Research Article

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Positive coincidence points for a class of nonlinear operators and their applications to matrix equations

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Abstract: Consider an ordered Banach space and f, g two self-operators defined on the interior of its positive cone. In this article, we prove that the equation $f(X) = g(X)$ has a positive solution, whenever f is strictly α -concave g -monotone or strictly $(-\alpha)$ -convex g -antitone with g super-homogeneous and surjective. As applications, we show the existence of positive definite solutions to new classes of nonlinear matrix equations.

Keywords: coincidence points, Banach spaces, monotone operators, matrix equations

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1 Introduction

The coincidence point theory is a powerful tool in nonlinear analysis for solving a wide range of nonlinear equations arising from various applications in engineering, economics and mechanics, see for instance [1–8]. In particular, nonlinear equations in Banach spaces involving α -concave and $(-\alpha)$ -convex operators are considered in [9–15] and some references therein.

The main aim of this article is to show the existence and uniqueness of coincidence points for two operators f and g using new iterative methods, where f is strictly g -monotone α -concave or strictly g -antitone $(-\alpha)$ -convex with g super-homogeneous and surjective. As applications, we show the existence of positive definite solutions to a new family of nonlinear matrix equations of type

$$g(X) = Q + \sum_{i=1}^k A_i^* f_i(X) A_i,$$

where Q is a positive semi definite matrix, A_i are arbitrary square matrices, f_i are Löwner-Heinz monotone operators and g is an Uchiyama operator [16–18]. Our results improve, extend and generalize some existing ones in the literature [11,15,19,20]. Furthermore, we provide two algorithms involving the Newton-Raphson method for solving new classes of nonlinear matrix equations. Next, we present some numerical experiments to illustrate their efficiency. Finally, we apply our algorithms to compute a positive root to certain polynomials. For recently developed methods to solve more nonlinear equations, we refer the reader to [21–23] and some references therein. The remainder of the article is structured as follows. Section

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2 contains the notation and basic definitions. The main theorems of coincidence points are stated and proved in Section 3. Some particular classes of operators are investigated in Section 4. The applications and the numerical experiments are presented in Section 5.

2 Preliminaries

Let $(E, \|\cdot\|)$ be a real Banach space and denote by 0_E its zero element. A nonempty subset P of E is called a cone if it satisfies $\lambda P \subseteq P$ for any positive real λ . A cone P is called pointed if $P \cap (-P) = \{0_E\}$, solid if $\overset{\circ}{P}$, the interior of P , is nonempty, and it is called normal if there exists a positive constant c_N , called the normal constant with respect to $\|\cdot\|$, such that $0_E \leq x \leq y$ implies $\|x\| \leq c_N \|y\|$. If P is a convex cone, then E is partially ordered with respect to P , that is, $x \leq y$ if and only if $y - x \in P$. In the rest of the article, we assume that P is normal, solid, closed, convex and pointed cone of E . If $x \leq y$, the set

$$[x, y] := \{z \in E : x \leq z \leq y\} = (x + P) \cap (y - P),$$

is called an order-interval, where $x, y \in E$. The notation $x \ll y$ is used if $y - x \in \overset{\circ}{P}$, and we define the set

$$[x|y] := \{z \in E : x \ll z \ll y\} = (x + \overset{\circ}{P}) \cap (y - \overset{\circ}{P}).$$

The open ball centered at $z \in E$ with radius $\varepsilon > 0$ is denoted by $B(z, \varepsilon)$.

Lemma 2.1. Let $x, y \in \overset{\circ}{P}$. If $x \ll y$, then for all $\varepsilon > 0$, $B(y, \varepsilon) \cap [x|y]$ is a nonempty open set.

Proof. The element $\lambda x + (1 - \lambda)y$ belongs to the set $B(y, \varepsilon) \cap [x|y]$, if $\lambda \in (0, 1)$ and $\lambda\|x - y\| < \varepsilon$. □

Definition 2.2. Let $f, g : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be two operators. We say that:

- (i) f is strictly α -concave if there exists a constant $\alpha \in (0, 1)$ such that $f(tx) \gg t^\alpha fx$, or equivalently $f(t^{-1}x) \ll t^{-\alpha} fx$, for all $t \in (0, 1)$ and $x \in \overset{\circ}{P}$.
- (ii) f is strictly $(-\alpha)$ -convex if there exists a constant $\alpha \in (0, 1)$ such that $f(tx) \ll t^{-\alpha} fx$, or equivalently $f(t^{-1}x) \gg t^\alpha fx$, for all $t \in (0, 1)$ and $x \in \overset{\circ}{P}$.

Definition 2.3. Let $g : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be an operator. We say that g is strictly super-homogeneous if $g(tx) \ll t g x$, or equivalently $g(t^{-1}x) \gg t^{-1} g x$, for all $t \in (0, 1)$ and $x \in \overset{\circ}{P}$.

Definition 2.4. Let $f : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be an operator. We say that:

- (i) f is strictly monotone (resp. antitone) if $x \ll y \Rightarrow fx \ll fy$ (resp. $fy \ll fx$).
- (ii) f is strictly reverse monotone (resp. antitone) if $fx \ll fy \Rightarrow x \ll y$ (resp. $y \ll x$).
- (iii) f is strictly inverse monotone if f is invertible and the inverse of f is strictly monotone.

Definition 2.5. Let $f, g : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be two operators. We say that:

- (i) f is strictly g -monotone if $gx \ll gy \Rightarrow fx \ll fy$.
- (ii) f is strictly g -antitone if $gx \ll gy \Rightarrow fy \ll fx$.

Remark 2.6. Every strictly inverse monotone (resp. antitone) operator is strictly reverse monotone (resp. antitone). The concept of strictly monotone (resp. antitone) is a particular case of the notion of strictly g -monotone (resp. strictly g -antitone), where g is the identity operator.

As an immediate consequence, we have the following result.

Lemma 2.7. Let $f, g : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be two operators. Assume that g is invertible and its inverse is strictly monotone. Then, if f is strictly monotone (resp. strictly antitone), then f is strictly g -monotone (resp. strictly g -antitone).

Further examples of strictly monotone operators are presented in Section 5.

3 Coincidence point theorems

The set of coincidence points of f and g in $\overset{\circ}{P}$ will be denoted by $C(f, g) := \{x \in \overset{\circ}{P} : fx = gx\}$, and it will be identified with its only element if it is a singleton set. A point $x \in \overset{\circ}{P}$ will be called a *common fixed point* if $fx = gx = x$. Now, we are ready to state our main results.

Theorem 3.1. Let $f, g : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be operators. Assume that g is surjective and strictly super-homogeneous and that f is strictly α -concave and strictly g -monotone. Then, $C(f, g)$ is nonempty and $g(C(f, g))$ is a singleton. More precisely, if $\{\varepsilon_n\}$ is a decreasing real sequence convergent to zero, then there exist two sequences $\{u_n\}$ and $\{v_n\}$ in $\overset{\circ}{P}$ such that for all $n \geq 1$, we have

$$\|gu_{n+1} - fu_n\| \leq \varepsilon_n, \|gv_{n+1} - fv_n\| \leq \varepsilon_n, \quad (1)$$

$$fu_{n-1} \ll gu_{n+1} \ll fu_n \ll fv_n \ll gv_{n+1} \ll fv_{n-1}, \quad (2)$$

then $\{gu_n\}$ and $\{gv_n\}$ converge to $g(C(f, g))$. In addition, f and g have a unique common fixed point if and only if $g(C(f, g)) \in C(f, g)$. In particular, if $C(f, g) \subseteq C(fg, gf)$, then f and g have a unique common fixed point.

