n . $K_{m,n}$ is a complete bipartite graph with two parts V_1, V_2 and $|V_1| = m, |V_2| = n$.

A cycle of G is called the *Hamiltonian cycle*, if it contains all vertices of G . The graph G is said to be *Hamiltonian* if it has a Hamiltonian cycle. It is well known that the problem to check whether a graph has a Hamiltonian cycle or not is NP-complete. The various kinds of sufficient conditions to imply a graph to be Hamiltonian have been studied widely. More details can be found in [2–5].

Let G be a Hamiltonian graph with n vertices. For a subset $X \subseteq V(G)$, an X -cycle of G is a cycle containing all vertices of X . A nonempty vertex set $X \subseteq V(G)$ is called a *Hamiltonian cycle enforcing set* (in short, an *H-force set*) of G if every X -cycle of G is a Hamiltonian cycle. For the graph G , $h(G)$ denotes the smallest cardinality of an H-force set of G and is called the *H-force number* of G . The subset X is called the *minimum*

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H-force set of G if X is an H -force set with $|X| = h(G)$, i.e., X is an H -force set and there is no H -force set X' with $|X'| < |X|$.

The H -force set and H -force number were introduced by Fabrici et al. in [6]. These concepts play an important role in the Hamiltonian problems. In [6], the authors also investigated the H -force numbers of the bipartite graphs, the outerplanar Hamiltonian graphs and so on. Recently, these concepts were generalized to digraphs by Zhang et al. in [7] and to hypertournaments by Li et al. in [8], and they gave a characterization of the minimum H -force sets of locally semicomplete digraphs and hypertournaments and obtained their H -force numbers.

If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices u and v of G , we say that G satisfies the condition of Ore's theorem. For convenience, we call a graph satisfying the condition of Ore's theorem an OTG. In [9], Ore proved that any OTG is Hamiltonian. In this article, we study the H -force sets and H -force number of the OTGs.

The following is an easy observation on the H -force sets of a graph.

Proposition 1.1. [6] *Let G be a Hamiltonian graph. If C is a non-Hamiltonian cycle of G , then any H -force set of G contains a vertex of $V(G) \setminus V(C)$.*

To present Theorem 1.2, we consider a special class of graphs, namely,

$$\psi_{2m+1} = \{Z_m \vee (K_m^c + \{u\}) \mid Z_m \text{ is a graph with } m \text{ vertices}\},$$

where K_m^c is a set of m vertices (also as the complement of the complete graph K_m) and u is another single vertex; furthermore, the edge set of $Z_m \vee (K_m^c + \{u\})$ consists of $E(Z_m)$ and $\{xy \mid x \in V(Z_m) \text{ and } y \in K_m^c \cup \{u\}\}$ (Figure 1). In [2], Li et al. investigated the graphs satisfying $d(u) + d(v) \geq n - 1$ for every pair of vertices u, v with $d(u, v) = 2$ and proved the following.

Theorem 1.2. [2] *Let G be a 2-connected graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of vertices u, v with $d(u, v) = 2$, then G is a Hamiltonian graph, unless n is odd and $G \in \psi_n$.*

By Theorem 1.2, we obtain immediately the following corollary.

Corollary 1.3. *Let G be a 2-connected graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of nonadjacent vertices u, v of G , then G is a Hamiltonian graph, unless n is odd and $G \in \psi_n$.*

Figure 1: $G \in \psi_{2m+1}$.

2 H-force sets of the OTGs

Lemma 2.1. *Let G be an OTG with n vertices and $X \subseteq V(G)$ be a vertex subset. If $d(u) + d(v) \geq n + 1$ for some pairs of nonadjacent vertices u and v , then*

- (1) *X is an H -force set of G if and only if X is an H -force set of $G + uv$, and*
- (2) *X is the minimum H -force set of G if and only if X is the minimum H -force set of $G + uv$, namely, $h(G) = h(G + uv)$.*

Proof. First, we prove (1). Note that a cycle of the graph G is always a cycle of the graph $G + uv$. It is easy to see that if X is an H-force set of $G + uv$, then X is an H-force set of G .

