Research Article

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A boundedness result for Marcinkiewicz integral operator

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Abstract: We extend a boundedness result for Marcinkiewicz integral operator. We find a new space of radial functions for which this class of singular integral operators remains L^p -bounded when its kernel satisfies only the sole integrability condition.

Keywords: Marcinkiewicz integral operator, L_p -bounded integral operator, maximal integral operator

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1 Introduction

For $n \geq 2$, let $\mathbb{R}^n$ be the n -dimensional Euclidean space and $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ equipped with the normalized surface measure $d\sigma$. For any nonzero $y \in \mathbb{R}^n$, let y' be the projection of y on $\mathbb{S}^{n-1}$, that is, $y' = \frac{y}{|y|}$. For $\alpha > 0$, the Zygmund class $L(\log L)^\alpha(\mathbb{S}^{n-1})$ consists of all measurable functions Ω on $\mathbb{S}^{n-1}$ which satisfy

$$\int_{\mathbb{S}^{n-1}} |\Omega(x)| \log^\alpha(e + |\Omega(x)|) d\sigma(x) < \infty.$$

The following inclusions among Zygmund classes and Lebesgue spaces hold and are proper. For $q > 1$ and $0 < \alpha < \beta$, we have

$$\begin{aligned} L^q(\mathbb{S}^{n-1}) &\subset L(\log L)(\mathbb{S}^{n-1}) \subset L^1(\mathbb{S}^{n-1}), \\ L(\log L)^\beta(\mathbb{S}^{n-1}) &\subset L(\log L)^\alpha(\mathbb{S}^{n-1}). \end{aligned}$$

Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero and satisfy

$$\int_{\mathbb{S}^{n-1}} \Omega(y) d\sigma(y) = 0. \quad (1.1)$$

Condition (1.1) is referred to as the cancellation property. It was shown that Ω must satisfy (1.1) for the existence of the singular integrals in our work, see [9]. For $\rho > 0$ and a measurable radial function h , consider the integral operator

$$\mathcal{M}_{\Omega, h}^\rho f(x) = \left(\int_0^\infty \left| t^{-\rho} \int_{|y| \leq t} f(x-y) K_{\Omega, h}(y) dy \right|^2 \frac{dt}{t} \right)^{1/2},$$

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where

$$K_{\Omega,h,\rho}(y) = \frac{\Omega(y)}{|y|^{n-\rho}} h(|y|).$$

When $n = 1$, $\rho = 1$, $h \equiv 1$ and $\Omega(y) = \text{sign}(y)$, we denote $\mathcal{M}_{\Omega,h}^\rho$ by $\mathcal{M}$. In 1938, Marcinkiewicz introduced the operator $\mathcal{M}$ to give an analogue of certain Littlewood-Paley g -functions [7]. In [8], Stein introduced the integral operator $\mathcal{M}_{\Omega,1}^1$ as a higher dimensional version of $\mathcal{M}$. Since then there was much work on the L^p -boundedness of the operator $\mathcal{M}_{\Omega,1}^1$ whenever Ω belongs to a variety of function spaces. For instance, Stein proved that $\mathcal{M}_{\Omega,1}^1$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p \leq 2$ whenever $\Omega \in \text{Lip}_\alpha(\mathbb{S}^{n-1})$, $0 < \alpha \leq 1$. In [10], Walsh proved the boundedness of $\mathcal{M}_{\Omega,1}^1$ when $p = 2$ and $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{d-1})$. Moreover, he showed that we may not attain the boundedness of $\mathcal{M}_{\Omega,1}^1$ on L^2 if we reduce the power $1/2$ by a smaller one in the Zygmund class $L(\log L)^{1/2}(\mathbb{S}^{d-1})$. In a recent study, Al-Salman et al. [3] proved that $\mathcal{M}_{\Omega,1}^1$ is bounded for $p \in (1, \infty)$ and $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{d-1})$. We refer the readers to [1–4, 11–14] for more background information and related results.