Theorem 3.2. Let $f, g : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be operators. Assume that g is surjective and strictly super-homogeneous, and that f is strictly $(-\alpha)$ -convex and strictly g -antitone. Then, $C(f, g)$ is nonempty and $g(C(f, g))$ is a singleton. More precisely, if $\{\varepsilon_n\}$ is a decreasing real sequence convergent to zero, then there exist two sequences $\{u_n\}$ and $\{v_n\}$ in $\overset{\circ}{P}$ such that for all $n \geq 1$, we have

$$\|gu_{n+1} - fv_n\| \leq \varepsilon_n, \|gv_{n+1} - fu_n\| \leq \varepsilon_n, \quad (3)$$

$$fv_{n-1} \ll gu_{n+1} \ll fv_n \ll fu_n \ll gv_{n+1} \ll fu_{n-1}, \quad (4)$$

then $\{gu_n\}$ and $\{gv_n\}$ converge to $g(C(f, g))$. In addition, f and g have a unique common fixed point if and only if $g(C(f, g)) \in C(f, g)$. In particular, if $C(f, g) \subseteq C(fg, gf)$, then f and g have a unique common fixed point.

To prove the theorem, we need the following lemmas. The next lemma give a sufficient condition to the existence of sequences satisfying (1) and (2).

Lemma 3.3. Let $f, g : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be two operators such that g is surjective and f is strictly g -monotone. Assume that there exist x_0 and y_0 in $\overset{\circ}{P}$ such that

$$gx_0 \ll fx_0 \ll fy_0 \ll gy_0. \quad (5)$$

Then, for all decreasing real sequences $\{\varepsilon_n\}$ converging to zero, there exist two sequences $\{u_n\}$ and $\{v_n\}$ in $\overset{\circ}{P}$ such that (1) and (2) hold.

Proof. Let $u_0 = x_0$ and $v_0 = y_0$. By hypothesis, using the surjectivity of g and Lemma 2.1, there exist $u_1, v_1 \in \overset{\circ}{P}$ such that $gu_1 \in B(fu_0, \varepsilon_0) \cap [gu_0|fu_0]$ and $gv_1 \in B(fv_0, \varepsilon_0) \cap [fv_0|gv_0]$, then the elements u_0, u_1, v_0, v_1 satisfy $gu_0 \ll gu_1 \ll fu_0 \ll fv_0 \ll gv_1 \ll gv_0$. As f is strictly g -monotone, we obtain $gu_0 \ll gu_1 \ll fu_0 \ll fu_1 \ll fv_1 \ll fv_0 \ll gv_1 \ll gv_0$. Using again the surjectivity of g there exist u_2, v_2 such that $gu_2 \in B(fu_1, \varepsilon_1) \cap [fu_0|fu_1]$ and $gv_2 \in B(fv_1, \varepsilon_1) \cap [fv_1|fv_0]$. By construction, the elements $u_0, u_1, u_2, v_0, v_1, v_2$ satisfy (1) and (2) for $n = 1$.

For the induction step, assume that until some positive integer n and for all $k \leq n$, there exist $\{u_k\}, \{v_k\}$ in $\mathring{P}$ satisfying (1) and (2). Hence, it follows that

$$gu_{n-1} \ll fu_{n-2} \ll gu_n \ll fu_{n-1} \ll fv_{n-1} \ll gv_n \ll fv_{n-2} \ll gv_{n-1}. \quad (6)$$

As f is strictly g -monotone, we get

$$fu_{n-1} \ll fu_n \ll fv_n \ll fv_{n-1}. \quad (7)$$

Combining (6) and (7), we obtain

$$gu_n \ll fu_{n-1} \ll fu_n \ll fv_n \ll fv_{n-1} \ll gv_n. \quad (8)$$

As before, from the surjectivity of g , there exist $u_{n+1}, v_{n+1} \in \mathring{P}$ such that $gu_{n+1} \in B(fu_n, \varepsilon_n) \cap [fu_{n-1}|fu_n]$ and $gv_{n+1} \in B(fv_n, \varepsilon_n) \cap [fv_n|fv_{n-1}]$. Consequently, (1) and (2) hold for all n . $\square$

The following lemma gives a sufficient condition to the existence of sequences satisfying (3) and (4).

Lemma 3.4. Let $f, g : \mathring{P} \rightarrow \mathring{P}$ be two operators such that g is surjective and f is g -antitone. Assume that there exist x_0 and y_0 in $\mathring{P}$ such that

$$gx_0 \ll fy_0 \leq fx_0 \ll gy_0. \quad (9)$$

Then, for all decreasing real sequences $\{\varepsilon_n\}$ convergent to zero, there exist two sequences $\{u_n\}, \{v_n\}$ in $\mathring{P}$ such that (3) and (4) hold.

Proof. Let $u_0 = x_0$ and $v_0 = y_0$. By hypothesis, using the surjectivity of g and Lemma 2.1, there exist $u_1, v_1 \in \mathring{P}$ such that $gu_1 \in B(fv_0, \varepsilon_0) \cap [gu_0|fv_0]$ and $gv_1 \in B(fu_0, \varepsilon_0) \cap [fu_0|gv_0]$, then we have u_0, u_1, v_0, v_1 satisfy $gu_0 \ll gu_1 \ll fv_0 \ll fu_0 \ll gv_1 \ll gv_0$. As f is strictly g -antitone, we obtain $gu_0 \ll gu_1 \ll fv_0 \ll fv_1 \ll fu_1 \ll fu_0 \ll gv_1 \ll gv_0$. Using again the surjectivity of g there exist u_2, v_2 such that $gu_2 \in B(fv_1, \varepsilon_1) \cap [fv_0|fv_1]$ and $gv_2 \in B(fu_1, \varepsilon_1) \cap [fu_1|fu_0]$. By construction, the elements $u_0, u_1, u_2, v_0, v_1, v_2$ satisfy (3) and (4) for $n = 1$. For the induction step, assume that until some positive integer n and for all $k \leq n - 1$, there exist $\{u_k\}, \{v_k\}$ in $\mathring{P}$ satisfying (3) and (4). Hence, it follows that

$$gu_{n-1} \ll fv_{n-2} \ll gu_n \ll fv_{n-1} \ll fu_{n-1} \ll gv_n \ll fu_{n-2} \ll gv_{n-1}. \quad (10)$$

As f is strictly g -antitone, we get

$$fv_{n-1} \ll fv_n \ll fu_n \ll fu_{n-1}. \quad (11)$$

Combining (10) and (11), we obtain

$$gu_n \ll fv_{n-1} \ll fv_n \ll fu_n \ll fu_{n-1} \ll gv_n. \quad (12)$$

As before, from the surjectivity of g , there exist $u_{n+1}, v_{n+1} \in \mathring{P}$ such that $gu_{n+1} \in B(fv_n, \varepsilon_n) \cap [fv_{n-1}|fv_n]$ and $gv_{n+1} \in B(fu_n, \varepsilon_n) \cap [fu_n|fu_{n-1}]$. Consequently, (3) and (4) hold for all n . $\square$

Condition (5) (resp. (9)) is obtained by the following lemma for our classes of operators.

