Research Article

Awad A. Bakery*

The spectrum generated by s -numbers and pre-quasi normed Orlicz-Cesáro mean sequence spaces

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Abstract: In this article, we study some topological properties of the multiplication operator on Orlicz-Cesáro mean sequence spaces equipped with the pre-quasi norm and the pre-quasi operator ideal constructed by this sequence space and s -numbers.

Keywords: pre-quasi norm, Orlicz-Cesáro mean, multiplication operator, Fredholm operator, approximable operator, simple Banach space

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1 Introduction

Throughout the article, we signify the space of all bounded linear operators from a Banach space X into a Banach space Y by $L(X, Y)$ and if $X = Y$, we write $L(X)$, the space of all real sequences by w , the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, $\mathbb{N} = \{0, 1, 2, \dots\}$, the space of convergent sequences to zero by c_0 , the space of bounded sequences by ℓ_∞ and all sequences whose elements are complex by $\mathbb{C}^\mathbb{N}$.

Definition 1.1. [1] An Orlicz function is a function $\varphi : [0, \infty) \rightarrow [0, \infty)$, which is non-decreasing, convex and continuous with $\varphi(0) = 0$, $\varphi(x) > 0$ for $x > 0$ and $\lim_{x \rightarrow \infty} \varphi(x) = \infty$.

Definition 1.2. [2] An Orlicz function φ is said to fulfill Δ_2 -condition, for each value of $x \geq 0$, if there is $a > 0$, such that $\varphi(2x) \leq a\varphi(x)$. The Δ_2 -condition is compared to $\varphi(mx) \leq am\varphi(x)$, for all values of $m > 1$ and x .

Lindenstrauss and Tzafriri [3] used the idea of an Orlicz function to define the Orlicz sequence space

$$\ell_\varphi = \{x \in \omega : \varrho(\lambda x) < \infty, \text{ for some } \lambda > 0\}, \text{ where } \varrho(x) = \sum_{k=0}^{\infty} \varphi(|x_k|).$$

$(\ell_\varphi, \|\cdot\|)$ is a Banach space with the Luxemburg norm

$$\|x\|_{\ell_\varphi} = \inf\{\lambda > 0 : \varrho(\lambda^{-1}x) \leq 1\}.$$

In recent years, lot of work has been carried out on sequence spaces characterized by Orlicz functions by Altin et al. [4], Et et al. [5,6], Tripathy et al. [7–9] and Mohiuddine et al. [10–13].

* **Corresponding author: Awad A. Bakery**, University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia; Department of Mathematics, Faculty of Science, Ain Shams University, P.O. Box 1156, Cairo, 11566, Abbassia, Egypt, e-mail: aabhassan@uj.edu.sa, awad_bakery@yahoo.com, awad_bakry@hotmail.com

For an Orlicz function φ , the Orlicz-Cesàro mean sequence spaces are defined by

$$ces_{\varphi} = \{u = (u_i) \in \omega : \varrho(\beta u) < \infty \text{ for some } \beta > 0\} \text{ and } \varrho(u) = \sum_{i=0}^{\infty} \varphi\left(\frac{\sum_{j=0}^i |u_j|}{i+1}\right).$$

$(ces_{\varphi}, \|\cdot\|)$ is a Banach space with the Luxemburg norm given by

$$\|u\|_{ces_{\varphi}} = \inf\{\beta > 0 : \varrho(\beta^{-1}u) \leq 1\}.$$

It appears that Orlicz-Cesàro mean sequence spaces ces_{φ} showed up without precedent for 1988, when Lim and Lee studied their dual spaces [14]. Recently, Cui et al. [15] obtained important properties of spaces ces_{φ} . In 2007, Maligranda et al. [16] proved that ces_{φ} is not B-convex, if $\varphi \in \Delta_2$ and $ces_{\varphi} \neq 0$. The extreme points and strong X -points of ces_{φ} have been portrayed by Foralewski et al. [17]. When $\varphi(u) = u^p$, $1 \leq p < \infty$, the space ces_{φ} is a Cesàro sequence space ces_p , with the norm given by

$$\|u\|_{ces_p} = \left[\sum_{i=0}^{\infty} \left(\frac{\sum_{j=0}^i |u_j|}{i+1} \right)^p \right]^{\frac{1}{p}}.$$

It is notable that $ces_1 = \{0\}$ [18].

Definition 1.3. [19] The Matuszewska-Orlicz lower index α_{φ} of an Orlicz function φ is defined as follows

$$\alpha_{\varphi} = \sup\{p > 0 : \exists_{K>0} \forall_{0<\lambda, t \leq 1} \varphi(\lambda t) \leq Kt^p \varphi(\lambda)\}.$$

Theorem 1.4. [19] For any Orlicz function φ , we have $\alpha_{\varphi} > 1$ if and only if $\ell_{\varphi} \subset ces_{\varphi}$. Specifically, if $\alpha_{\varphi} > 1$, then $ces_{\varphi} \neq \{0\}$.

Theorem 1.5. [19] Let φ_1 and φ_2 be two Orlicz functions. If there exist $b, t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$, for every $t \in [0, t_0]$, then $ces_{\varphi_1} \subset ces_{\varphi_2}$.

Theorem 1.6. [19] Let φ_1 and φ_2 be two Orlicz functions and $\alpha_{\varphi_1} > 1$, then $ces_{\varphi_1} \subset ces_{\varphi_2}$ if and only if there exist $b, t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$, for all $t \in [0, t_0]$.