In the following, we discuss some results that are relevant to our work. For $1 \leq \gamma < \infty$, let

$$l^\infty(L^\gamma)(\mathbb{R}^+) = \left\{ h : \mathbb{R}^n \rightarrow \mathbb{C} : \|h\|_{l^\infty(L^\gamma)(\mathbb{R}^+)} := \sup_{j \in \mathbb{Z}} \left(\int_{2^{j-1}}^{2^j} |h(r)|^\gamma \frac{dr}{r} \right)^{1/\gamma} < \infty \right\}.$$

If $\gamma = \infty$, we set

$$l^\infty(L^\gamma)(\mathbb{R}^+) = L^\infty(\mathbb{R}^+).$$

For $1 < \gamma < \beta < \infty$ and $t > 0$, the spaces $l^\infty(L^\gamma)(\mathbb{R}^+)$ have the following interesting properties:

$$L^\infty(\mathbb{R}^+) \subset l^\infty(L^\beta)(\mathbb{R}^+) \subset l^\infty(L^\gamma)(\mathbb{R}^+) \subset l^\infty(L^1)(\mathbb{R}^+), \quad (1.2)$$

$$\left(\int_{2^{j-1}}^{2^j} |h(tr)|^\gamma \frac{dr}{r} \right)^{1/\gamma} = \left(\int_{2^{j-1}t}^{2^j t} |h(r)|^\gamma \frac{dr}{r} \right)^{1/\gamma} \leq 2 \|h\|_{l^\infty(L^\gamma)(\mathbb{R}^+)}. \quad (1.3)$$

Ding, Lu and Yabuta [4] studied the L^2 -boundedness of $\mathcal{M}_{\Omega,h}^\rho$ when $h \in l^\infty(L^\gamma)(\mathbb{R}^+)$ and proved that $\mathcal{M}_{\Omega,h}^\rho$ is bounded on $L^2(\mathbb{R}^n)$ for $\Omega \in L(\log L)(\mathbb{S}^{n-1})$ and $h \in l^\infty(L^\gamma)(\mathbb{R}^+)$, $1 < \gamma \leq \infty$. In 2007, Al-Salman and Al-Qassem [2] improved this result and proved that $\mathcal{M}_{\Omega,h}^\rho$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In a very recent work, Hawawsheh et al. [5] introduced a new subspace $\mathcal{D}_\gamma(\mathbb{R}^+)$ of $l^\infty(L^\gamma)(\mathbb{R}^+)$. More precisely, $\mathcal{D}_\gamma(\mathbb{R}^+)$ consists of all measurable radial functions on $\mathbb{R}^n$, which satisfy the following conditions:

$$\frac{h(r)}{r^{1/\gamma'}} \in l^\infty(L^\gamma)(\mathbb{R}^+), \quad (1.4)$$

$$\sum_{j=1}^{\infty} \left(\int_{2^{j-1}}^{2^j} |h(r)|^\gamma \frac{dr}{r} \right)^{1/\gamma} < \infty. \quad (1.5)$$

The space $\mathcal{D}_\gamma(\mathbb{R}^+)$ has many remarkable properties. For instance, $\mathcal{D}_\gamma \not\subset L^\infty(\mathbb{R}^+)$ and $\mathcal{D}_\beta(\mathbb{R}^+) \subset \mathcal{D}_\gamma(\mathbb{R}^+)$ for $1 < \gamma < \beta < \infty$. Hawawsheh and Al-Salman [6] proved that taking $h \in \mathcal{D}_\gamma(\mathbb{R}^+)$ allows $\mathcal{M}_{\Omega,h}^\rho$ to be bounded whenever $\Omega \in L^1(\mathbb{S}^{n-1})$, which is known as the sole integrability condition. In fact, they obtained the following result.

Theorem 1.1. *If $h \in \mathcal{D}_\gamma$ for some $\gamma > 1$ and $\Omega \in L^1(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$, then $\mathcal{M}_{\Omega,h}^\rho$ is bounded on $L^p(\mathbb{R}^n)$ for $\frac{2\gamma}{2\gamma-1} < p < \infty$.*

In view of Theorem 1.1, the natural question that arises is: what is the largest subspace of $l^\infty(L^Y)(\mathbb{R}^+)$ for which the result of Theorem 1.1 holds? In this work, we aim to improve the result of the previous theorem by finding a subspace which contains $\mathcal{D}_Y$.