Lemma 3.5. Let $f, g : \mathring{P} \rightarrow \mathring{P}$ be two operators such that g is surjective and strictly super-homogeneous and f is strictly α -concave and strictly g -monotone (resp. strictly $(-\alpha)$ -convex and strictly g -antitone). Then, there exist $x_0, y_0 \in \mathring{P}$ satisfying (5) (resp. (9)).

Proof. Let $h \in \mathring{P}$. Choose $t_0 \in (0, 1)$ sufficiently small such that

$$t_0^{(1-\alpha)}gh \leq fh \leq t_0^{(\alpha-1)}gh. \quad (13)$$

Assume that g is strictly super-homogeneous. For $x_0 = t_0 h$ and $y_0 = t_0^{-1} h$, we deduce that

$$g y_0 - g x_0 = g y_0 - g(t_0^2 y_0) \geq g y_0 - t_0^2 g y_0 = (1 - t_0^2) g y_0,$$

which implies that

$$g x_0 \ll g y_0. \quad (14)$$

Since f is strictly α -concave, it follows from (13) that

$$\begin{aligned} f x_0 &= f(t_0 h) \gg t_0^\alpha f h \geq t_0 g h \geq g(t_0 h) = g x_0, \\ f y_0 &= f(t_0^{-1} h) \ll t_0^{-\alpha} f h \leq t_0^{-1} g h \leq g(t_0^{-1} h) = g y_0. \end{aligned}$$

Now, using (14) combined with the fact that f is strictly g -monotone, we obtain (5). Assume now that f is strictly g -antitone and strictly $(-\alpha)$ -convex. Then, it follows from (13) that

$$\begin{aligned} f x_0 &= f(t_0 h) \ll t_0^{-\alpha} f h \leq t_0^{-1} g h \leq g(t_0^{-1} h) = g y_0, \\ f y_0 &= f(t_0^{-1} h) \gg t_0^\alpha f h \geq t_0 g h \geq g(t_0 h) = g x_0. \end{aligned}$$

Now, from (14) and the fact that f is strictly g -antitone, we obtain (9). $\square$

The following two technical lemmas are needed to prove the convergence of the sequences.

Lemma 3.6. Let $f, g : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be two operators. Assume that g is strictly super-homogeneous and f is strictly α -concave and strictly g -monotone. Let $\{u_n\}, \{v_n\}$ be sequences satisfying (1) and (2) and $t_n := \sup\{0 < t \leq 1 : g u_n \geq t g v_n\}$, for all $n \geq 0$. Then, $\lim_{n \rightarrow \infty} t_n = 1$.

Proof. Observe first that from definition of t_n and (2) it follows that $g u_{n+1} \geq g u_n \geq t_n g v_n \geq t_n g v_{n+1}$, so we have $t_n \leq t_{n+1}$ for all n , which implies $0 < t_0 \leq t_1 \leq \dots \leq t_n \leq \dots \leq 1$. Then, there is $t_* \in (0, 1]$ such that $\lim_{n \rightarrow \infty} t_n = t_*$. We claim that $t_* = 1$. Assume by contradiction that $t_* < 1$.

Observe first that from the super-homogeneity of g , we have

$$g u_n \geq t_n g v_n \gg g(t_n v_n). \quad (15)$$

Since f is strictly α -concave and strictly g -monotone, it follows from (15) that

$$g u_{n+2} \gg f u_n \gg f(t_n v_n) = f\left(\frac{2t_n}{1+t_*} \frac{1+t_*}{2} v_n\right) \gg \left(\frac{2t_n}{1+t_*}\right) \left(\frac{1+t_*}{2}\right)^\alpha f v_n \gg \left(\frac{2t_n}{1+t_*}\right) \left(\frac{1+t_*}{2}\right)^\alpha g v_{n+2}.$$

By definition of t_n , it follows that

$$t_{n+2} \geq \left(\frac{2t_n}{1+t_*}\right) \left(\frac{1+t_*}{2}\right)^\alpha,$$

which yields a contradiction when n tends to infinity. Consequently, we have $\lim_{n \rightarrow \infty} t_n = 1$. $\square$

Lemma 3.7. Let $f, g : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be two operators. Assume that g is strictly super-homogeneous and f is strictly $(-\alpha)$ -convex and g -antitone. Let $\{u_n\}, \{v_n\}$ be sequences satisfying (3) and (4) and $t_n := \sup\{0 < t \leq 1 : g u_n \geq t g v_n\}$, for all $n \geq 0$. Then, $\lim_{n \rightarrow \infty} t_n = 1$.

Proof. Observe first that from definition of t_n , we have $g u_n \geq t_n g v_n$ for all $n \geq 0$. In particular, from (4) follows that $g u_{n+1} \geq g u_n \geq t_n g v_n \geq t_n g v_{n+1}$, so we have $t_n \leq t_{n+1}$ for all n , which implies $0 < t_0 \leq t_1 \leq \dots \leq t_n \leq \dots \leq 1$. Then, there is $t_* \in [t_n, 1]$ for all $n \geq 0$ such that $\lim_{n \rightarrow \infty} t_n = t_*$. We claim that $t_* = 1$. Assume by contradiction that $t_* < 1$.

Observe first that from the super-homogeneity of g , we have

$$g(t_n^{-1}u_n) \gg t_n^{-1}gu_n \geq gv_n. \quad (16)$$

Since f is strictly $(-\alpha)$ -convex and g -antitone, it follows from (16) that

$$gu_{n+2} \gg fv_n \gg f(t_n^{-1}u_n) = f\left(\left(\frac{2t_n}{1+t_*}\right)^{-1}\left(\frac{1+t_*}{2}\right)^{-1}u_n\right) \gg \left(\frac{2t_n}{1+t_*}\right)\left(\frac{1+t_*}{2}\right)^\alpha fu_n \gg \left(\frac{2t_n}{1+t_*}\right)\left(\frac{1+t_*}{2}\right)^\alpha gv_{n+2}.$$

By definition of t_{n+2} , it follows that

$$t_{n+2} \gg \left(\frac{2t_n}{1+t_*}\right)\left(\frac{1+t_*}{2}\right)^\alpha,$$

which yields a contradiction when n tends to infinity. Consequently, we have $\lim_{n \rightarrow \infty} t_n = 1$. $\square$

Next, we show that our sequences converge, and we deduce that the set of coincidence points is nonempty.