The multiplication operator and operator ideal theorems have been give importance in functional analysis, since they have various applications in the fixed point theorem, geometry of Banach spaces, spectral theory, eigenvalue distribution theorem etc. For more details see [20–26]. On sequence spaces, Mursaleen and Noman [27] studied the Compact operators on some difference sequence spaces, Komal and Gupta [28] investigated the multiplication operators on Orlicz spaces equipped with the Luxemburg norm and Komal et al. [29] examined the multiplication operators on Cesàro sequence spaces equipped with the Luxemburg norm. Some of the operator ideals in the class of Hilbert spaces or Banach spaces are constructed by numeric sequence spaces. For example, the ideal of compact operators is generated by Kolmogorov numbers and c_0 . Pietsch [30] studied the quasi-ideals generated by ℓ_p ($0 < p < \infty$) and the approximation numbers. He proved that the ideals of nuclear operators and of Hilbert Schmidt operators between Hilbert spaces are defined by ℓ_1 and ℓ_2 , respectively. He showed that the class of all finite rank operators are dense in the Banach quasi-ideal and the algebra $L(\ell_p)$, where $(1 \leq p < \infty)$ is the simple Banach space. Pietsch [31] proved that the quasi Banach operator ideal formed by the sequence of approximation numbers is small. Makarov and Faried [32] showed that if for any infinite dimensional Banach spaces X, Y and for any $q > p > 0$, then $S_{\ell_p}^{\text{app}}(X, Y)$ is strictly contained in $S_{\ell_q}^{\text{app}}(X, Y)$. In [33], Bakery and Mohammed investigated the operator ideals generated by s -numbers and Orlicz-Cesàro mean sequence spaces. In [34], Faried and Bakery introduced the concept of pre-quasi operator ideal which is more general than the quasi operator ideal, studied the operator ideals formed by s -numbers, generalized Cesàro and Orlicz sequence spaces ℓ_M and showed that the operator ideal formed by approximation numbers and the previous sequence spaces is small under certain conditions. Many articles have been

published on the above topics, sequence spaces defined by Orlicz function, multiplier sequences, spectra of matrix operators etc in the recent past by Tripathy et al. [35–41]. The aim of this article is to study the concept of pre-quasi norm on Orlicz-Cesàro mean sequence spaces, which is more general than the quasi norm, and give the conditions on this sequence space equipped with the pre-quasi norm to be Banach space. We give the necessity and sufficient conditions on this sequence space equipped with the pre-quasi norm such that the multiplication operator defined on this sequence space is bounded, approximable, invertible, Fredholm and closed range operator. The inclusion relation of the components of pre-quasi operator ideal formed by the sequence of s -numbers and this sequence space for different Orlicz functions has been determined. Furthermore, we give the sufficient conditions on this sequence space equipped with a pre-quasi norm such that the pre-quasi Banach operator ideal constructed by s -numbers and this sequence space is simple and its components are closed. Finally, the pre-quasi operator ideal formed by the sequence of s -numbers and this sequence space is strictly contained in the class of all bounded linear operators, whose sequence of eigenvalues belongs to this sequence space.

2 Definitions and preliminaries

Definition 2.1. [30] A finite rank operator is a bounded linear operator whose dimension of the range space is finite. The space of all finite rank operators on E is denoted by $F(E)$.

Definition 2.2. [30] A bounded linear operator $A : E \rightarrow E$ (where E is a Banach space) is called approximable if there are $S_n \in F(E)$, for all $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} \|A - S_n\| = 0$. The space of all approximable operators on E is denoted by $\Psi(E)$, and the space of all approximable operators from E to F is denoted by $\Psi(E, F)$.

Throughout the article, we define $e_i = (0, 0, \dots, 1, 0, 0, \dots)$, where 1 appears at the i th place, for all $i \in \mathbb{N}$.

Lemma 2.3. [30] Let $T \in L(X, Y)$. If T is not approximable, then there are operators $G \in L(X)$ and $B \in L(Y)$ such that $B T G e_k = e_k$, for all $k \in \mathbb{N}$.

Definition 2.4. [30] A Banach space E is called simple if the algebra $L(E)$ contains one and only one non-trivial closed ideal.

Definition 2.5. [30] A bounded linear operator $A : E \rightarrow E$ (where E is a Banach space) is called compact if $A(B)$ has compact closure, where B denotes the closed unit ball of E . The space of all compact operators on E is denoted by $L_c(E)$.

Theorem 2.6. [30] If E is an infinite dimensional Banach space, we have

$$F(E) \subsetneq \Psi(E) \subsetneq L_c(E) \subsetneq L(E).$$

Definition 2.7. [42] A bounded linear operator $A : E \rightarrow E$ is called Fredholm if A has closed range, $\dim(\ker A)$ and $\text{co-dim}(\text{range } A)$ are finite.

Definition 2.8. [34] A class of linear sequence spaces $\mathbb{E}$ is called a special space of sequences (sss) if

- (1) $e_i \in \mathbb{E}$, for all $i \in \mathbb{N}$,
- (2) if $u = (u_i) \in w$, $v = (v_i) \in \mathbb{E}$ and $|u_i| \leq |v_i|$, for every $i \in \mathbb{N}$, then $u \in \mathbb{E}$ “i.e. $\mathbb{E}$ is solid,”
- (3) if $(u_i)_{i=0}^\infty \in \mathbb{E}$, then $\left(u_{\left[\frac{i}{2}\right]}\right)_{i=0}^\infty \in \mathbb{E}$, wherever $\left[\frac{i}{2}\right]$ means the integral part of $\frac{i}{2}$.

Theorem 2.9. [33] If φ is an Orlicz function satisfying Δ_2 -condition and $\alpha_\varphi > 1$, then ces_φ is (sss).

The concept of pre-quasi operator ideal (Ω) is more general than the quasi operator ideal.

Definition 2.10. [33] An operator ideal

$$\Omega := \{\Omega(X, Y); X \text{ and } Y \text{ are Banach spaces}\} \subseteq L$$

is called a pre-quasi operator ideal, if there is a function $g : \Omega \rightarrow [0, \infty)$ satisfying the following conditions:

- (1) for all $T \in \Omega(X, Y)$, $g(T) \geq 0$ and $g(T) = 0$ if and only if $T = 0$,
- (2) there exists a constant $M \geq 1$ such that $g(\lambda T) \leq M|\lambda|g(T)$, for all $T \in \Omega(X, Y)$ and $\lambda \in \mathbb{R}$,
- (3) there exists a constant $K \geq 1$ such that $g(T_1 + T_2) \leq K[g(T_1) + g(T_2)]$, for all $T_1, T_2 \in \Omega(X, Y)$,
- (4) there exists a constant $C \geq 1$ such that if $T \in L(X_0, X)$, $P \in \Omega(X, Y)$ and $R \in L(Y, Y_0)$, then $g(RPT) \leq C\|R\|g(P)\|T\|$, where X_0 and Y_0 are normed spaces.