Suppose that X is an H-force set of G , but not an H-force set of $G + uv$. Let C be the longest non-Hamiltonian cycle containing all vertices of X in $G + uv$ and H be the subgraph induced by $V(C)$ in G , i.e., $H = G[V(C)]$. Clearly, the cycle C contains the edge uv . Assume that there are $a(\geq 1)$ vertices apart from $V(C)$. Then, H contains $n - a$ vertices. Let $C = v_1 v_2 \dots v_{n-a} v_1$ with $u = v_1$ and $v = v_{n-a}$. Then, $v_1 v_2 \dots v_{n-a}$ is a (u, v) -path of G containing all vertices of X . Let $S = \{v_i | uv_{i+1} \in E(H)\}$, $T = \{v_i | v_i v \in E(H)\}$, where the subscripts are taken modulo $n - a$. Since $v_{n-a} \notin S \cup T$, we have $|S \cup T| < n - a = |V(H)|$.

Recall that C is the longest non-Hamiltonian cycle containing all vertices of X in $G + uv$. For $a > 1$, the vertices u and v have no common adjacent vertex apart from $V(C)$. So,

$$d_H(u) + d_H(v) \geq d(u) + d(v) - a \geq n + 1 - a = n - a + 1 = |V(H)| + 1.$$

For $a = 1$, the common adjacent vertex of u, v in $G - C$ is possibly the unique vertex of $G - C$. So,

$$d_H(u) + d_H(v) \geq d(u) + d(v) - 2 \geq n + 1 - 2 = n - 1 = |V(H)|.$$

In both cases, $d_H(u) + d_H(v) \geq |V(H)|$. Since

$$d_H(u) + d_H(v) = |S| + |T| = |S \cup T| + |S \cap T| \geq |V(H)|$$

and $|S \cup T| < |V(H)|$, then $|S \cap T| > 0$. Let $v_i \in S \cap T$. Then, there is a non-Hamiltonian cycle $v_1 v_2 \dots v_i v_{i+1} \dots v_{n-a-1} v_{n-a} v_1$ containing all vertices of X in G (Figure 2). This contradicts with the fact that X is an H-force set of G . Thus, if the subset X is an H-force set of G , X is an H-force set of $G + uv$.

Figure 2: A nonhamiltonian cycle $v_1 v_2 \dots v_i v_{i+1} \dots v_{n-a-1} v_{n-a} v_1$.

By (1), it is easy to see that (2) holds. ■

Lemma 2.1 motivates the following definition. The *weak closure* of an OTG G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $n + 1$ until no such pair remains. We denote the weak closure of G by $C_w(G)$. The idea of the weak closure follows the well-known Bondy-Chvátal's *closure* of a graph. If we join pairs of nonadjacent vertices with the degree sum at least n until no such pair remains, then we obtain the closure of a graph, denoted by $C(G)$. Bondy and Chvátal proved that $C(G)$ is well defined. That means, if G_1 and G_2 are two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains, then $G_1 = G_2$. By the similar argument, we can see that $C_w(G)$ is well defined.

Lemma 2.1 implies that the H-force set and H-force number of an OTG G are the H-force set and H-force number of $C_w(G)$, respectively. We can obtain the H-force set and H-force number of G by studying the weak closure $C_w(G)$ of G . Clearly, for an OTG G , the weak closure $C_w(G)$ is a complete graph or a graph satisfying $d_{C_w(G)}(u) + d_{C_w(G)}(v) = n$ for every pair of nonadjacent vertices u and v .

Theorem 2.2. Let G be an OTG with $n \geq 5$ vertices and X be the minimum H-force set of G . Let $C_w(G)$ be the weak closure of G and S be the largest independent set of $C_w(G)$. Then,

(1) the H-force number $h(G) \in \left\{ \frac{n}{2}, n - 2, n \right\}$, and

$$(2) X = \begin{cases} V(G) - \{x, y\}, & \text{if } h(G) = n - 2, \\ S, & \text{if } h(G) = \frac{n}{2}, \\ V(G), & \text{if } h(G) = n, \end{cases}$$

where $x, y \in V(G)$ with $d_{C_w(G)}(x) = n - 1$ and $d_{C_w(G)}(y) = n - 1$.