Lemma 3.8. *Under the hypotheses of Lemma 3.6 (resp. Lemma 3.7). If $\{u_n\}$, $\{v_n\}$ be sequences satisfying (2) (resp. (4)), then the sequences $\{gu_n\}$ and $\{gv_n\}$ converge to gv_* , where $v_* \in C(f, g)$.*

Proof. For any positive integers n and k , it follows from (2) (resp. (4)) and Lemma 3.6 (resp. Lemma 3.7) that

$$\begin{aligned} 0_E &\leq gu_{n+k} - gu_n \leq gv_n - gu_n \leq (1 - t_n)gv_n \leq (1 - t_n)gv_0, \\ 0_E &\leq gv_n - gv_{n+k} \leq gv_n - gu_n \leq (1 - t_n)gv_n \leq (1 - t_n)gv_0. \end{aligned}$$

Thus, we deduce from the normality of P that

$$\|gu_{n+k} - gu_n\| \leq c_N(1 - t_n)\|gv_0\| \quad \text{and} \quad \|gv_{n+k} - gv_n\| \leq c_N(1 - t_n)\|gv_0\|,$$

where $c_N > 0$ is the constant of normality of P . Hence, by using Lemma 3.6 (resp. Lemma 3.7), we deduce that $\{gu_n\}$ and $\{gv_n\}$ are Cauchy sequences. Therefore, the sequences $\{gu_n\}$ and $\{gv_n\}$ converge to some $w_1, w_2 \in E$, respectively. As the sequences $\{gu_n\}$ and $\{gv_n\}$ are included in the closed order interval $[gu_0, gv_0]$, we deduce that their limits are in $[gu_0, gv_0] \subset P$. Therefore, there exist $u_*, v_* \in \overset{\circ}{P}$ such that $w_1 = gu_*$ and $w_2 = gv_*$. Now, since $gu_*, gv_* \in [gu_n, gv_n]$ and $gu_n \geq t_n gv_n$ for all $n \geq 0$, it follows that $0_E \leq gv_* - gu_* \leq gv_n - gu_n \leq (1 - t_n)gv_0$. So, by the normality of P , we get $\|gv_* - gu_*\| \leq c_N(1 - t_n)\|gv_0\|$, and then, $gv_* = gu_*$. Hence, for all $n \geq 0$ we have $gu_n \ll gv_* \ll gv_n$, then from the fact that f is strictly g -monotone (resp. f is strictly g -antitone), we obtain

$$gu_n \ll fu_n \ll fv_* \ll fv_n \ll gv_n \text{ (resp. } gu_n \ll fv_n \ll fv_* \ll fu_n \ll gv_n).$$

Consequently, as n tends to infinity, we obtain $gv_* \leq fv_* \leq gv_*$. Therefore, $gv_* = fv_*$ which proves that $v_* \in C(f, g)$. $\square$

The following lemma shows that the image by g of the set of coincidence points is reduced to a singleton.

Lemma 3.9. *Let $f, g : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be two operators such that g is surjective and strictly super-homogeneous, and f is strictly α -concave and strictly g -monotone (resp. strictly $(-\alpha)$ -convex and strictly g -antitone). Then, $g(C(f, g))$ is a singleton.*

Proof. From Lemma 3.5, there exist $x_0, y_0 \in \overset{\circ}{P}$ satisfying (5) (resp. (9)). Then by Lemma 3.3 (resp. Lemma 3.4), there exist two sequences $\{u_n\}$ and $\{v_n\}$ satisfying (1) and (2) (resp. (3) and (4)). Then using Lemma 3.8, we see that $C(f, g)$ is nonempty. Let $x_*, y_* \in C(f, g)$ and define $t_* := \sup\{0 < t \leq 1 : gy_* \in [tgx_*, t^{-1}gx_*]\}$. Observe first that we have $gy_* \in [t_*gx_*, t_*^{-1}gx_*]$ with $t_* \in (0, 1]$. Next, we claim that $t_* = 1$. To this purpose assume that $0 < t_* < 1$. Since g is strictly super-homogeneous, then we get

$$g(t_*x_*) \ll t_*gx_* \leq gy_* \leq t_*^{-1}gx_* \ll g(t_*^{-1}x_*).$$

So, if we assume that f is strictly α -concave and strictly g -monotone, we have

$$t_*^\alpha g x_* = t_*^\alpha f x_* \ll f(t_* x_*) \ll f y_* \ll f(t_*^{-1} x_*) \ll t_*^{-\alpha} f x_* = t_*^{-\alpha} g x_*.$$

However, if we assume that f is strictly $(-\alpha)$ -convex and strictly g -antitone, we obtain

$$t_*^\alpha g x_* = t_*^\alpha f x_* \ll f(t_*^{-1} x_*) \ll f y_* \ll f(t_* x_*) \ll t_*^{-\alpha} f x_* = t_*^{-\alpha} g x_*.$$

Consequently, from the definition of t_* , we obtain $t_*^\alpha \leq t_*$ where $\alpha \in (0, 1)$, which is a contradiction. Therefore, $t_* = 1$ and so $g x_* = g y_*$, which implies $g(C(f, g))$ is a singleton. $\square$

Using some commutativity conditions, we show next the existence of common fixed point of f and g .

Lemma 3.10. *Under the hypotheses of Lemma 3.9, f and g have a unique common fixed point z in $\overset{\circ}{P}$ if and only if $z = g(C(f, g)) \in C(f, g)$. In particular, if $C(f, g) \subseteq C(fg, gf)$, then f and g have a unique common fixed point.*

Proof. If f and g have a unique common fixed point z in $\overset{\circ}{P}$, then we have $f(z) = g(z) = z$ which obviously means that the unique point $z = g(C(f, g)) \in C(f, g)$. Conversely, if $g(C(f, g)) \in C(f, g)$, then $fz = gz = g(C(f, g)) = z$. If z_1, z_2 are two common fixed points, then $z_1 = gz_1, z_2 = gz_2 \in g(C(f, g))$, which is a singleton. In particular, if $C(f, g) \subseteq C(fg, gf)$, for $z_* \in C(f, g)$, we have $gz_* = fz_*$. It follows from $C(f, g) \subseteq C(fg, gf)$ that $g^2 z_* = gfz_* = fgz_*$. Thus, $gz_* \in C(f, g)$. $\square$

Proof of Theorem 3.1. (resp. Theorem 3.2). At first, observe that $C(f, g)$ is nonempty follows from Lemma 3.8 and $g(C(f, g))$ is a singleton follows from Lemma 3.9. More precisely, we deduce, by lemmas and $\{v_n\}$ in $\overset{\circ}{P}$ such that (1) and (2) (resp. (3) and (4)) and 3.5, that there exist two sequences $\{u_n\}$ and $\{v_n\}$ satisfying (1) and (2) (resp. (3) and (4)). Moreover, by lemmas 3.8 and 3.9, the sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ converge to $g(C(f, g))$. Finally, the uniqueness result of the common fixed point follows from Lemma 3.10. $\square$

4 Coincidence point theorems for inverse monotone operators

Consider $\mathcal{H}$ the finite dimensional real Hilbert space. Assume that E is the Banach space of self-adjoint linear operators on $\mathcal{H}$, and P its positive cone of semi-definite operators (i.e., $P := \{A \in E : \langle Av, v \rangle \geq 0 \text{ for all } v \in \mathcal{H}\}$). An operator $A \in P$ is said to be positive definite, if $\langle Av, v \rangle > 0$ for all $v \in \mathcal{H} \setminus \{0\}$, and the set of all positive definite operators is denoted by P^+ . It is well known that $\overset{\circ}{P} = P^+$ (see, for instance, [24]). Generally, the monotonic dependence of the operator f with g may be difficult to perform. An important subclass of such operators has been investigated in the literature, when g is an inverse monotone operator and f is a monotone or an antitone operator. Therefore, in this section, we give a particular attention to study such operators and establish our first results for this subclass. First, we recall the theorem of Furuta [18], and we show some preliminary results needed in the sequel.

Theorem 4.1. [18, Theorem 3.1] *Let X and Y be strictly positive operators on $\mathcal{H}$. If $X \ll Y$, then $f(X) \ll f(Y)$ for any nonconstant operator monotone function f on $(0, \infty)$.*

In order to extend Furuta's theorem we need the following lemmas.