The function g is called a pre-quasi norm on the ideal Ω .

Definition 2.11. [43] An s -number function is a map defined on $L(X, Y)$, which associates with each operator $T \in L(X, Y)$ a non-negative scalar sequence $(s_n(T))_{n=0}^\infty$ assuming that the taking after states are verified:

- (a) $\|T\| = s_0(T) \geq s_1(T) \geq s_2(T) \geq \dots \geq 0$, for $T \in L(X, Y)$,
- (b) $s_{m+n-1}(T_1 + T_2) \leq s_m(T_1) + s_n(T_2)$, for all $T_1, T_2 \in L(X, Y)$, $m, n \in \mathbb{N}$,
- (c) ideal property: $s_n(RVT) \leq \|R\|s_n(V)\|T\|$, for all $T \in L(X_0, X)$, $V \in L(X, Y)$ and $R \in L(Y, Y_0)$, where X_0 and Y_0 are arbitrary Banach spaces,
- (d) if $G \in L(X, Y)$ and $\lambda \in \mathbb{R}$, we obtain $s_n(\lambda G) = |\lambda|s_n(G)$,
- (e) rank property: if $\text{rank}(T) \leq n$, then $s_n(T) = 0$ for each $T \in L(X, Y)$ and
- (f) norming property: $s_{r \geq n}(I_n) = 0$ or $s_{r < n}(I_n) = 1$, where I_n represents the unit operator on the n -dimensional Hilbert space ℓ_2^n .

Notations 2.12. [34]

$$S_E := \{S_E(X, Y); X \text{ and } Y \text{ are Banach spaces}\}, \text{ where } S_E(X, Y) := \{T \in L(X, Y) : ((s_i(T))_{i=0}^\infty \in E)\}.$$

Theorem 2.13. [33] The function $g(P) = \sum_{i=0}^\infty \varphi\left(\frac{\sum_{j=0}^i |s_j(P)|}{i+1}\right)$ is a pre-quasi norm on S_{ces_φ} , if φ is an Orlicz function satisfying Δ_2 -condition and $\alpha_\varphi > 1$.

Theorem 2.14. [33] If X and Y are Banach spaces, φ is an Orlicz function satisfying Δ_2 -condition and $\alpha_\varphi > 1$, then $(S_{ces_\varphi}(X, Y), g)$ is a pre-quasi Banach operator ideal.

3 Main results

In this section, we give the concept of pre-quasi norm on Orlicz-Cesáro mean sequence spaces, which is more general than the usual norm, and give the conditions on Orlicz-Cesáro mean sequence spaces $(ces_\varphi)_\varrho$ equipped with the pre-quasi norm ϱ , where $\varrho(x) = \sum_{n=0}^\infty \varphi\left(\frac{\sum_{k=0}^n |x_k|}{n+1}\right)$, for all $x \in ces_\varphi$, to be a Banach space.

Definition 3.1. Let $\mathbb{E}$ be (sss). If there is a function $\varrho : \mathbb{E} \rightarrow [0, \infty)$ fulfilling the following conditions:

- (i) $\varrho(x) \geq 0$, for every $x \in \mathbb{E}$ and $\varrho(x) = 0 \Leftrightarrow x = \theta$, where θ is the zero element of $\mathbb{E}$,
- (ii) there exists $L \geq 1$ such that $\varrho(\lambda x) \leq L|\lambda|\varrho(x)$, for all $x \in \mathbb{E}$, and for any scalar λ ,
- (iii) for some $K \geq 1$, we have $\varrho(x + y) \leq K(\varrho(x) + \varrho(y))$ for every $x, y \in \mathbb{E}$.

The space $\mathbb{E}$ with ϱ is called pre-quasi normed (sss) and denoted by $\mathbb{E}_\varrho$, which gives a class more general than the quasi normed space. If the space $\mathbb{E}$ is complete with ϱ , then $\mathbb{E}_\varrho$ is called a pre-quasi Banach (sss).

Theorem 3.2. Every quasi-norm is pre-quasi norm.

Theorem 3.3. If φ is an Orlicz function satisfying Δ_2 -condition and $\alpha_\varphi > 1$, then $(ces_\varphi)_\varrho$ is a pre-quasi Banach (sss).

Proof. The function $\varrho : ces_\varphi \rightarrow [0, \infty)$, where $\varrho(x) = \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |x_k|}{n+1}\right)$, for all $x \in ces_\varphi$, is satisfying the following conditions:

- (i) $\varrho(x) \geq 0$, for each $x \in ces_\varphi$ and $\varrho(x) = 0 \Leftrightarrow x = \theta$.
- (ii) Suppose $\lambda \in \mathbb{R}$, $x \in ces_\varphi$ and since φ is satisfying Δ_2 -condition and $\alpha_\varphi > 1$, we have a number $b > 0$ such that

$$\varrho(\lambda x) = \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |\lambda x_k|}{n+1}\right) \leq |\lambda|b \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |x_k|}{n+1}\right) = L|\lambda|\varrho(x),$$

where $L = \max\{1, b\}$.