Proof. According to Lemma 2.1, it is sufficient to consider the minimum H-force set and H-force number of the weak closure $C_w(G)$. For convenience, let $G_w = C_w(G)$.

Obviously, the minimum H-force set of G is $V(G)$ and $h(G) = n$ if G_w is a complete graph.

So assume that G_w is not a complete graph and satisfies $d_{G_w}(u) + d_{G_w}(v) = n$ for every pair of non-adjacent vertices u and v in G_w . Clearly, $2 \leq \delta(G_w) \leq \frac{n}{2}$.

We consider the following two cases.

Case 1: $2 \leq \delta(G_w) < \frac{n}{2}$.

Let $\delta(G_w) = a$ and $d_{G_w}(u) = \delta(G_w) = a$ for some $u \in V(G_w)$. Then, $d_{G_w}(u) < \frac{n}{2}$. We use the following notations:

$$U = \{v \in V(G_w) - u : uv \in E(G_w)\}, \quad W = V(G_w) - U - \{u\}.$$

Obviously, $|U| = a$, $|W| = n - a - 1$ (Figure 3).

Figure 3: G_w and $G_{11} \in \varphi_1$.

Claim A.

- (i) For each vertex $w \in W$, $d_{G_w}(w) = n - a$.
 - (ii) W induces a complete subgraph.
 - (iii) For each vertex $w \in W$, $d_U(w) = 2$.
- Let $w \in W$ and $x, y \in U$ be the vertices adjacent to w in U .
- (iv) If $U - \{x, y\} \neq \emptyset$, then $d_{G_w}(p) = a$ for each vertex $p \in U - \{x, y\}$.
 - (v) If $U - \{x, y\} \neq \emptyset$, then x and y are adjacent to every vertex of $U - \{x, y\}$.
 - (vi) x and y are adjacent to every vertex of $U - \{x, y\}$ if $U - \{x, y\} \neq \emptyset$.
 - (vii) For any vertex $q \in U - \{x, y\}$, q has no neighbour in W .
 - (viii) x and y are adjacent to every vertex of W .
 - (ix) $d_{G_w}(x) = d_{G_w}(y) = n - 1$ and U induces a complete subgraph.

Proof.

- (i) For the vertex $w \in W$, since w is not adjacent to u , we see that $d_{G_w}(w) = n - d_{G_w}(u) = n - a$.
- (ii) For the distinct vertices $w, w' \in W$, we have $d_{G_w}(w) = d_{G_w}(w') = n - a > \frac{n}{2}$, and hence $d_{G_w}(w) + d_{G_w}(w') > n$. So, w and w' are adjacent. By the choice of w, w' , W induces a complete subgraph.
- (iii) Since $|W| = n - a - 1$ and W is a complete subgraph, we have

$$d_U(w) = d_{G_w}(w) - (n - a - 2) = (n - a) - (n - a - 2) = 2.$$

- (iv) Since the vertex p is not adjacent to w and $d_{G_w}(w) = n - a$, we have $d_{G_w}(p) = n - d_{G_w}(w) = n - (n - a) = a$.
- (v) If p, q are a pair of nonadjacent vertices in $U - \{x, y\}$, then $d_{G_w}(p) + d_{G_w}(q) = n$. But $d_{G_w}(p) + d_{G_w}(q) = 2a < n$. So, p and q are adjacent and $U - \{x, y\}$ induces a complete subgraph.
- (vi) By contradiction. Let p be a vertex of $U - \{x, y\}$ such that x is not adjacent to p . By (iv), we have $d_{G_w}(x) = n - d_{G_w}(p) = n - a$. For a vertex $w' \in W$, $d_{G_w}(w') = n - a$ by (i). Then, $d_{G_w}(x) + d_{G_w}(w') > n$. So, x is adjacent to every vertex of W .