Lemma 4.2. *If $X \ll Y$, then $A^* X A \ll A^* Y A$ for every invertible operator A .*

Proof. The result follows from the fact that for all nonzero vector u , if A is invertible, then

$$Y \gg X \Leftrightarrow \langle u, Yu \rangle > \langle u, Xu \rangle \Leftrightarrow \langle Au, YAu \rangle > \langle Au, XAu \rangle.$$

Therefore,

$$\langle u, A^*YAu \rangle = \langle Au, YAu \rangle > \langle Au, XAu \rangle = \langle u, A^*XAU \rangle.$$

□

Lemma 4.3. *Let X and Y be strictly positive operators on $\mathcal{H}$. If $X \ll Y$, then $Y^{-1} \ll X^{-1}$.*

Proof. At first, note that if X is strictly positive operator, then $\langle Xv, v \rangle > 0$ for all $v \neq 0$, which means that X is invertible. We show next that $T \mapsto T^{-1}$ is a strictly antitone on commuting positive operators. Let $0 \ll X \ll Y$ such that X and Y commute, then $X^{1/2}, Y^{1/2}$, X and Y also commute. Since, for all nonzero vector v , we have

$$\langle (Y - X)v, v \rangle > 0.$$

Then, for all nonzero vector v and $w = Y^{-1/2}X^{-1/2}v$, we have

$$\begin{aligned} \langle (X^{-1} - Y^{-1})v, v \rangle &= \langle (X^{-1} - Y^{-1})X^{1/2}Y^{1/2}w, X^{1/2}Y^{1/2}w \rangle \\ &= \langle Y^{-1}(Y - X)X^{-1}X^{1/2}Y^{1/2}w, X^{1/2}Y^{1/2}w \rangle \\ &= \langle (Y - X)w, w \rangle > 0. \end{aligned}$$

Now, for non-necessarily commuting operators X and Y which satisfy $0 \ll X \ll Y$, we have, by Lemma 4.2, $Y^{-1/2}XY^{-1/2} \ll I$. As $Y^{-1/2}XY^{-1/2}$ and I commute, then $I \ll Y^{1/2}X^{-1}Y^{1/2}$. Using again Lemma 4.2, we obtain $Y^{-1} \ll X^{-1}$. □

Corollary 4.4. *Let X and Y be strictly positive operators on $\mathcal{H}$. If $X \ll Y$, then $g(Y) \ll g(X)$ for any nonconstant operator antitone function g on $(0, \infty)$.*

Proof. The function $f = 1/g$ is operator monotone on $\overset{\circ}{P}$. Then, the result is a direct consequence of Theorem 4.1 and Lemma 4.3. □

Theorem 4.5. *Let $F, G : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ be nonconstant operators. Assume that*

- (i) *F is strictly α -concave (resp. strictly $(-\alpha)$ -convex) and strictly monotone (resp. strictly antitone) operator.*
- (ii) *G is surjective, strictly super-homogeneous and strictly inverse monotone operator.*

Then, the equation

$$G(X) = F(X), \quad X \in \overset{\circ}{P}. \quad (17)$$

has a solution.

Proof. Since from (i) and (ii), F is a G -strictly monotone (resp. antitone) operator, then all hypotheses of Theorem 3.1 are fulfilled. □

Lemma 4.6. *Let X, Y be two commuting strictly positive operators such that $X \ll Y$, and $g : (0, \infty) \rightarrow (0, \infty)$ be a strictly monotone function. Then, $g(X) \ll g(Y)$.*

Proof. The commutativity hypothesis implies that there exists a unitary matrix Q such that $X = Q^*D_XQ$ and $Y = Q^*D_YQ$, where $D_X = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $D_Y = \text{diag}(\mu_1, \dots, \mu_n)$ are diagonal matrices. If $X \ll Y$, then using Lemma 4.2, we deduce that $\lambda_i < \mu_i$ for all i , and as g is monotone it follows that $g(\lambda_i) < g(\mu_i)$ for all i . Therefore, $g(D_X) \ll g(D_Y)$, which implies that $g(X) \ll g(Y)$. □

Lemma 4.7. *Let X, Y be two commuting operators and f a real function. Then, the operators $f(X)$ and $f(Y)$ commute.*

Proof. The proof is straightforward, hence omitted. □

Proposition 4.8. *If $g, g' : (0, \infty) \rightarrow (0, \infty)$ are two surjective strictly super-homogeneous inverse monotone operators, then so is $g \circ g'$.*

Proof. Since g and g' are surjective and inverse monotone, then we deduce that $g \circ g'$ is surjective and inverse monotone operator. It remains to prove that $g \circ g'$ is strictly super-homogeneous. By hypothesis, g' is strictly super-homogeneous and by Lemma 4.7, $g'(tX)$ and $g'(X)$ commute. Then using Lemma 4.6, we deduce that $(g \circ g')(tX) = g(g'(tX)) \ll g(tg'(X)) \ll t(g \circ g')(X)$, for all $t \in (0, 1)$. $\square$

Proposition 4.9. *Let $F, F' : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ are two strictly monotone operators. Assume that F is strictly α -concave (resp. $(-\alpha)$ -convex) and F' is strictly β -concave (resp. $(-\beta)$ -convex), then for all $a, b > 0$, we have:*

- (i) $F \circ F'$ and $aF + bF'$ are strictly monotone operators.
- (ii) $aF + bF'$ is strictly $\max\{\alpha, \beta\}$ -concave (resp. $(-\max\{\alpha, \beta\})$ -convex).
- (iii) $F \circ F'$ is strictly $\alpha\beta$ -concave (resp. $(-\alpha\beta)$ -convex).

From Propositions 4.8 and 4.9, as a consequence of Theorem 4.5, we deduce immediately the following result.

Theorem 4.10. *Let $\{g_i\}_{i=1}^m$ be a family of nonconstant surjective strictly super-homogeneous inverse monotone operator functions on $(0, \infty)$. Let $\{F_j\}_{j=1}^m$ be a family of strictly α -concave (resp. strictly $(-\alpha)$ -convex) monotone (resp. antitone) operator on $\overset{\circ}{P}$, and J a family of nonempty subset of $[1, m]$. Then, the nonlinear equation,*

$$g_1 \circ \dots \circ g_k(X) = Q + \sum_{\substack{I \in J \\ \sigma \in S_I}} A_{I,\sigma}^* F_{I,\sigma}(X) A_{I,\sigma},$$

has a positive solution, where Q is a semi-definite positive matrix, S_I is the permutation group of the set $I = \{i_1, \dots, i_k\}$, $A_{I,\sigma}$ are invertible matrices and $F_{I,\sigma} = F_{\sigma(i_1)} \circ \dots \circ F_{\sigma(i_k)}$.

5 Applications

In this section, we show at first that some existing monotone operators in the literature are already strictly super-homogeneous and surjective. Next, we derive a corollary from a previous result and solve nonlinear equations involving Uchiyama's operator functions [16,17]. We construct two algorithms for monotone and antitone operators, and we apply them to solve several nonlinear matrix equations. We also investigate the unidimensional case and we show that our approach may be used to compute polynomial roots satisfying the Descartes condition. We present some numerical experiments confirming the efficiency of the constructed algorithm.