- (iii) Let $x, y \in ces_\varphi$. Since φ is non-decreasing, convex and satisfying Δ_2 -condition and $\alpha_\varphi > 1$, then there exists a number $b > 0$ such that

$$\begin{aligned} \varrho(x+y) &= \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |x_k + y_k|}{n+1}\right) \\ &\leq \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |x_k|}{n+1}\right) + \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |y_k|}{n+1}\right) \\ &\leq \frac{1}{2} \left(\sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |2x_k|}{n+1}\right) + \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |2y_k|}{n+1}\right) \right) \\ &\leq \frac{b}{2} \left(\sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |x_k|}{n+1}\right) + \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |y_k|}{n+1}\right) \right) = K(\varrho(x) + \varrho(y)), \end{aligned}$$

for some $K = \max\left\{1, \frac{b}{2}\right\}$. Hence, $(ces_\varphi)_\varrho$ is a pre-quasi normed (sss). Since φ is continuous and non-decreasing, hence φ^{-1} exists, to prove that $(ces_\varphi)_\varrho$ is a pre-quasi Banach (sss), let $x^n = (x_k^n)_{k=0}^{\infty}$ be a Cauchy sequence in $(ces_\varphi)_\varrho$, then for every $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, one has

$$\sum_{k=0}^{\infty} \varphi\left(\frac{\sum_{i=0}^k |x_i^n - x_i^m|}{k+1}\right) < \varepsilon \Rightarrow \varphi(|x_i^n - x_i^m|) < \varepsilon \Rightarrow |x_i^n - x_i^m| < \varphi^{-1}(\varepsilon).$$

So, (x_i^m) is a Cauchy sequence in $\mathbb{R}$ for fixed $i \in \mathbb{N}$, this gives $\lim_{i \rightarrow \infty} x_i^m = x_i^0$ for fixed $i \in \mathbb{N}$. Hence $\varrho(x^n - x^0) < \varepsilon$. Finally, to prove that $x^0 \in ces_\varphi$, we have

$$\begin{aligned} \varrho(x^0) &= \sum_{k=0}^{\infty} \varphi\left(\frac{\sum_{i=0}^k |x_i^0 - x_i^n + x_i^n|}{k+1}\right) \\ &\leq K \left(\sum_{k=0}^{\infty} \varphi\left(\frac{\sum_{i=0}^k |x_i^0 - x_i^n|}{k+1}\right) + \sum_{k=0}^{\infty} \varphi\left(\frac{\sum_{i=0}^k |x_i^n|}{k+1}\right) \right) \\ &= K(\varrho(x^n - x^0) + \varrho(x^n)) < \infty, \end{aligned}$$

so $x^0 \in ces_\varphi$. This means that $(ces_\varphi)_\varrho$ is a pre-quasi Banach (sss). $\square$

Corollary 3.4. $(ces_p)_\varrho$, where $\varrho(x) = \sum_{n=0}^{\infty} \left(\frac{\sum_{k=0}^n |x_k|}{n+1}\right)^p$, for all $x \in ces_p$, is a pre-quasi Banach (sss), if $1 < p < \infty$.

4 Multiplication operator on pre-quasi normed (sss)

In this section, we define a multiplication operator on Orlicz-Cesàro mean sequence spaces with a pre-quasi norm and give the necessity and sufficient conditions on Orlicz-Cesàro mean sequence spaces equipped with the pre-quasi norm such that the multiplication operator defined on Orlicz-Cesàro mean sequence spaces is bounded, approximable, invertible, Fredholm and closed range operator.

Definition 4.1. Let $\alpha = (\alpha_k)$ be a sequence of complex numbers and X_φ be a pre-quasi normed (sss). The multiplication operator on X is defined as $T_\alpha : X \rightarrow X$, $T_\alpha x = (\alpha_k x_k)$, for all $x \in X$. If the multiplication operator T_α is continuous, then it is called a multiplication operator induced by α .

Theorem 4.2. If $\alpha \in \mathbb{C}^\mathbb{N}$, φ is an Orlicz function satisfying Δ_2 -condition and $\alpha_\varphi > 1$, then $\alpha \in \ell_\infty$ if and only if $T_\alpha \in L((ces_\varphi)_\varphi)$.

Proof. Let $\alpha \in \ell_\infty$. Then, there exists $C > 0$ such that $|\alpha_n| \leq C$, for all $n \in \mathbb{N}$. For $x \in (ces_\varphi)_\varphi$, since φ is an Orlicz function satisfying Δ_2 -condition and $\alpha_\varphi > 1$, we have

$$\varrho(T_\alpha x) = \varrho(\alpha x) = \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |\alpha_k| |x_k|}{n+1}\right) \leq \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n C |x_k|}{n+1}\right) \leq D \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |x_k|}{n+1}\right) = D \varrho(x),$$

where D is a constant that depends on C , which gives that $T_\alpha \in L((ces_\varphi)_\varphi)$.

Conversely, suppose that $T_\alpha \in L((ces_\varphi)_\varphi)$. We prove that $\alpha \in \ell_\infty$. If $\alpha \notin \ell_\infty$, then for every $n \in \mathbb{N}$, there exists some $i_n \in \mathbb{N}$ such that $\alpha_{i_n} > n$. Since φ is non-decreasing and continuous, we obtain

$$\varrho(T_\alpha e_{i_n}) = \varrho(\alpha e_{i_n}) = \sum_{j=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^j |\alpha_k| |(e_{i_n})_k|}{j+1}\right) = \sum_{j=0}^{\infty} \varphi\left(\frac{|\alpha_{i_n}|}{j+1}\right) > \sum_{j=0}^{\infty} \varphi\left(\frac{n}{j+1}\right) = \varrho(ne_{i_n}).$$

This proves that T_α is not a bounded operator. Hence, α must be a bounded function. $\square$

Theorem 4.3. Let $\alpha \in \mathbb{C}^\mathbb{N}$ and $(ces_\varphi)_\varphi$ be a pre-quasi normed (sss), then $|\alpha_n| = 1$, for all $n \in \mathbb{N}$ if and only if T_α is an isometry.

Proof. Let $|\alpha_n| = 1$, for all $n \in \mathbb{N}$. Then,

$$\varrho(T_\alpha x) = \varrho(\alpha x) = \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |\alpha_k| |x_k|}{n+1}\right) = \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |x_k|}{n+1}\right) = \varrho(x),$$

for all $x \in (ces_\varphi)_\varphi$. Hence, T_α is an isometry.