2 Lemmas

This section prepares for the proof of our main result by using a variation of the technique that was used to prove Theorem 1.1.

Let $\mathcal{D}_Y^*(\mathbb{R}^+)$ consist of all measurable radial functions on $\mathbb{R}^n$, which satisfy the following conditions:

$$r^{1/Y'}h(r) \in l^\infty(L^Y)(\mathbb{R}^+), \quad (2.1)$$

$$\sum_{j=0}^{\infty} \left(\int_{2^{j-1}}^{2^j} |h(r)|^Y \frac{dr}{r} \right)^{1/Y} < \infty. \quad (2.2)$$

It can be readily seen that $\mathcal{D}_Y^*(\mathbb{R}^+) \subset l^\infty(L^Y)(\mathbb{R}^+)$ and $\mathcal{D}_Y^*(\mathbb{R}^+) \not\subset \mathcal{D}_Y(\mathbb{R}^+)$. In fact, one can consider the function $h(r) = \frac{1}{r^{1/2Y'}} \mathbb{1}_{(0,1)}(r)$.

Lemma 2.1. *Let $h \in \mathcal{D}_Y^*(\mathbb{R}^+)$ for some $Y > 1$ and $\Omega \in L^1(\mathbb{S}^{n-1})$. Then,*

$$\int_{t/2 < |y| \leq t} |K_{\Omega,h}(y)| dy \leq 2^{1+1/Y'} (\log 2)^{1/Y'} t^{\rho-1/Y'} \|\Omega\|_1 \|r^{1/Y'} h\|_{l^\infty(L^Y)(\mathbb{R}^+)}.$$

Proof. By switching to polar coordinates, Hölder's inequality and using (1.3), we have

$$\begin{aligned} \int_{t/2 < |y| \leq t} |K_{\Omega,h}(y)| dy &= \int_{t/2}^t r^\rho |h(r)| \left(\int_{\mathbb{S}^{n-1}} |\Omega(y')| d\sigma(y') \right) \frac{dr}{r} \leq 2^{1/Y'} t^{\rho-1/Y'} \|\Omega\|_1 \left(\int_{t/2}^t |r^{1/Y'} h(r)| \frac{dr}{r} \right) \\ &\leq 2^{1+1/Y'} (\log 2)^{1/Y'} t^{\rho-1/Y'} \|\Omega\|_1 \|r^{1/Y'} h\|_{l^\infty(L^Y)(\mathbb{R}^+)}. \end{aligned} \quad \square$$

Lemma 2.2. *Let Ω and h be as in Lemma 2.1. If*

$$F(x, t) = t^{-\rho} \int_{t/2 < |y| \leq t} f(x-y) K_{\Omega,h}(y) dy,$$

then

$$\|F(x, t)\|_p \leq 2^{1+1/Y'} (\log 2)^{1/Y'} t^{-1/Y'} \|\Omega\|_1 \|r^{1/Y'} h\|_{l^\infty(L^Y)(\mathbb{R}^+)} \|f\|_p.$$

Proof. An application of Minkowski's inequality for integrals yields

$$\begin{aligned} \|F(x, t)\|_p &= \left(\int_{\mathbb{R}^n} |F(x, t)|^p dx \right)^{1/p} = \left(\int_{\mathbb{R}^n} \left| t^{-\rho} \int_{t/2 < |y| \leq t} f(x-y) K_{\Omega,h}(y) dy \right|^p dx \right)^{1/p} \\ &\leq t^{-\rho} \int_{t/2 < |y| \leq t} |K_{\Omega,h}(y)| \left(\int_{\mathbb{R}^n} |f(x-y)|^p dx \right)^{1/p} dy. \end{aligned}$$

Now, applying Lemma 2.1 and recalling that the Lebesgue integral is invariant under translation we have the desired result. $\square$

Definition 2.3. For a function u defined on $\mathbb{R}^n$, the maximal integral operator of u is defined as

$$\sigma^*(u)(x) = \sup_{1 < t < \infty} t^{-\rho} \int_{t/2 < |y| \leq t} |u(x-y)| \frac{|\Omega(y)|}{|y|^{n-\rho}} |h(|y|)| dy.$$

The following lemma constitutes the boundedness of the maximal integral operator $\sigma^*(u)$.