Theorem 5.1. (Löwner-Heinz inequality). *The operator function t^α is monotone for all $\alpha \in [0, 1]$.*

Theorem 5.2. (Uchiyama [16,17]) *Let $g_1(x)$ and $g_2(x)$ be functions defined on $[0, \infty)$ by*

$$g_1(x) = \prod_{i=1}^k (x + a_i)^{\gamma_i} \quad \text{and} \quad g_2(x) = x^\alpha \exp(x), \quad (18)$$

where $\gamma_i > 0$, $\gamma_1 \geq 1$, $0 = a_1 < a_2 < \dots < a_k$ and $\alpha \geq 1$. Then, the inverse of the operator functions g_1 and g_2 are operator monotone functions on $[0, \infty)$.

Lemma 5.3. *If $g : (0, \infty) \rightarrow (0, \infty)$ is a surjective function, then the operator $g : P^+ \rightarrow P^+$ is surjective. In particular, g_1 and g_2 defined in (18) are surjective.*

Proof. Let $g : (0, \infty) \rightarrow (0, \infty)$ be a surjective function. It is clear that $g : P^+ \rightarrow P^+$ is well defined. Let $A \in P^+$ and let Q be a unitary matrix such that $A = Q^*DQ$ with $D = \text{diag}(\lambda_1, \dots, \lambda_n)$. By the surjectivity of the function f , there exist $\mu_1, \dots, \mu_n \in (0, \infty)$ such that $\lambda_i = g(\mu_i)$. For $B = Q^*D'Q$, where $D' = \text{diag}(\mu_1, \dots, \mu_n)$, we have $g(B) = Q^*f(D')Q = Q^*DQ = A$. $\square$

Lemma 5.4. Let X_1, X_2, X_3 and X_4 be four commuting strictly positive operators such that $X_1 \leq X_2$ and $X_3 \ll X_4$. Then, $X_1^r X_3^s \ll X_2^r X_4^s$ for all $r, s > 0$.

Proof. By the commutativity hypothesis, there exists a unitary matrix Q such that $Q^*X_iQ = D_i$, where D_i is a diagonal matrix for $i = 1, \dots, 4$. Since $X_1 \leq X_2$ and $X_3 \ll X_4$, then by Lemma 4.2, we obtain $D_1 \leq D_2$ and $D_3 \ll D_4$, which implies that $D_1^r D_3^s \ll D_2^r D_4^s$. Using again Lemma 4.2, we infer the result. $\square$

Lemma 5.5. The operator functions g_1 and g_2 defined in (18) are strictly super-homogeneous.

Proof. Note that, for all reals t and a , we have $tX + a \ll X + a$ and $t^\gamma X \ll tX$, whenever $t \in (0, 1)$ and $\gamma \geq 1$. Now, as the operators $X + a$ and $tX + a$ commute, then using Lemma 5.4, we get

$$g_1(tX) \ll tX \prod_{i=2}^k (X + a_i)^{y_i} = t g_1(X).$$

Next, observe that $\exp(tX) \ll \exp(X)$ and $t^\gamma X^\gamma \leq tX^\gamma$ for all $t \in (0, 1)$ and $\gamma \geq 1$. We apply Lemma 5.4 again and obtain the super-homogeneity of g_2 . $\square$

Next, we give an example of strictly super-homogeneous inverse monotone operator of Riccati type.

Proposition 5.6. Let $A \in \mathring{P}$. The operator $g_3 : \mathring{P} \rightarrow \mathring{P}$, $X \mapsto XAX$ is strictly super-homogeneous and bijective where its inverse is the monotone operator $X \mapsto A^{-1/2}(A^{1/2}XA^{1/2})^{1/2}A^{-1/2}$.

Proof. The super-homogeneity of the operator $X \mapsto XAX$ and the monotony of the operator $X \mapsto A^{-1/2}(A^{1/2}XA^{1/2})^{1/2}A^{-1/2}$ are straightforward. For the remainder of the proof see [25, Exercise 1.2.13]. $\square$

Corollary 5.7. Let g_1 and g_2 be the functions defined in Theorem 5.2. Let $f_k, \hat{f}_k : \mathring{P} \rightarrow \mathring{P}$ be the following operators

$$f_1(X) = \sum_{i=1}^{r_1} (I + A_i^* X A_i)^{a_i}, f_2(X) = \sum_{j=1}^{r_2} (I + B_j^* X^{-1} B_j)^{-b_j}, f_3(X) = \left(\sum_{p=1}^{r_3} \left(\sum_{q=1}^{r_4} C_{p,q}^* X^{c_{p,q}} C_{p,q} \right) \right)^{-1},$$

$$\hat{f}_1(X) = \sum_{i=1}^{r_1} (I + A_i^* X A_i)^{-a_i}, \hat{f}_2(X) = \sum_{j=1}^{r_2} (I + B_j^* X^{-1} B_j)^{b_j}, \hat{f}_3(X) = \sum_{p=1}^{r_3} \left(\sum_{q=1}^{r_4} C_{p,q}^* X^{c_{p,q}} C_{p,q} \right)^{-1},$$

where r_1, r_2, r_3 are positive integers, I is the identity matrix, A_i, B_i and $C_{p,q}$ are arbitrary invertible matrices. Let $Q \in \mathring{P}$ and M_k be arbitrary square matrices for $k = 1, 2, 3$. If $a_i, b_j, c_{p,q} \in (0, 1)$ for all $i \in [1, r_1]$, $j \in [1, r_2]$, $p \in [1, r_3]$ and $q \in [1, r_4]$, then each of the following nonlinear matrix equations

$$g_1^{n_1} \circ g_2^{m_1} \circ \dots \circ g_1^{n_k} \circ g_2^{m_k}(X) = Q + M_1^* f_1(X) M_1 + M_2^* f_2(X) M_2 + M_3^* f_3(X) M_3, \quad (19)$$

$$g_1^{n_1} \circ g_2^{m_1} \circ \dots \circ g_1^{n_k} \circ g_2^{m_k}(X) = Q + M_1^* \hat{f}_1(X) M_1 + M_2^* \hat{f}_2(X) M_2 + M_3^* \hat{f}_3(X) M_3, \quad (20)$$

has a positive solution, where k, n_i and m_i are arbitrary integers for all $i \in [1, k]$.

Proof. Observed first that g_1 and g_2 are nonconstant, surjective, strictly super-homogeneous and inverse monotone operator functions on $(0, \infty)$. Next, using the Löwner-Hans theorem and Lemmas 4.3 and 4.2,

we show easily that f_k , (resp. $\widehat{f_k}$) are strictly α_k -concave (resp. $(-\alpha_k)$ -convex), where $\alpha_1 = \alpha_2 = \max_i \{r_i\}$ and $\alpha_3 = \max_{i,j} \{s_{i,j}\}$. Consequently, the result follows from Theorem 4.10. $\square$

Example 5.8. System (19) includes the following nonlinear matrix equations as particular cases:

- (1) $X = Q + \sum_{i=1}^n A_i^* X^{q_i} A_i$, $0 < q < 1$ (see Duan et al. [10], Li et al. [15] and Ameer et al. [26]).
- (2) $X - A^* X^{-q} A = Q$, $q \in (0, 1)$ (see Hasanov [19]).
- (3) $X^s - A^* X^{-q} A = Q$, $s \geq q$ (see Meng and Kim [20]).