Recall that $|W| = n - a - 1$ and $d_{G_w}(x) = n - a$. Hence, x has no neighbour in $U - \{x\}$. In particular, x is not adjacent to y . So, $d_{G_w}(y) = n - d_{G_w}(x) = a$. Since the degree of any vertex of $U - \{x, y\}$ is also a , the vertex y is adjacent to every vertex of $U - \{x, y\}$. So, w is the only adjacent vertex of y in W . Therefore,

$$N_{G_w}(x) = \{u\} \cup W, \quad N_{G_w}(y) = \{u\} \cup (U - \{x, y\}) \cup \{w\}.$$

In other words, $\{x\} \cup (W - \{w'\}) \subseteq N_{G_w}(w')$ for any vertex $w' \in W - \{w\}$. By $|\{x\} \cup (W - \{w'\})| = n - a - 1$ and $d_{G_w}(w') = n - a$, we see that every vertex of $W - \{w\}$ has exactly one adjacent vertex in $U - \{x, y\}$.

Let $q \in U - \{x, y\}$ be arbitrary. By the arguments above, we see that $\{u\} \cup (U - \{x, q\}) \subseteq N_{G_w}(q)$. Since $d_{G_w}(q) = a$ (see (iv)) and $|\{u\} \cup (U - \{x, q\})| = a - 1$, we have that q has an adjacent vertex in $W - \{w\}$. This means every vertex of $U - \{x, y\}$ has exactly one adjacent vertex in $W - \{w\}$.

So, $|U - \{x, y\}| = |W - \{w\}|$. But $a - 2 < n - a - 2$, a contradiction. Thus, x is adjacent to every vertex of $U - \{x, y\}$. Similarly, y is adjacent to every vertex of $U - \{x, y\}$.

- (vii) For $q \in U - \{x, y\}$, claims (iv), (v) and (vi) imply that q is not adjacent to any vertex of W .
- (viii) Claims (iii) and (vii) show that every vertex of W has two adjacent vertices in U , which are just x and y .
- (ix) By (vi) and (viii), $d_{G_w}(x) + d_{G_w}(y) \geq 2(|U - \{x, y\}| + |W| + |\{u\}|) = 2(n - 2) > n$. Then, x and y are adjacent and hence (ix) holds. $\square$

Now, G_w is isomorphic to $G_{11} = K_2 \vee (K_{a-1} + K_{n-a-1})$ (Figure 3), where $U' = (U - \{x, y\}) \cup \{u\}$. In particular, $d_{G_w}(x) = d_{G_w}(y) = n - 1$, U' and W induce complete subgraphs with $|U'| = a - 1$ and $|W| = n - a - 1$.

We shall show that $X = V - \{x, y\} = U' \cup W$ is the minimum H-force set and hence $h(G_w) = n - 2$.

Obviously, X is an H-force set because a cycle containing all vertices of X must encounter x and y . Suppose to the contrary that X' is an H-force set of G_w with $|X'| < |X|$. Clearly, there exists a vertex $z \in U'$ or $z \in W$ such that $z \notin X'$. Without loss of generality, assume $z \in U'$. Consider the subgraph $G_w - z$. For any pair of nonadjacent vertices $u \in U' - z$, $v \in W$ of $G_w - z$, we have

$$d_{G_w-z}(u) + d_{G_w-z}(v) = d_{G_w}(u) - 1 + d_{G_w}(v) = n - 1.$$

Then, $G_w - z$ is an OTG and hence $G_w - z$ is a Hamiltonian graph. In other words, there is a non-Hamiltonian cycle containing all vertices of X' in G_w , a contradiction. Thus, there is no such an H-force set X' with $|X'| < |X|$. So, X is the minimum H-force set and $h(G_w) = h(G) = n - 2$.

Case 2: $\delta(G_w) = \frac{n}{2}$ and n is even.

Subcase 2.1: There is $v \in V(G_w)$ such that $G_w - v$ is not a Hamiltonian graph.

Let $H = G_w - v$. For arbitrary $x, y \in V(H)$, we have

$$d_H(x) + d_H(y) \geq d_{G_w}(x) - 1 + d_{G_w}(y) - 1 = n - 2 = |V(H)| - 1.$$

By Corollary 1.3, either H is not a 2-connected graph or $H \in \psi_{n-1}$ (Figure 1).

Subcase 2.1.1: $H = G_w - v$ is not a 2-connected graph.