Conversely, suppose that $|\alpha_i| < 1$ for some $i = i_0$. Since φ is non-decreasing, we have

$$\varrho(T_\alpha e_{i_0}) = \varrho(\alpha e_{i_0}) = \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |\alpha_k| |(e_{i_0})_k|}{n+1}\right) = \sum_{n=0}^{\infty} \varphi\left(\frac{|\alpha_{i_0}|}{n+1}\right) < \varrho(e_{n_0}).$$

Similarly, if $|\alpha_{n_0}| > 1$, then we can show that $\varrho(T_\alpha e_{n_0}) > \varrho(e_{n_0})$. In the two cases, we obtain contradiction. Hence, $|\alpha_n| = 1$, for each $n \in \mathbb{N}$. $\square$

Theorem 4.4. Let $\alpha \in \mathbb{C}^\mathbb{N}$, φ be an Orlicz function satisfying Δ_2 -condition, $\alpha_\varphi > 1$ and $T_\alpha \in L((ces_\varphi)_\varphi)$. Then, $T_\alpha \in \Psi((ces_\varphi)_\varphi)$ if and only if $(\alpha_n)_{n=0}^\infty \in c_0$.

Proof. Suppose T_α is an approximable operator, hence T_α is a compact operator. We show that $\lim_{n \rightarrow \infty} \alpha_n = 0$. If this were not valid, then there exists $\delta > 0$ such that the set $B_\delta = \{r \in \mathbb{N} : |\alpha_r| \geq \delta\}$ is an infinite set.

Let $d_1, d_2, \dots, d_n, \dots$ be in B_δ . Then, $\{e_{d_i} : d_i \in B_\delta\}$ is an infinite bounded set in $(ces_\varphi)_\mathbb{Q}$. Consider

$$\begin{aligned} \varrho(T_\alpha e_{d_i} - T_\alpha e_{d_j}) &= \varrho(\alpha e_{d_i} - \alpha e_{d_j}) = \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n |\alpha_k((e_{d_i})_k - (e_{d_j})_k)|}{n+1}\right) \geq \sum_{n=0}^{\infty} \varphi\left(\frac{\sum_{k=0}^n \delta |(e_{d_i})_k - (e_{d_j})_k|}{n+1}\right) \\ &= \varrho(\delta e_{d_i} - \delta e_{d_j}), \end{aligned}$$

for all $d_i, d_j \in B_\delta$. This proves $\{e_{d_i} : d_i \in B_\delta\}$ is a bounded sequence which cannot have a convergent subsequence under T_α . This shows that T_α cannot be a compact, hence is not approximable operator, which is a contradiction. Hence, $\lim \alpha_n = 0$. Conversely, suppose $\lim \alpha_n = 0$. Then, for every $\delta > 0$, the set $B_\delta = \{n \in \mathbb{N} : |\alpha_n| \geq \delta\}$ is a finite set. Then, $((ces_\varphi)_\mathbb{Q})_{B_\delta} = \{x = (x_n) \in \mathbb{C}^\mathbb{N} : x_n = x_n, \text{ if } n \in B_\delta \text{ or } x_n = 0, \text{ otherwise}\}$ is a finite dimensional space for each $\delta > 0$. Therefore, $T_\alpha|_{((ces_\varphi)_\mathbb{Q})_{B_\delta}}$ is a finite rank operator. For each $n \in \mathbb{N}$, define $\alpha_n \in \mathbb{C}^\mathbb{N}$ by

$$(\alpha_n)_m = \begin{cases} \alpha_m, & m \in B_{\frac{1}{n}}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, T_{α_n} is a finite rank operator as the space $((ces_\varphi)_\mathbb{Q})_{B_{\frac{1}{n}}}$ is finite dimensional for each $n \in \mathbb{N}$. Now, since φ is convex and non-decreasing, we have

$$\begin{aligned} \varrho((T_\alpha - T_{\alpha_n})x) &= \varrho(((\alpha_m - (\alpha_n)_m)x_m)_{m=0}^\infty) = \sum_{i=0}^{\infty} \varphi\left(\frac{\sum_{m=0}^i |(\alpha_m - (\alpha_n)_m)x_m|}{i+1}\right) \\ &= \sum_{i=0, i \notin B_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\sum_{m=0}^i |(\alpha_m - (\alpha_n)_m)x_m|}{i+1}\right) + \sum_{i=0, i \in B_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\sum_{m=0}^i |(\alpha_m - (\alpha_n)_m)x_m|}{i+1}\right) \\ &= \sum_{i=0, i \notin B_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\sum_{m=0}^i |\alpha_m x_m|}{i+1}\right) \leq \frac{1}{n} \sum_{i=0, i \notin B_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\sum_{m=0}^i |x_m|}{i+1}\right) \\ &< \frac{1}{n} \sum_{i=0}^{\infty} \varphi\left(\frac{\sum_{m=0}^i |x_m|}{i+1}\right) = \frac{1}{n} \varrho(x). \end{aligned}$$

This proves that $\|T_\alpha - T_{\alpha_n}\| \leq \frac{1}{n}$ and that T_α is a limit of finite rank operators and, hence, T_α is an approximable operator. $\square$

Theorem 4.5. Let $\alpha \in \mathbb{C}^\mathbb{N}$, φ be an Orlicz function satisfying Δ_2 -condition, $\alpha_\varphi > 1$ and $T_\alpha \in L((ces_\varphi)_\mathbb{Q})$. Then, $T_\alpha \in L_c((ces_\varphi)_\mathbb{Q})$ if and only if $(\alpha_n)_{n=0}^\infty \in c_0$.

Proof. It is easy so omitted. $\square$

Corollary 4.6. Let φ be an Orlicz function satisfying Δ_2 -condition and $\alpha_\varphi > 1$, we have $L_c((ces_\varphi)_\mathbb{Q}) \subsetneq L((ces_\varphi)_\mathbb{Q})$.