Next, we give two algorithms for solving nonlinear equations.

Algorithm 1. Case of the matrix Eq. (19).

- 1: Input: the operators f and g , the tolerance tol and the number of iteration iMax .
 - 2: Find t_0 and u_0 satisfying (5)
 - 3: Solve $gu = (1 - t_0)fu_0 + t_0gu_0$ and compute the error $= \|fu - gu\|$
 - 4: Set $u_1 = u$
 - 5: **while** error $> \text{tol}$ **do** ▷ Newton-Raphson method
 - 6: Solve $gu = (1 - t_0)fu_1 + t_0fu_0$ and compute the error $= \|fu - gu\|$
 - 7: Exit if iMax is reached or the condition (2) is not satisfied
 - 8: Set $fu_0 = fu_1$ and $fu_1 = fu$
 - 9: **end while**
 - 10: Output: u the positive definite solution of Eq. (19).
-

Algorithm 2. Case of the matrix Eq. (20).

- 1: Input: the operators f and g , the tolerance tol and the number of iterations iMax .
 - 2: Find t_0 , u_0 and v_0 satisfying (9)
 - 3: Solve $gu = (1 - t_0)fv_0 + t_0gu_0$ and $gv = (1 - t_0)fu_0 + t_0gv_0$
 - 4: Compute the error $= \max \{\|fu - gu\|, \|fv - gv\|\}$
 - 5: Set $u_1 = u$ and $v_1 = v$
 - 6: **while** error $> \text{tol}$ **do** ▷ Newton-Raphson method
 - 7: Solve $gu = (1 - t_0)fv_1 + t_0fv_0$ and $gv = (1 - t_0)fu_1 + t_0fu_0$
 - 8: Compute the error $= \max \{\|fu - gu\|, \|fv - gv\|\}$
 - 9: Exit if iMax is reached or condition (4) is not satisfied
 - 10: Set $fu_0 = fu_1$, $fv_0 = fv_1$, $fu_1 = fu$ and $fv_1 = fv$
 - 11: **end while**
 - 12: Output: u the positive definite solution of Eq. (20).
-

Remark 5.9. Note that equations $gu = (1 - t_0)fu_0 + t_0gu_0$ and $gu = (1 - t_0)fu_1 + t_0fu_0$ of Algorithm 5 and equations $gu = (1 - t_0)fv_0 + t_0gu_0$, $gv = (1 - t_0)fu_0 + t_0gv_0$, $gu = (1 - t_0)fv_1 + t_0fv_0$ and $gv = (1 - t_0)fu_1 + t_0fu_0$ of Algorithm 5 are solved by the Newton-Raphson method with initial guess close to zero.

The following numerical experiments are carried out for the initial guesses $X_0 = Y_0 = 10^{-3}I$ and the constant matrices

$$U = \begin{pmatrix} 2 & -2 & -1 \\ -3 & -2 & -3 \\ -1 & -2 & 2 \end{pmatrix}, \quad V = \begin{pmatrix} 2 & -4 & -8 \\ -4 & 4 & -4 \\ -7 & -3 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 7 \end{pmatrix}. \quad (21)$$

Example 5.10. Consider the following nonlinear matrix equation

$$X^2 + X = Q + U^* X^{1/2} U + V^* X^{1/3} V, \quad (22)$$

resp.

$$X^2 + X = Q + U^*X^{-1/2}U + V^*X^{-1/3}V. \quad (23)$$

By Corollary 5.7, Eq. (22), resp. (23), has a positive definite solution, since (22) follows from Eq. (19), resp. (20), by taking $k = 1$, $n_1 = 1$, $m_1 = 0$, $g_1(x) = x(x + 1)$, $M_1 = M_2 = 0$, $M_3 = I$, $r_3 = 1$, $r_4 = 2$, $c_{1,1} = \frac{1}{2}$, $c_{1,2} = \frac{1}{3}$, resp. $c_{1,1} = -\frac{1}{2}$, $c_{1,2} = -\frac{1}{3}$, $C_{1,1} = U$ and $C_{1,2} = V$. Using Algorithm 5, resp. Algorithm 5, we obtain the following approximate solution of Eq. (22), resp. Eq. (23), after 22 iterations

$$X_{22} = \begin{pmatrix} 13.992350147612 & 0.451964873128 & -0.122298155191 \\ 0.451964873128 & 10.875503975397 & 1.884559066264 \\ -0.122298155191 & 1.884559066264 & 14.597646330214 \end{pmatrix},$$

resp.

$$X_{22} = Y_{22} = \begin{pmatrix} 5.87748108914077 & 0.10066316313069 & -0.02098827705064 \\ 0.10066316313069 & 4.91430570936858 & 0.82493481690477 \\ -0.02098827705064 & 0.82493481690477 & 6.69047330454621 \end{pmatrix},$$

with the error estimation $\|X_{22}^2 + X_{22} - Q - U^*X_{22}^{1/2}U - V^*X_{22}^{1/3}V\|_2 = 2.0 \times 10^{-13}$, resp. $\|X_{22}^2 + X_{22} - Q - M_1^*X_{22}^{-1/2}M_1 - M_2^*X_{22}^{-1/3}M_2\|_2 = 4.4 \times 10^{-14}$.

Example 5.11. The nonlinear matrix equation

$$\prod_{j=0}^6 (X + jI) = Q + U^*(I + X)^{1/2}U + V^*(I + X^{-1})^{-1/3}V, \quad (24)$$

resp.

$$\prod_{j=0}^6 (X + jI) = Q + U^*(I + X)^{-1/2}U + V^*(I + X^{-1})^{1/3}V. \quad (25)$$

Eq. (24), resp. Eq. (25), follows from Eq. (19), resp. Eq. (20), by taking $k = 1$, $n_1 = 1$, $m_1 = 0$, $g_1(x) = \prod_{j=0}^6 (x + j)$, $M_1 = M_2 = I$, $M_3 = 0$, $r_1 = r_2 = 1$, $a_1 = \frac{1}{2}$, $b_1 = \frac{1}{3}$, $A_1 = U$ and $B_1 = V$. Using Algorithm 5, resp. Algorithm 5, we obtain the following approximate solution of Eq. (24), resp. Eq. (25), after 19 iterations, resp. 22 iterations

$$X_{19} = \begin{pmatrix} 0.05492899089403 & 0.00637836705832 & 0.00438239263599 \\ 0.00637836705832 & 0.03981342625499 & 0.01419254290757 \\ 0.00438239263599 & 0.01419254290757 & 0.06103245551386 \end{pmatrix},$$

resp.

$$X_{22} = Y_{22} = \begin{pmatrix} 0.14554684990904 & -0.00218735236784 & -0.00398188778218 \\ -0.00218735236784 & 0.10940957866430 & 0.02158828596070 \\ -0.00398188778218 & 0.02158828596070 & 0.17148995243330 \end{pmatrix},$$

with the error estimation $\|\prod_{k=0}^6 (X_{19} + kI) - Q - U^*(I + X_{19})^{1/2} - V^*(I + X_{19}^{-1})^{-1/3}V\|_2 = 5.8 \times 10^{-12}$, resp. $\|\prod_{k=0}^6 (X_{22} + kI) - Q - U^*(I + X_{22})^{-1/2}U - V^*(I + X_{22})^{1/3}V\|_2 = 1.2 \times 10^{-13}$.