Then, there exists a vertex u such that $H - u$ is not connected.

Claim B.

- (i) For any vertex $x \in V(G_w)$, if $d_{G_w}(x) > \frac{n}{2}$, then $d_{G_w}(x) = n - 1$.
- (ii) $H - u$ has exactly two components, say U and W .
- (iii) For any vertices $x \in U$, $y \in W$, $d_{G_w}(x) = d_{G_w}(y) = \frac{n}{2}$.
- (iv) $|U| = |W| = \frac{n}{2} - 1$.
- (v) Both U and W induce complete subgraphs, and $d_{G_w}(u) = d_{G_w}(v) = n - 1$.

Proof.

- (i) Recall that $\delta(G_w) = \frac{n}{2}$ and $d_{G_w}(u) + d_{G_w}(v) = n$ for nonadjacent vertices u, v . It is clear that (i) holds.
- (ii) Suppose to the contrary that there are three components U , W and U' in $H - u$. So, $|U| + |W| \leq n - 3$. For the vertices $x \in U$, $y \in W$, we see that

$$\begin{aligned} n &= d_{G_w}(x) + d_{G_w}(y) = d_U(x) + d_W(y) + d_{\{u,v\}}(x) + d_{\{u,v\}}(y) \\ &\leq |U| - 1 + |W| - 1 + 4 = |U| + |W| + 2 \leq n - 3 + 2 = n - 1, \end{aligned}$$

a contradiction. Thus, $H - u$ has two components.

- (iii) For $x \in U, y \in W$, we have $d_{G_w}(x) \neq n-1$ and $d_{G_w}(y) \neq n-1$. By (i), $d_{G_w}(x) = d_{G_w}(y) = \frac{n}{2}$.
- (iv) Note that $|U \cup W| = n-2$. Assume without loss of generality that $|U| \leq \frac{n}{2} - 1$. For any vertex $x \in U$, we have

$$\frac{n}{2} = d_{G_w}(x) \leq |U| - 1 + 2 = |U| + 1.$$

Then, $|U| \geq \frac{n}{2} - 1$. So, $|U| = |W| = \frac{n}{2} - 1$.

- (v) For any vertex $x \in U$, $d_{G_w}(x) = \frac{n}{2} \leq d_U(x) + 2 \leq |U| - 1 + 2 = \frac{n}{2} - 1 + 1 = \frac{n}{2}$. Then, $d_{G_w}(x) = d_U(x) + 2 = (|U| - 1) + 2$. This implies that U induces a complete subgraph and x is adjacent to u, v . Similarly, W induces a complete subgraph and every vertex of W is adjacent to u, v .

Furthermore, all vertices of U and W are the neighbours of u and v . So, $d_{G_w}(u) \geq n-2$, $d_{G_w}(v) \geq n-2$ and hence $d_{G_w}(u) + d_{G_w}(v) > n$. It means that u and v are adjacent. Thus, $d_{G_w}(u) = d_{G_w}(v) = n-1$. $\square$

Figure 4: $G_{12} \in \varphi_1$.

In subcase 2.1.1, $G_w \cong G_{12}$ (Figure 4), in which U and W are complete subgraphs with $|U| = |W| = \frac{n}{2} - 1$. By the same argument of the case when $2 \leq \delta(G_w) < \frac{n}{2}$ and $G_w \cong G_{11}$, we also have $V - \{u, v\}$ is the minimum H-force set and $h(G_w) = h(G) = n-2$.

Subcase 2.1.2: $H = G_w - v \in \psi_{n-1}$.

Let $H = G_w - v = Z_m \vee (K_{m+1}^c)$, where Z_m is a graph with m vertices. Clearly, $|V(H)| = n-1 = 2m+1$, namely, $n = 2m+2$. Recall that $n \geq 5$. So, $m \geq 2$.

Claim C. If there is at least one edge in Z_m , then

- Z_m is a complete subgraph.
- The vertex v is adjacent to all vertices of Z_m .
- The vertex v is adjacent to all vertices of K_{m+1}^c .
- $G_w \cong G_{21} = K_{m+1}^c \vee K_{m+1}$ (Figure 5). Furthermore, $X = V(K_{m+1}^c)$ is the minimum H-force set and $h(G) = h(G_w) = \frac{n}{2}$.

