Research Article

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Bipartite graphs with close domination and k -domination numbers

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Abstract: Let k be a positive integer and let G be a graph with vertex set $V(G)$. A subset $D \subseteq V(G)$ is a k -dominating set if every vertex outside D is adjacent to at least k vertices in D . The k -domination number $\gamma_k(G)$ is the minimum cardinality of a k -dominating set in G . For any graph G , we know that $\gamma_k(G) \geq \gamma(G) + k - 2$ where $\Delta(G) \geq k \geq 2$ and this bound is sharp for every $k \geq 2$. In this paper, we characterize bipartite graphs satisfying the equality for $k \geq 3$ and present a necessary and sufficient condition for a bipartite graph to satisfy the equality hereditarily when $k = 3$. We also prove that the problem of deciding whether a graph satisfies the given equality is NP-hard in general.

Keywords: domination number, k -domination number, hereditary property, vertex-edge cover, TC-number, computational complexity

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1 Introduction

Let G be an undirected simple graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G , respectively. For two vertices $u, v \in V(G)$, u and v are neighbors if they are adjacent, that is, if there is an edge $e = uv \in E(G)$. The open neighborhood of a vertex v is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the degree of v is given by the cardinality of $N_G(v)$. Two vertices $u, v \in V(G)$ are false twins if $N_G(u) = N_G(v)$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and the maximum degree of G , respectively. For a subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by S . An edge $e = uv \in E(G)$ is subdivided when it is deleted from G and a new vertex x is added with two new edges xu and xv . Here, the vertex x is a subdivision vertex. Let $[r]$ denote the set of integers $\{1, \dots, r\}$ throughout the paper.

A subset $D \subseteq V(G)$ is dominating in G if every vertex of $V(G) \setminus D$ has at least one neighbor in D . Similarly, a subset $D \subseteq V(G)$ is k -dominating in G if every vertex of $V(G) \setminus D$ has at least k neighbors in D . The domination number $\gamma(G)$ and the k -domination number $\gamma_k(G)$ of G are the minimum cardinalities of a dominating and a k -dominating set of G , respectively.

We say that a connected graph G is a (γ, γ_k) -graph if $\gamma_k(G) = \gamma(G) + k - 2$ and $\Delta(G) \geq k$. A connected graph G is (γ, γ_k) -perfect if $\delta(G) \geq k$ and every connected induced subgraph H of G with $\delta(H) \geq k$ satisfies the equality $\gamma_k(H) = \gamma(H) + k - 2$.

Hypergraphs are set systems that are conceived as a natural generalization of graphs. A hypergraph $H = (V, E)$ contains a finite set V of vertices together with a collection E of nonempty subsets of V , called hyperedges or simply edges. The number of vertices, that is $|V|$, is called the order of H . Throughout this paper, we suppose that $|e| \geq 2$ holds for every $e \in E$. The degree of a vertex v in H , denoted by $d_H(v)$, is the

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number of edges containing the vertex v . The hypergraph H is k -uniform if every edge contains exactly k vertices. Thus, every (simple) graph is a 2-uniform hypergraph. A hypergraph $H' = (V', E')$ is an induced subhypergraph of $H = (V, E)$ if $V' \subseteq V$ and E' contains all edges $e \in E$ satisfying $e \subseteq V'$. We also use the notation $H[V']$ for the subhypergraph induced by V' . Given a collection F of k -uniform hypergraphs, we say that a k -uniform hypergraph H is F -free if no induced subhypergraph of H is isomorphic to any hypergraph contained in F . The complete k -uniform hypergraph of order n is the hypergraph $\mathcal{K}_n^k = (V, E)$, where $|V| = n \geq k$ and E contains all k -element subsets of V .

A set $T \subseteq V$ is a transversal (or vertex cover) in the hypergraph $H = (V, E)$ if $|e \cap T| \geq 1$ holds for every edge $e \in E$. The complement $V \setminus T$ of a transversal T is called a (weakly) independent vertex set. The minimum cardinality of a transversal and the maximum cardinality of a weakly independent vertex set in H are denoted by $\tau(H)$ and $\alpha_w(H)$, respectively. The minimum number of edges that together cover every vertex of H is the edge cover number of H and denoted by $\rho(H)$. For instance, in a complete k -uniform hypergraph $\mathcal{K}_n^k$, any $k - 1$ vertices form a maximum weakly independent vertex set and any $n - k + 1$ vertices form a minimum transversal. In particular, we have $\tau(\mathcal{K}_n^k) = n - k + 1$, $\alpha_w(\mathcal{K}_n^k) = k - 1$, and $\rho(\mathcal{K}_n^k) = \lceil \frac{n}{k} \rceil$.