Proof. Since the identity operator I on $(ces_\varphi)_\mathbb{Q}$ is a multiplication operator induced by the sequence $\alpha = (1, 1, \dots)$, hence $I \notin L_c((ces_\varphi)_\mathbb{Q})$ and $I \in L((ces_\varphi)_\mathbb{Q})$. $\square$

Theorem 4.7. Let $(ces_\varphi)_\mathbb{Q}$ be a pre-quasi Banach (sss) and $T_\alpha \in L((ces_\varphi)_\mathbb{Q})$. Then, there exist $a > 0$ and $A > 0$ such that $a < |\alpha_n| < A$, for all $n \in \mathbb{N} \setminus \ker(\alpha) := \ker(\alpha)^\complement$ if and only if T_α has closed range.

Proof. Suppose α is bounded away from zero on $\ker(\alpha)^\complement$. Then, there exists $\varepsilon > 0$ such that $|\alpha_n| \geq \varepsilon$, for all $n \in \ker(\alpha)^\complement$. We have to prove that $\text{range}(T_\alpha)$ is closed. Let h be a limit point of $\text{range}(T_\alpha)$. Then, there exists a sequence $T_\alpha x_n$ in $(ces_\varphi)_\mathbb{Q}$, for all $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} T_\alpha x_n = h$. Clearly, the sequence $T_\alpha x_n$ is a Cauchy sequence. Now, since φ is non-decreasing, we have

$$\begin{aligned}
\varrho(T_\alpha x_n - T_\alpha x_m) &= \sum_{i=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^i |\alpha_k(x_n)_k - \alpha_k(x_m)_k|}{i+1} \right) \\
&= \sum_{i=0, i \in \ker(\alpha)^c}^{\infty} \varphi \left(\frac{\sum_{k=0}^i |\alpha_k(x_n)_k - \alpha_k(x_m)_k|}{i+1} \right) + \sum_{i=0, i \notin \ker(\alpha)^c}^{\infty} \varphi \left(\frac{\sum_{k=0}^i |\alpha_k(x_n)_k - \alpha_k(x_m)_k|}{i+1} \right) \\
&\geq \sum_{i=0, i \in \ker(\alpha)^c}^{\infty} \varphi \left(\frac{\sum_{k=0}^i |\alpha_k(x_n)_k - \alpha_k(x_m)_k|}{i+1} \right) = \sum_{i=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^i |\alpha_k(y_n)_k - \alpha_k(y_m)_k|}{i+1} \right) \\
&> \sum_{i=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^i |\varepsilon(y_n)_k - \varepsilon(y_m)_k|}{i+1} \right) = \varrho(\varepsilon(y_n - y_m)),
\end{aligned}$$

where

$$(y_n)_k = \begin{cases} (x_n)_k, & k \in \ker(\alpha)^c, \\ 0, & k \notin \ker(\alpha)^c. \end{cases}$$

This proves that $\{y_n\}$ is a Cauchy sequence in $(ces_\varphi)_\varrho$. But $(ces_\varphi)_\varrho$ is complete. Therefore, there exists $x \in (ces_\varphi)_\varrho$ such that $\lim_{n \rightarrow \infty} y_n = x$. In view of continuity of T_α , hence $\lim_{n \rightarrow \infty} T_\alpha y_n = T_\alpha x$. But $\lim_{n \rightarrow \infty} T_\alpha x_n = \lim_{n \rightarrow \infty} T_\alpha y_n = h$. Therefore, $T_\alpha x = h$. Hence, $h \in \text{range}(T_\alpha)$. This proves that T_α has closed range. Conversely, suppose that T_α has closed range. Then, T_α is bounded away from zero on $((ces_\varphi)_\varrho)_{\ker(\alpha)^c}$. That is, there exists $\varepsilon > 0$ such that $\varrho(T_\alpha x) \geq \varrho(\varepsilon x)$, for all $x \in ((ces_\varphi)_\varrho)_{\ker(\alpha)^c}$. Let $D = \{k \in \ker(\alpha)^c : |\alpha_k| < \varepsilon\}$. If $D \neq \emptyset$, then for $n_0 \in D$, we have

$$\varrho(T_\alpha e_{n_0}) = (\varrho((\alpha_k(e_{n_0})_k))_{k=0}^\infty) = \sum_{i=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^i |\alpha_k(e_{n_0})_k|}{i+1} \right) < \sum_{i=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^i |\varepsilon(e_{n_0})_k|}{i+1} \right) = \varrho(\varepsilon e_{n_0}),$$

which is a contradiction. Hence, $D = \emptyset$ so that $|\alpha_k| \geq \varepsilon$, for all $k \in \ker(\alpha)^c$. This proves the theorem. $\square$

Theorem 4.8. Let $\alpha \in \mathbb{C}^\mathbb{N}$ and $(ces_\varphi)_\varrho$ be a pre-quasi Banach (sss). There exist $a > 0$ and $A > 0$ such that $a < \alpha_n < A$, for all $n \in \mathbb{N}$ if and only if $T_\alpha \in L((ces_\varphi)_\varrho)$ is invertible.

Proof. Suppose that the condition is true. Define $\beta \in \mathbb{C}^\mathbb{N}$ by $\beta_n = \frac{1}{\alpha_n}$. Then, T_α and T_β are bounded linear operators from Theorem 4.2. Also, $T_\alpha T_\beta = T_\beta T_\alpha = I$. Hence, T_β is the inverse of T_α . Alternately, suppose that T_α is invertible. Then, $\text{range}(T_\alpha) = ((ces_\varphi)_\varrho)_\mathbb{N}$. Therefore, $\text{range}(T_\alpha)$ is closed. Hence, by Theorem 4.7, there exists $a > 0$ such that $|\alpha_n| \geq a$, for all $n \in \ker(\alpha)^c$. Now $\ker(\alpha) = \emptyset$; otherwise $\alpha_{n_0} = 0$, for some $n_0 \in \mathbb{N}$, in which case $e_{n_0} \in \ker(T_\alpha)$ which is a contradiction, since $\ker(T_\alpha)$ is trivial. Hence, $|\alpha_n| \geq a$, for all $n \in \mathbb{N}$. Since T_α is bounded, so by Theorem 4.2, there exists $A > 0$ such that $|\alpha_n| \leq A$, for all $n \in \mathbb{N}$. Thus, we have proved that $a \leq |\alpha_n| \leq A$, for all $n \in \mathbb{N}$. $\square$

Theorem 4.9. Let $(ces_\varphi)_\varrho$ be a pre-quasi Banach (sss) and $T_\alpha \in L((ces_\varphi)_\varrho)$. Then, T_α is a Fredholm operator if and only if

- (i) $\ker(\alpha)$ is a finite subset of $\mathbb{N}$;
- (ii) $|\alpha_n| \geq \varepsilon$, for all $n \in \ker(\alpha)^c$.