Proof.

- (i) Let x be an endpoint of the edge of Z_m . Then, $d_{G_w}(x) \geq m+2$. For arbitrary $y \in Z_m - x$, since $d_{G_w}(y) \geq m+1$, we have $d_{G_w}(x) + d_{G_w}(y) \geq 2m+3 > n$. This implies that x is adjacent to the remaining vertices of Z_m .

Furthermore, for any pair of $p, q \in Z_m - x$, we have $d_{G_w}(p) \geq m+2$, $d_{G_w}(q) \geq m+2$, and hence $d_{G_w}(p) + d_{G_w}(q) \geq 2m+4 > n$, which implies that p and q are adjacent. Thus, Z_m is a complete subgraph.

- (ii) For a vertex $z \in Z_m$,

$$d_{G_w}(v) + d_{G_w}(z) \geq \frac{n}{2} + (n-2) = m+1 + 2m > 2m+2 = n.$$

Thus, v is adjacent to all vertices of Z_m .

- (iii) Since $\delta(G_w) = \frac{n}{2}$, $d_{G_w}(x) \geq \frac{n}{2} = m+1 = |V(Z_m)| + 1$ for any vertex $x \in K_{m+1}^c$. So, x is adjacent to v . Thus, v is adjacent to all vertices of K_{m+1}^c .

- (iv) By the arguments above, it is obvious $G_w \cong G_{21} = K_{m+1}^c \vee K_{m+1}$.

Figure 5: $G_{21} \in \varphi_2$ and $K_{\frac{n}{2}, \frac{n}{2}}^c \in \varphi_2$.

Let $X = V(K_{m+1}^c)$. For any $x \in X$, it is obvious that $G_w - x$ is an OTG and hence it is Hamiltonian. By Proposition 1.1, the minimum H-force set contains all vertices of $V(K_{m+1}^c)$. In addition, it is clear that $X = V(K_{m+1}^c)$ is an H-force set of G_w . So, $X = V(K_{m+1}^c)$ is the minimum H-force set and $h(G) = h(G_w) = \frac{n}{2}$. $\square$

Claim D. If there is no edge in Z_m , namely, $Z_m = K_m^c$, then

- (i) v has no neighbour in Z_m .
- (ii) v is adjacent to all vertices of K_{m+1}^c .
- (iii) $G_w \cong K_{\frac{n}{2}, \frac{n}{2}}$ (Figure 5). Furthermore, $X = V(K_{m+1}^c)$ is the minimum H-force set and $h(G) = h(G_w) = \frac{n}{2}$.

Proof.

- (i) By contradiction. Let v be adjacent to some $x \in V(Z_m)$. Let y be another vertex of Z_m . Then, $d_{G_w}(x) + d_{G_w}(y) \geq m + 2 + m + 1 > n$ and hence x and y are adjacent, which contradicts with $Z_m = K_m^c$.
- (ii) Since $\delta(G_w) = \frac{n}{2}$, we have v is adjacent to all vertices of K_{m+1}^c .
- (iii) By (i) and (ii), it is easy to see that $G_w \cong K_{\frac{n}{2}, \frac{n}{2}}$. Clearly, the largest independent set $V(K_{m+1}^c)$ is the minimum H-force set and $h(G) = h(G_w) = \frac{n}{2}$. $\square$

Subcase 2.2: For any vertex $v \in V(G_w)$, the subgraph $G_w - v$ is Hamiltonian.

In this subcase, Proposition 1.1 yields that the minimum H-force set is $V(G)$ and $h(G) = h(G_w) = n$. $\blacksquare$

In [10], Dirac proved Theorem 2.3.

Theorem 2.3. [10] *If the minimum degree $\delta(G)$ of G is at least $\frac{n}{2}$, then G is Hamiltonian.*

Clearly, the graph satisfying the condition of Dirac's theorem is an OTG. By Theorem 2.2, we obtain immediately the following results on the H-force sets of graphs satisfying the condition of Dirac's theorem.