Lemma 2.4. Let Ω and h be as in Lemma 2.1 and $\gamma > 1$. Then,

$$\|\sigma^*(u)\|_q \leq C \|u\|_q$$

for $q > \gamma'$.

Proof. With the aid of Hölder's inequality we get

$$\begin{aligned} \sigma^*(u)(x) &= \sup_{1 < t < \infty} t^{-\rho} \int_{t/2 < |y| \leq t} |u(x-y)| \frac{|\Omega(y)|}{|y|^{n-\rho+1/\gamma'}} |y|^{1/\gamma'} h(|y|) dy \\ &\leq \sup_{1 < t < \infty} \left(t^{-\rho} \int_{t/2 < |y| \leq t} |u(x-y)|^{\gamma'} \frac{|\Omega(y)|}{|y|^{n-\rho+1}} dy \right)^{1/\gamma'} \left(t^{-\rho} \int_{t/2 < |y| \leq t} \frac{|\Omega(y)|}{|y|^{n-\rho}} |y|^{1/\gamma'} h(|y|)^{\gamma'} dy \right)^{1/\gamma} \\ &\leq \|\Omega\|_1^{1/\gamma} \|r^{1/\gamma'} \cdot h\|_{r^{\infty}(L^{\gamma'})(\mathbb{R}^+)} \times \sup_{1 < t < \infty} \left(t^{-\rho} \int_{t/2 < |y| \leq t} |u(x-y)|^{\gamma'} \frac{|\Omega(y)|}{|y|^{n-\rho+1}} dy \right)^{1/\gamma'} \\ &\leq C \sup_{1 < t < \infty} \left(\int_{t/2}^t \int_{\mathbb{S}^{n-1}} |u(x-ry')|^{\gamma'} |\Omega(y')| d\sigma(y') \frac{dr}{r^2} \right)^{1/\gamma'} \\ &\leq C \left(\int_{1/2}^{\infty} \int_{\mathbb{S}^{n-1}} |u(x-ry')|^{\gamma'} |\Omega(y')| d\sigma(y') \frac{dr}{r^2} \right)^{1/\gamma'}. \end{aligned} \quad (2.3)$$

Finally, by using (2.3) and Minkowski's inequality for integrals we obtain

$$\begin{aligned} \|\sigma^*(u)\|_q^{\gamma'} &\leq C \left(\int_{\mathbb{R}^n} \left(\int_{1/2}^{\infty} \int_{\mathbb{S}^{n-1}} |u(x-ry')|^{\gamma'} |\Omega(y')| d\sigma(y') \frac{dr}{r^2} \right)^{q/\gamma'} dx \right)^{\gamma'/q} \\ &\leq C \int_{1/2}^{\infty} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |u(x-ry')|^{\gamma'} |\Omega(y')| d\sigma(y') \right)^{q/\gamma'} dx \right)^{\gamma'/q} \frac{dr}{r^2} \\ &\leq C \int_{1/2}^{\infty} \int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{R}^n} |u(x-ry')|^q dx \right)^{\gamma'/q} |\Omega(y')| d\sigma(y') \frac{dr}{r^2} \\ &\leq C \|u\|_q^{\gamma'}. \end{aligned} \quad (2.4)$$

□

Now, we are in a position to study the Dyadic operators

$$S_{\Omega, h}^{\rho, j} g(x) = \left(\int_{2^j}^{2^{j+1}} \left| t^{-\rho} \int_{t/2 < |y| \leq t} K_{\Omega, h}(y) g(x+y, t) dy \right|^2 \frac{dt}{t} \right)^{1/2}.$$