Proof. Suppose T_α is Fredholm. If $\ker(\alpha)$ is an infinite subset of $\mathbb{N}$, then $e_n \in \ker(T_\alpha)$, for all $n \in \ker(\alpha)$. But e_n are linearly independent, which shows that $\ker(T_\alpha)$ is an infinite dimensional, which is a contradiction. Hence, $\ker(\alpha)$ must be a finite subset of $\mathbb{N}$. Condition (ii) follows from Theorem 4.7. Conversely, if conditions (i) and (ii) are true, then we prove that T_α is Fredholm. By Theorem 4.7, condition (ii) gives that T_α has a closed range. Condition (i) implies that $\ker(T_\alpha)$ and $(\text{range}(T_\alpha))^c$ are finite dimensional. This proves that T_α is Fredholm. $\square$

5 Pre-quasi closed ideal components

In this section, we answer the following question. For which Orlicz-Cesàro mean sequence spaces ces_φ , are the components of pre-quasi operator ideal S_{ces_φ} closed?

Theorem 5.1. $(S_{(ces_\varphi)_0}, g)$ is a pre-quasi closed operator ideal, where $g(T) = \varrho((s_n(T))_{n=0}^\infty)$, if X, Y are normed spaces, φ is an Orlicz function satisfying Δ_2 -condition and $\alpha_\varphi > 1$.

Proof. From Theorem 2.13, the function $g(T) = \varrho((s_n(T))_{n=0}^\infty)$ is a pre-quasi norm on $S_{(ces_\varphi)_0}$. Let $T_m \in S_{(ces_\varphi)_0}(X, Y)$, for all $m \in \mathbb{N}$ and $\lim_{m \rightarrow \infty} g(T_m - T) = 0$, since φ is continuous, $\alpha_\varphi > 1$ and $L(X, Y) \supseteq S_{(ces_\varphi)_0}(X, Y)$, we have

$$g(T - T_m) = \sum_{n=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^n |s_k(T - T_m)|}{n+1} \right) \geq \sum_{n=0}^{\infty} \varphi \left(\frac{\|T - T_m\|}{n+1} \right),$$

then $(T_m)_{m \in \mathbb{N}}$ is a convergent sequence in $L(X, Y)$. While $(s_n(T_m))_{n=0}^\infty \in (ces_\varphi)_0$, for each $m \in \mathbb{N}$, since φ is non-decreasing, convex and φ is continuous from right at 0, one can find two numbers $K, K_0 \geq 1$ such that

$$\begin{aligned} g(T) &= \sum_{n=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^n s_k(T)}{n+1} \right) \\ &= \sum_{n=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^n s_k(T - T_m + T_m)}{n+1} \right) \\ &\leq \sum_{n=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^n \left(s_{\left[\frac{k}{2}\right]}(T - T_m) + s_{\left[\frac{k}{2}\right]}(T_m) \right)}{n+1} \right) \\ &\leq K \left(\sum_{n=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^n s_{\left[\frac{k}{2}\right]}(T - T_m)}{n+1} \right) + \sum_{n=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^n s_{\left[\frac{k}{2}\right]}(T_m)}{n+1} \right) \right) \\ &\leq K \left(\sum_{n=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^n \|T - T_m\|}{n+1} \right) + K_0 \sum_{n=0}^{\infty} \varphi \left(\frac{\sum_{k=0}^n s_k(T_m)}{n+1} \right) \right) < \varepsilon, \end{aligned}$$

we have $(s_n(T))_{n=0}^\infty \in (ces_\varphi)_0$, then $T \in S_{(ces_\varphi)_0}(X, Y)$. □

6 Pre-quasi simple Banach operator ideal

We give here the sufficient conditions on Orlicz-Cesàro mean sequence spaces such that the pre-quasi operator ideal formed by the sequence of s -numbers and this sequence space is strictly contained for different Orlicz functions.

Theorem 6.1. Let φ_1, φ_2 be two Orlicz functions and $\alpha_{\varphi_1} > 1$. For any infinite dimensional Banach spaces X, Y and if there exist $b, t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$, for all $t \in [0, t_0]$, then

$$S_{ces_{\varphi_1}}(X, Y) \subsetneq S_{ces_{\varphi_2}}(X, Y) \subsetneq L(X, Y).$$

Proof. Let the conditions be satisfied, if $T \in S_{ces_{\varphi_1}}(X, Y)$, then $(s_n(T)) \in ces_{\varphi_1}$. We have

$$\sum_{n=0}^{\infty} \varphi_2 \left(\frac{\sum_{k=0}^n s_k(T)}{n+1} \right) < \sum_{n=0}^{\infty} \varphi_1 \left(\frac{\sum_{k=0}^n s_k(T)}{n+1} \right) < \infty,$$

hence $T \in S_{ces_{\varphi_2}}(X, Y)$. Next, if we take $(s_n(T))_{n=0}^{\infty}$ such that $\varphi_1\left(\frac{\sum_{k=0}^n s_k(T)}{n+1}\right) = \frac{1}{q/n+1}$ and $\varphi_2\left(\frac{\sum_{k=0}^n s_k(T)}{n+1}\right) = \frac{1}{(n+1)\frac{q}{p}}$, for any $q > p > 0$. We can find $T \in L(X, Y)$ with $T \notin S_{ces_{\varphi_1}}(X, Y)$ and $T \in S_{ces_{\varphi_2}}(X, Y)$. It is easy to verify that $S_{ces_{\varphi_2}}(X, Y) \subset L(X, Y)$. Next, if we take $(s_n(T))_{n=0}^{\infty}$ such that $\varphi_2\left(\frac{\sum_{k=0}^n s_k(T)}{n+1}\right) = \frac{1}{n+1}$. One can find $T \in L(X, Y)$ such that $T \notin S_{ces_{\varphi_2}}(X, Y)$. This completes the proof. $\square$

Corollary 6.2. For any infinite dimensional Banach spaces X, Y and $1 < p < q < \infty$, then

$$S_{ces_p}(X, Y) \subsetneq S_{ces_q}(X, Y) \subsetneq L(X, Y).$$

It is natural to raise the question “for which Orlicz-Cesàro mean sequence space, is the pre-quasi Banach ideal simple?”