Corollary 2.4. *Let G be a graph satisfying $\delta(G) \geq \frac{n}{2}$ and X be the minimum H-force set of G . Let $G_w = C_w(G)$ be the weak closure of G and S be the largest independent set of G_w . Then,*

- (1) *The H-force number $h(G) \in \{\frac{n}{2}, n - 2, n\}$.*
- (2)
$$X = \begin{cases} V(G) - \{x, y\}, & \text{if } h(G) = n - 2, \\ S, & \text{if } h(G) = \frac{n}{2}, \\ V(G), & \text{if } h(G) = n, \end{cases}$$

where $x, y \in V(G)$ with $d_{G_w}(x) = n - 1$ and $d_{G_w}(y) = n - 1$.

3 Classification of the OTGs

To present Theorem 3.1, we define several special classes of graphs, namely,

$$\varphi_1 = \left\{ K_2 \vee (K_m + K_{n-m-2}) \mid 1 \leq m < \frac{n}{2} \right\} \text{ (Figures 3 and 4),}$$

$$\varphi_2 = \left\{ G_{21}, K_{\frac{n}{2}, \frac{n}{2}} \right\} \text{ (Figure 5),}$$

$$\varphi_3 = \left\{ \left\{ Z_{n-m} \vee K_m \mid n \text{ is even, } 0 \leq m < \frac{n}{2} \text{ and } d(x) = \frac{n}{2} \text{ for } x \in Z_{n-m} \right\} - \left\{ G_{12}, K_{\frac{n}{2}, \frac{n}{2}} \right\} \right\} \\ \cup \{G \mid G \text{ is a complete graph}\} \text{ (Figure 6),}$$

where Z_{n-m} is a graph with $n - m$ vertices.

Figure 6: $G_{31} \in \varphi_3$ and $G_{32} \in \varphi_3$.

Let C_w be the class of the weak closures of all OTGs. Clearly, $\varphi_1, \varphi_2, \varphi_3$ are all the subsets of C_w . By Theorem 2.2, we can give a partition of the class C_w and obtain the following theorem.

Theorem 3.1. *The class C_w of the weak closures of all OTGs has a partition*

$$C_w = \varphi_1 \dot{\cup} \varphi_2 \dot{\cup} \varphi_3$$

due to the H -force set, where $\varphi_1, \varphi_2, \varphi_3$ are defined above. Let $G_w \in C_w$ be arbitrary. Then,

- (1) $h(G_w) = n - 2$ if and only if $G_w \in \varphi_1$.
- (2) $h(G_w) = \frac{n}{2}$ if and only if $G_w \in \varphi_2$.
- (3) $h(G_w) = n$ if and only if $G_w \in \varphi_3$.

Proof. By case 1 and subcase 2.1.1 of Theorem 2.2, statement (1) holds. Also, by subcase 2.1.2, statement (2) holds. We show (3) as follows.

“ $\Rightarrow$ ” By the proof of Theorem 2.2, we know that $h(G_w) = n - 2$ if $2 \leq \delta(G_w) < \frac{n}{2}$. So, if $h(G_w) = n$, then $\delta(G_w) = \frac{n}{2}$ (n is even) or G_w is a complete graph. In the latter case, $G_w \in \varphi_3$ and we are done.

Assume that G_w is not a complete graph. Then, $\delta(G_w) = \frac{n}{2}$ and $d_{G_w}(u) + d_{G_w}(v) = n$ for every pair of non-adjacent vertices u, v . So, the degree of any vertex $u \in V(G_w)$ is either $\frac{n}{2}$ or $n - 1$. Set $U = \{v \mid d_{G_w}(v) = n - 1\}$ and $m = |U|$.

When $U \neq \emptyset$, we claim that $1 \leq m < \frac{n}{2}$. Clearly, $1 \leq m \leq \frac{n}{2}$. Note that $d_{G_w}(w) = \frac{n}{2}$ for any vertex $w \in V(G_w) - U$. If $m = \frac{n}{2}$, then $G_w \cong G_{21}$ and $h(G_w) = \frac{n}{2}$, a contradiction. So, $1 \leq m < \frac{n}{2}$. Since $h(G_{12}) = n - 2$, we have $G_w \not\cong G_{12}$. Thus, $G_w \in \varphi_3$.