Example 5.12. Consider the nonlinear matrix equation

$$X^{3/2} \exp(X) = Q + U^*X^{1/2}U + U^*(I + X)^{1/3}U, \quad (26)$$

resp.

$$X^{3/2} \exp(X) = Q + U^*X^{-1/2}U + U^*(I + X)^{-1/3}U. \quad (27)$$

Eq. (26), resp. Eq. (27), follows from Eq. (19), resp. Eq. (20), by taking $k = 1$, $n_1 = 0$, $m_1 = 1$, $g_2(x) = x^{3/2}\exp(x)$, $M_1 = M_3 = I$, $M_2 = 0$, $r_1 = r_3 = r_4 = 1$, $a_1 = \frac{1}{3}$, $c_{1,1} = \frac{1}{2}$ and $A_1 = C_{1,1} = U$. Using Algorithm 5, resp. Algorithm 5, we obtain the following approximate solution Eq. (26), resp. Eq. (27), after 15 iterations

$$X_{15} = \begin{pmatrix} 2.41259163868507 & 0.18595925781130 & 0.19682687712695 \\ 0.18595925781130 & 2.39839598599567 & 0.19404230898730 \\ 0.19682687712695 & 0.19404230898730 & 2.45169488926669 \end{pmatrix},$$

resp.

$$X_{15} = Y_{15} = \begin{pmatrix} 2.03394095971354 & 0.16618750281669 & 0.17364353452140 \\ 0.16618750281669 & 1.96913461734563 & 0.18897565538307 \\ 0.17364353452140 & 0.18897565538307 & 2.10078599153675 \end{pmatrix},$$

with the error estimation $\|X_{15}^{3/2} \exp(X_{15}) - Q - U^*X_{15}^{1/2}U - U^*(I + X_{15})^{1/3}U\|_2 = 7.2 \times 10^{-13}$, resp. $\|X_{15}^{3/2} \exp(X_{15}) - Q - U^*X_{15}^{-1/2}U - U^*(I + X_{15})^{-1/3}U\|_2 = 7.2 \times 10^{-14}$.

Example 5.13. Consider the nonlinear matrix equation

$$(XAX)^3 + 3(XAX)^2 + 2(XAX) = I + B^*(I + X)^{1/2}B + C^*(I + X^{-1})^{-1/3}C + D^*X^{1/4}D, \quad (28)$$

resp.

$$(XAX)^3 + 3(XAX)^2 + 2(XAX) = I + B^*(I + X)^{-1/2}B + C^*(I + X^{-1})^{1/3}C + D^*X^{-1/4}D, \quad (29)$$

where

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ -5 & 3 & 0 \\ -3 & -10 & -10 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -3 & 1 & 1 \\ -3 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Eq. (28), resp. (29), is equivalent to the operator equation $g(X) = f(X)$, with $g = g_1 \circ g_3$, where $g_1(x) = x(x+1)(x+2)$ an operator function of Uchiyama and g_3 the operator given in Proposition 5.6. The operator f is given by

$$f(X) = I + B^*(I + X)^{1/2}B + C^*(I + X^{-1})^{-1/3}C + D^*X^{1/4}D,$$

resp.

$$f(X) = I + B^*(I + X)^{-1/2}B + C^*(I + X^{-1})^{1/3}C + D^*X^{-1/4}D.$$

It is not difficult to show that g is surjective, strictly super-homogeneous and inverse monotone, and that f is strictly $\frac{1}{4}$ -concave, resp. $\left(-\frac{1}{4}\right)$ -convex and monotone, resp. antitone. Hence, the existence of the positive solution follows from Theorem 4.5. Using Algorithm 5, resp. Algorithm 5, we obtain the following approximate solution of Eq. (28), resp. Eq. (29), after 12 iterations

$$X_{12} = \begin{pmatrix} 1.09659903724799 & -0.33165797383082 & 0.02892330222728 \\ -0.33165797383082 & 0.84635677329539 & 0.09427312566765 \\ 0.02892330222728 & 0.09427312566765 & 0.59229346812813 \end{pmatrix},$$

resp.

$$X_{12} = \begin{pmatrix} 1.14822024828508 & -0.35572457927850 & 0.05666726902198 \\ -0.35572457927850 & 0.89667499030598 & 0.15669691360091 \\ 0.05666726902198 & 0.15669691360091 & 0.61401608713953 \end{pmatrix},$$

with the error estimation $\|g(X_{12}) - f(X_{12})\|_2 = 1.4 \times 10^{-12}$, resp. $\|g(X_{12}) - f(X_{12})\|_2 = 8.6 \times 10^{-13}$.

Finally, we show that Algorithm 5 may be applied to compute a positive root of polynomials of the form

$$P(x) = a_n x^n + \cdots + a_{m+1} x^{m+1} - a_m x^m - \cdots - a_1 x - a_0, \quad (30)$$

where $a_i \geq 0$ for $i = 0, \dots, n$. It is well known that according to Descartes' rule of signs, Eq. (30) has only one positive solution, provided at least two coefficients a_i and a_j are nonzeros, where $0 \leq j \leq m < i \leq n$ and m, n are two given integers. Another proof of this result can be obtained by taking $P = P_g - P_f$, where

$$P_g(x) = a_n x^n + \cdots + a_{m+1} x^{m+1} \quad \text{and} \quad P_f(x) = a_m x^m + \cdots + a_1 x + a_0.$$

Proposition 5.14. *Polynomial (30) has a positive root.*

Proof. Since we look for the positive solution of $P(x) = 0$, we pose $y = x^{m+1}$ for $x \in (0, +\infty)$ and substitute $y^{1/(m+1)}$ in place of x in Eq. (30). So, we can define $f, g : (0, +\infty) \rightarrow (0, +\infty)$ as follows

$$f(y) = P_f(y^{1/(m+1)}) \quad \text{and} \quad g(y) = P_g(y^{1/(m+1)}).$$

The function f is obviously strictly $\frac{m}{m+1}$ -concave. On the other hand, the function g is strictly increasing and bijective on $(0, +\infty)$, which implies that the inverse of g is strictly increasing. Finally, it is not difficult to see that g is strictly super-homogeneous. Consequently, the result follows from Theorem 4.5. $\square$

Example 5.15. We apply Algorithm 5 to compute the positive root of the polynomial

$$P(x) = 4x^5 + 3x^4 + 7x^3 - 2x^2 - 2x - 6,$$

we obtain the following results with respect to the iteration number n :

n	x_n	$P(x_n)$
1	0.8767017517334639	7.298×10^{-1}
5	0.8995646487790856	1.217×10^{-3}
9	0.8996012127218075	2.000×10^{-6}
13	0.8996012728107530	3.288×10^{-9}
17	0.8996012729095380	5.405×10^{-12}
21	0.8996012729097004	7.105×10^{-15}

We conclude this work with the following questions that arise naturally:

Question 1: It is interesting to determine under which hypotheses the monotony results in Section 4 remain true in infinite dimensional Hilbert space.

Question 2: A huge amount of literature uses coincidence Picard iteration technique $gx_{n+1} = fx_n$ to study the coincidence problem of f and g . The convergence of such sequences should be less sensitive to the computational approximations, unlike our method, which take into consideration the approximation constraints. Is it possible to apply or to extend our method to these coincidence problems.

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