Lemma 2.5. Let $\frac{2\gamma}{2\gamma-1} < p < 2$. If $g \in Z \equiv L^{p'}\left(\mathrm{d}x, L^2\left([2^j, 2^{j+1}], \frac{\mathrm{d}t}{t}\right)\right)$ for $j = 0, 1, 2, \dots$, then

$$\|S_{\Omega,h}^{\rho,j}g\|_{p'}^2 \leq C2^{-j/\gamma'}\|g\|_Z.$$

Proof. Since $p' > 2$, there exists a non-negative function $u \in L^{(p'/2)'}(\mathbb{R}^n)$ with $\|u\|_{L^{(p'/2)'}(\mathbb{R}^n)} = 1$ such that

$$\|S_{\Omega,h}^{\rho,j}g(x)\|_{p'}^2 = \int_{\mathbb{R}^n} \left(\int_{2^j}^{2^{j+1}} \left| t^{-\rho} \int_{t/2 < |y| \leq t} K_{\Omega,h}(y)g(x+y, t) \mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t} \right) u(x) \mathrm{d}x. \quad (2.5)$$

Let $G(x, t) = t^{-\rho} \int_{t/2 < |y| \leq t} K_{\Omega,h}(y)g(x+y, t) \mathrm{d}y$. Then, by applying the Cauchy-Schwartz inequality we have

$$\begin{aligned} |G(x, t)|^2 &\leq \left(t^{-\rho} \int_{t/2 < |y| \leq t} |K_{\Omega,h}(y)| |g(x+y, t)| \mathrm{d}y \right)^2 \\ &\leq \left[t^{-\rho} \int_{t/2 < |y| \leq t} |K_{\Omega,h}(y)| \mathrm{d}y \right] \left[t^{-\rho} \int_{t/2 < |y| \leq t} |K_{\Omega,h}(y)| |g(x+y, t)|^2 \mathrm{d}y \right]. \end{aligned} \quad (2.6)$$

Thus, by Lemma 2.1 and (2.6) we get

$$|G(x, t)|^2 \leq Ct^{-1/\gamma'} \left[t^{-\rho} \int_{t/2 < |y| \leq t} |K_{\Omega,h}(y)| |g(x+y, t)|^2 \mathrm{d}y \right]. \quad (2.7)$$

Now, combining (2.5) and (2.7) gives

$$\begin{aligned} \|S_{\Omega,h}^{\rho,j}g(x)\|_{p'}^2 &\leq C2^{-j/\gamma'} \int_{\mathbb{R}^n} \int_{2^j}^{2^{j+1}} \int_{t/2 < |y| \leq t} t^{-\rho} |K_{\Omega,h}(y)| |g(x+y, t)|^2 u(x) \mathrm{d}y \frac{\mathrm{d}t}{t} \mathrm{d}x \\ &= C2^{-j/\gamma'} \int_{2^j}^{2^{j+1}} \int_{t/2 < |y| \leq t} t^{-\rho} |K_{\Omega,h}(y)| \int_{\mathbb{R}^n} |g(x+y, t)|^2 u(x) \mathrm{d}x \mathrm{d}y \frac{\mathrm{d}t}{t} \\ &= C2^{-j/\gamma'} \int_{2^j}^{2^{j+1}} \int_{t/2 < |y| \leq t} t^{-\rho} |K_{\Omega,h}(y)| \int_{\mathbb{R}^n} |g(x, t)|^2 u(x-y) \mathrm{d}x \mathrm{d}y \frac{\mathrm{d}t}{t} \\ &= C2^{-j/\gamma'} \int_{\mathbb{R}^n} \int_{2^j}^{2^{j+1}} |g(x, t)|^2 \int_{t/2 < |y| \leq t} t^{-\rho} |K_{\Omega,h}(y)| u(x-y) \mathrm{d}y \frac{\mathrm{d}t}{t} \mathrm{d}x \\ &\leq C2^{-j/\gamma'} \int_{\mathbb{R}^n} \left(\int_{2^j}^{2^{j+1}} |g(x, t)|^2 \frac{\mathrm{d}t}{t} \right) \sigma^*(u)(x) \mathrm{d}x. \end{aligned} \quad (2.8)$$