When U is an empty set, then $m = 0$ and G_w is a $\frac{n}{2}$ -regular graph. Since $h\left(K_{\frac{n}{2}, \frac{n}{2}}\right) = \frac{n}{2}$, we have $G_w \not\cong K_{\frac{n}{2}, \frac{n}{2}}$. Thus, $G_w \in \varphi_3$.

“ $\Leftarrow$ ” It is not difficult to check that $\varphi_3 \cap \varphi_1 = \emptyset$ and $\varphi_3 \cap \varphi_2 = \emptyset$. Let $G_w \in \varphi_3$ be a weak closure of an OTG. So, $h(G_w) \neq n - 2$ or $\frac{n}{2}$. According to Theorem 2.2, $h(G_w) = n$. $\square$

To present Corollary 3.2, we define several special classes of graphs, namely,

$$\vartheta_i = \{G \text{ is an OTG, } G_w \in \varphi_i\}, \quad i = 1, 2, 3.$$

Let $\mathcal{G}$ be the class of all OTG's. Clearly, $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are all the subsets of $\mathcal{G}$. By Theorem 3.1, we can give a partition of the class $\mathcal{G}$ and obtain the following corollary.

Corollary 3.2. *The class $\mathcal{G}$ of all OTGs has a partition*

$$\mathcal{G} = \mathcal{G}_1 \dot{\cup} \mathcal{G}_2 \dot{\cup} \mathcal{G}_3$$

due to the weak closure, where $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are defined above. Let $G \in \mathcal{G}$ be arbitrary. Then,

$$h(G) = \begin{cases} n-2, & \text{if } G \in \mathcal{G}_1, \\ \frac{n}{2}, & \text{if } G \in \mathcal{G}_2, \\ n, & \text{if } G \in \mathcal{G}_3. \end{cases}$$

4 Algorithm to check the minimum H-force set of an OTG

Input: A graph $G = (V, E)$.

Output: If G is an OTG, return the H-force number $h(G)$ and the minimum H-force set X . If G is not an OTG, return “not an OTG”.

1. For each pair of nonadjacent vertices u, v , check $d(u) + d(v)$. If there is $uv \notin E$ with $d(u) + d(v) < n$, then return “not an OTG”.
2. Make the weak closure G_w of G .
3. For any $x \in V(G)$, check $d_{G_w}(x)$.
 - (1) If there are exactly two vertices u and v with $d_{G_w}(u) = d_{G_w}(v) = n-1$ and $G_w \in \varphi_1$, return $h(G) = n-2$ and $X = V(G) - \{u, v\}$.
If there are exactly two vertices u and v with $d_{G_w}(u) = d_{G_w}(v) = n-1$ and $G_w \in \varphi_3$, return $h(G) = n$ and $X = V(G)$.
 - (2) If there are exactly $\frac{n}{2}$ vertices with degree $n-1$, return $h(G) = \frac{n}{2}$. Set $X = \{x | d(x) < n-1\}$. Return X .
 - (3) If there are a vertices with degree $n-1$ where $2 < a < \frac{n}{2}$ or $a = 1$ or $a = n$, return $h(G) = n$ and $X = V(G)$.
 - (4) If for any $v \in V(G_w)$, $d_{G_w}(v) = \frac{n}{2}$ and $G_w \cong K_{\frac{n}{2}, \frac{n}{2}}$, return $h(G) = \frac{n}{2}$. Find a partition set X of $K_{\frac{n}{2}, \frac{n}{2}}$, return X .
 - (5) If for any $v \in V(G_w)$, $d_{G_w}(v) = \frac{n}{2}$ and $G_w \not\cong K_{\frac{n}{2}, \frac{n}{2}}$, return $h(G) = n$ and $X = V(G)$.

Theorem 4.1. *For an OTG, the minimum H-force set and the H-force number can be found in time $O(n^3)$.*

Proof. The complexity follows from the fact that Step 1 can be performed in time $O(n^2)$, Step 2 in time $O(n^3)$ and Step 3 in time $O(n^2)$. $\square$

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