Theorem 6.3. Let φ_1, φ_2 be two Orlicz functions and $\alpha_{\varphi_1} > 1$. For any infinite dimensional Banach spaces X, Y and if there exist $b, t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$, for all $t \in [0, t_0]$, then

$$L(S_{ces_{\varphi_2}}, S_{ces_{\varphi_1}}) = \Psi(S_{ces_{\varphi_2}}, S_{ces_{\varphi_1}}).$$

Proof. Suppose that there exists $T \in L(S_{ces_{\varphi_2}}, S_{ces_{\varphi_1}})$, which is not approximable. According to Lemma 2.3, we can find $D \in L(S_{ces_{\varphi_2}})$ and $B \in L(S_{ces_{\varphi_1}})$ with $BTDI_k = I_k$. Then it follows, for all $k \in \mathbb{N}$ that

$$\|I_k\|_{S_{ces_{\varphi_1}}} = \sum_{i=0}^{\infty} \varphi_1\left(\frac{\sum_{j=0}^i s_j(I_k)}{i+1}\right) \leq \|BTD\| \|I_k\|_{S_{ces_{\varphi_2}}} \leq \sum_{i=0}^{\infty} \varphi_2\left(\frac{\sum_{j=0}^i s_j(I_k)}{i+1}\right).$$

This contradicts Theorem 6.1, which is a contradiction. Hence $T \in \Psi(S_{ces_{\varphi_2}}, S_{ces_{\varphi_1}})$, which completes the proof. $\square$

Corollary 6.4. Let φ_1, φ_2 be two Orlicz functions and $\alpha_{\varphi_1} > 1$. For any infinite dimensional Banach spaces X, Y and if there exist $b, t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$, for all $t \in [0, t_0]$, then

$$L(S_{ces_{\varphi_2}}, S_{ces_{\varphi_1}}) = L_C(S_{ces_{\varphi_2}}, S_{ces_{\varphi_1}}).$$

Proof. Since every approximable operator is compact. $\square$

7 Eigenvalues of s-type operators

We give here the sufficient conditions on Orlicz-Cesàro mean sequence spaces such that the pre-quasi operator ideal formed by the sequence of s-numbers and this sequence space is strictly contained in the class of all bounded linear operators whose sequence of eigenvalues belongs to this sequence space.

Notations 7.1.

$S_E^\lambda := \{S_E^\lambda(X, Y); X \text{ and } Y \text{ are Banach spaces}\}$, where

$S_E^\lambda(X, Y) := \{T \in L(X, Y) : ((\lambda_i(T))_{n=0}^{\infty} \in E \text{ and } \|T - \lambda_n(T)\| \text{ is not invertible, for all } n \in \mathbb{N})\}.$

Theorem 7.2. For any infinite dimensional Banach spaces X and Y , if φ is an Orlicz function satisfying Δ_2 -condition and $\alpha_\varphi > 1$, then

$$S_{ces_\varphi}(X, Y) \subsetneq S_{ces_\varphi}^\lambda(X, Y).$$

Proof. Let $T \in S_{ces_\varphi}(X, Y)$, then $(s_n(T))_{n=0}^\infty \in ces_\varphi$, we have

$$\sum_{i=0}^{\infty} \varphi(s_i(T)) < \sum_{i=0}^{\infty} \varphi\left(\frac{\sum_{j=0}^i s_j(T)}{i+1}\right) < \infty,$$

and since φ is continuous, so $\lim_{n \rightarrow \infty} s_n(T) = 0$. Suppose $\|T - s_n(T)\|$ is invertible, for all $n \in \mathbb{N}$, then $\|T - s_n(T)\|^{-1}$ exists and bounded, for all $n \in \mathbb{N}$. This gives $\lim_{n \rightarrow \infty} \|T - s_n(T)\|^{-1} = \|T\|^{-1}$ exists and bounded. Since (S_{ces_φ}, g) is a pre-quasi operator ideal, we have

$$I = TT^{-1} \in S_{ces_\varphi}(X, Y) \Rightarrow (s_n(I))_{n=0}^\infty \in ces_\varphi \Rightarrow \lim_{n \rightarrow \infty} s_n(I) = 0.$$

But $\lim_{n \rightarrow \infty} s_n(I) = 1$. This is a contradiction, then $\|T - s_n(T)\|$ is not invertible, for all $n \in \mathbb{N}$. Therefore, the

sequence $(s_n(T))_{n=0}^\infty$ is the eigenvalue of T . Next, on considering $(s_i(T))_{i=0}^\infty$ such that $\varphi\left(\frac{\sum_{j=0}^i s_j(T)}{i+1}\right) = \frac{1}{i+1}$, we

find $T \in L(X, Y)$ with $T \notin S_{ces_\varphi}(X, Y)$ and if we take $(\lambda_i(T))_{i=0}^\infty$ such that $\varphi\left(\frac{\sum_{j=0}^i \lambda_j(T)}{i+1}\right) = \frac{1}{(i+1)^2}$. Hence, $T \in S_{ces_\varphi}^\lambda(X, Y)$. This finishes the proof. $\square$

Corollary 7.3. For any infinite dimensional Banach spaces X and Y , if $p > 1$, then

$$S_{ces_p}(X, Y) \subsetneq S_{ces_p}^\lambda(X, Y).$$

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