An application of Hölder's inequality yields

$$\|S_{\Omega,h}^{\rho,j}g(x)\|_{p'}^2 \leq C\|g\|_Z \|\sigma^*(u)\|_{L^{(p'/2)'}(\mathbb{R}^n)}. \quad (2.9)$$

Finally, by the observation that $(p'/2)' > \gamma'$ whenever $\frac{2\gamma}{2\gamma-1} < p < 2$ and Lemma 2.4 we get the required estimate. $\square$

3 Main result

Theorem 3.1. *If $h \in \mathcal{D}_\gamma + \mathcal{D}_\gamma^*$ for some $\gamma > 1$ and $\Omega \in L^1(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$, then $\mathcal{M}_{\Omega,h}^\rho$ is bounded on $L^p(\mathbb{R}^n)$ for $\frac{2\gamma}{2\gamma-1} < p < \infty$.*

Proof. Based on the observation that the operator $\mathcal{M}_{\Omega,h}^\rho$ is sublinear in h and Theorem 1.1, we need only to prove Theorem 3.1 for $h \in \mathcal{D}_\gamma^*$. First, note that

$$\begin{aligned} \mathcal{M}_{\Omega,h}^\rho f(x) &= \left(\int_0^\infty \left| t^{-p} \int_{|y|\leq t} f(x-y) K_{\Omega,h}(y) dy \right|^2 \frac{dt}{t} \right)^{1/2} \\ &= \left(\int_0^\infty \left| \sum_{k=0}^\infty t^{-p} \int_{t2^{-k-1} < |y| \leq t2^k} f(x-y) K_{\Omega,h}(y) dy \right|^2 \frac{dt}{t} \right)^{1/2} \\ &\leq \sum_{k=0}^\infty \left(\int_0^\infty \left| t^{-p} \int_{t2^{-k-1} < |y| \leq t2^k} f(x-y) K_{\Omega,h}(y) dy \right|^2 \frac{dt}{t} \right)^{1/2} \\ &= \left(\sum_{k=0}^\infty 2^{-kp} \right) \left(\int_0^\infty \left| t^{-p} \int_{t/2 < |y| \leq t} f(x-y) K_{\Omega,h}(y) dy \right|^2 \frac{dt}{t} \right)^{1/2} \\ &:= C_\rho \left(\int_0^\infty \left| t^{-p} \int_{t/2 < |y| \leq t} f(x-y) K_{\Omega,h}(y) dy \right|^2 \frac{dt}{t} \right)^{1/2} \\ &\leq C_\rho (\mathcal{R}_{\Omega,h}^\rho f(x) + \mathcal{S}_{\Omega,h}^\rho f(x)), \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} \mathcal{R}_{\Omega,h}^\rho f(x) &= \left(\int_0^1 \left| t^{-p} \int_{t/2 < |y| \leq t} f(x-y) K_{\Omega,h}(y) dy \right|^2 \frac{dt}{t} \right)^{1/2} = \left(\int_0^1 |F(x, t)|^2 \frac{dt}{t} \right)^{1/2}, \\ \mathcal{S}_{\Omega,h}^\rho f(x) &= \left(\int_1^\infty \left| t^{-p} \int_{t/2 < |y| \leq t} f(x-y) K_{\Omega,h}(y) dy \right|^2 \frac{dt}{t} \right)^{1/2} = \left(\int_1^\infty |F(x, t)|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

By using the homogeneity of L^p -norm and the classical Minkowski inequality we have

$$\|\mathcal{M}_{\Omega,h}^\rho f\|_p \leq C_\rho (\|\mathcal{R}_{\Omega,h}^\rho f\|_p + \|\mathcal{S}_{\Omega,h}^\rho f\|_p). \quad (3.2)$$

For $p \geq 2$, by Lemma 2.2 we have

$$\|\mathcal{S}_{\Omega,h}^\rho f\|_p^2 = \left(\int_{\mathbb{R}^n} \left(\int_1^\infty |F(x, t)|^2 \frac{dt}{t} \right)^{p/2} dx \right)^{2/p} \leq \int_1^\infty \|F(x, t)\|_p^2 \frac{dt}{t} \leq C \|f\|_p^2 \int_1^\infty \frac{1}{t^{1+\frac{2}{\gamma}}} dt = C \|f\|_p^2. \quad (3.3)$$

Next, we consider the case where $1 < p < 2$. By duality, there exists a function $g(x, t)$ defined on $\mathbb{R}^n \times \mathbb{R}^+$ in $Z = L^{p'}(dx, L^2([1, \infty], \frac{dt}{t}))$ with $\|g\|_Z \leq 1$ such that

$$\begin{aligned}
\|S_{\Omega,h}^\rho f\|_p &= \int_{\mathbb{R}^n} \int_1^\infty \left(t^{-\rho} \int_{t/2 < |y| \leq t} f(x-y) K_{\Omega,h}(y) dy \right) g(x,t) \frac{dt}{t} dx \\
&= \int_1^\infty \int_{t/2 < |y| \leq t} \left(\int_{\mathbb{R}^n} f(x-y) g(x,t) dx \right) t^{-\rho} K_{\Omega,h}(y) dy \frac{dt}{t} \\
&= \int_{\mathbb{R}^n} \left(\sum_{j=0}^\infty \int_{2^j}^{2^{j+1}} t^{-\rho} \int_{t/2 < |y| \leq t} K_{\Omega,h}(y) g(x+y,t) dy \frac{dt}{t} \right) f(x) dx \\
&\leq (\ln 2)^{1/2} \int_{\mathbb{R}^n} \left(\sum_{j=0}^\infty \left(\int_{2^j}^{2^{j+1}} \left| t^{-\rho} \int_{t/2 < |y| \leq t} K_{\Omega,h}(y) g(x+y,t) dy \right|^2 \frac{dt}{t} \right)^{1/2} \right) f(x) dx \\
&:= (\ln 2)^{1/2} \int_{\mathbb{R}^n} \left(\sum_{j=0}^\infty S_{\Omega,h}^{\rho,j} g(x) \right) f(x) dx.
\end{aligned} \tag{3.4}$$

By Hölder's inequality and Lemma 2.5, we get

$$\|S_{\Omega,h}^\rho f\|_p \leq (\ln 2)^{1/2} \|f\|_p \left(\sum_{j=0}^\infty \|S_{\Omega,h}^{\rho,j} g(x)\|_{p'} \right) \leq C \|f\|_p. \tag{3.5}$$

On the other hand, to prove the boundedness of the operator $\mathcal{R}_{\Omega,h}^\rho$, consider

$$\begin{aligned}
\mathcal{R}_{\Omega,h}^\rho f(x) &= \left(\int_0^1 \left| t^{-\rho} \int_{t/2 < |y| \leq t} f(x-y) K_{\Omega,h}(y) dy \right|^2 \frac{dt}{t} \right)^{1/2} \\
&\leq \sum_{j=0}^\infty \left(\int_{2^{j-1}}^{2^j} \left| t^{-\rho} \int_{t/2 < |y| \leq t} f(x-y) K_{\Omega,h}(y) dy \right|^2 \frac{dt}{t} \right)^{1/2} \\
&:= \sum_{j=1}^\infty \mathcal{R}_{\Omega,h}^{\rho,j} f(x).
\end{aligned} \tag{3.6}$$

Therefore, by Minkowski's inequality we have

$$\|\mathcal{R}_{\Omega,h}^\rho f(x)\|_p \leq \sum_{j=0}^\infty \|\mathcal{R}_{\Omega,h}^{\rho,j} f\|_p \tag{3.7}$$

for $1 < p < \infty$. Thus, we need to estimate $\|\mathcal{R}_{\Omega,h}^{\rho,j} f\|_p$ so that the series on the right-hand side of (3.7) converges. The proof follows the same lines of the one in [6]. In fact, by using (2.2) in the proofs of Lemmas 2.4 and 2.5 we achieve our goal. $\square$

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