Two vertices u and v of H are adjacent if there is an edge e of H such that $\{u, v\} \subseteq e$. A hypergraph is connected if for every two vertices x and y there exists a sequence $x = x_0, x_1, \dots, x_j = y$ such that x_{i-1} and x_i are adjacent vertices for every $i \in [j]$. The 2-section graph of $H = (V, E)$ is the graph G on the same vertex set V such that two vertices form an edge in G if and only if they are adjacent in H . The incidence graph of the hypergraph $H = (V, E)$ is the bipartite graph $G' = (A \cup B, E')$ where the vertices in A and B represent the vertices and edges of H , respectively. Moreover, two vertices, $a \in A$ and $b \in B$, are adjacent in G' if the vertex of H that is represented by a is contained in the hyperedge represented by b . Note that H is a connected hypergraph, if and only if, its 2-section graph is connected and, if and only if, its incidence graph is connected.

1.1 Structure of the paper

In Section 2, we first cite some previous results and prove some general lemmas. Then, for each $k \geq 2$, three graph classes $\mathcal{B}_k^*$, $\mathcal{B}_k$, and $\mathcal{G}_k$ are defined such that every connected (γ, γ_k) -graph belongs to $\mathcal{G}_k$. Moreover, in Section 3, we concentrate on the case of $k \geq 3$ and show that every connected bipartite (γ, γ_k) -graph is contained in $\mathcal{B}_k$. We also prove that a connected bipartite graph $G \in \mathcal{B}_k$ is a (γ, γ_k) -graph if and only if its γ_k -simplified graph $G^* \in \mathcal{B}_k^*$ is a (γ, γ_k) -graph. Then, we present a characterization of bipartite (γ, γ_k) -graphs in terms of the properties of the underlying hypergraph. In Section 4, we work on the hereditary version of the problem and give a characterization for all bipartite (γ, γ_3) -perfect graphs. Later, in Section 5, we prove that it can be decided in polynomial time whether a given bipartite graph is a (γ, γ_k) -graph, while the corresponding decision problem is NP-hard on $\mathcal{G}_k$.

2 Preliminary results on (γ, γ_k) -graphs

Fink and Jacobson [1,2] introduced k -domination in graphs as a generalization of the concept of domination. Motivated by this definition, related problems have been studied extensively by many researchers (see, for example, [3–10]). For more details, we refer the reader to the books on domination by Haynes, Hedetniemi, and Slater [11,12] and to the survey on k -domination and k -independence by Chellali et al. [13].

Fink and Jacobson [1] proved the following result on the relation between the domination number and the k -domination number of G .

Theorem 2.1. [1] For any graph G with $\Delta(G) \geq k \geq 2$, $\gamma_k(G) \geq \gamma(G) + k - 2$.

Although it is proved that the above inequality is sharp for every $k \geq 2$, the characterization of graphs attaining the equality is still open, even for the small values of k . The corresponding characterization problem was studied in [8,14,15]. Recently, we considered a large class of graphs and gave a characterization for the members satisfying the equality $\gamma_2(G) = \gamma(G)$. We also proved that it is NP-hard to decide whether this equality holds for a graph. Moreover, we gave a necessary and sufficient condition for a graph to satisfy $\gamma_2(G) = \gamma(G)$ hereditarily [16]. Some similar problems involving different domination-type graph and hypergraph invariants were considered, for example, in [17–22].

Lemma 2.2. *Let D be a minimum k -dominating set of a graph G . If $\gamma_k(G) = \gamma(G) + k - 2$, then $\gamma(G[D]) \geq \gamma_k(G) - (k - 2)$.*

Proof. Suppose, to the contrary, that $\gamma(G[D]) \leq \gamma_k(G) - k + 1$. Let S be a dominating set of cardinality $|D| - k + 1$ in $G[D]$. We claim that S is also a dominating set in G . Indeed, as we removed $k - 1$ vertices from the k -dominating set D , every vertex in $V(G) \setminus D$ is still dominated by at least one vertex of S . Note that S dominates all the vertices in D by the choice of S . Since $\gamma(G) \leq |S| = \gamma_k(G) - k + 1$, we get a contradiction. Thus, $\gamma(G[D]) \geq \gamma_k(G) - k + 2$. $\square$

Since $\gamma(G) \leq |V(G)| - \Delta(G)$ holds for every graph G , the previous lemma directly implies the following result obtained by Hansberg (see Theorem 5 in [8]).

Lemma 2.3. [8] *Let D be a minimum k -dominating set of a graph G . If $\gamma_k(G) = \gamma(G) + k - 2$, then $\Delta(G[D]) \leq k - 2$.*

For each $k \geq 2$, we define three graph classes which will have crucial role in our study.

Definition 2.4. Given an arbitrary connected k -uniform hypergraph F with vertex set $V(F) = D = \{v_1, \dots, v_s\}$ and edge set $E(F) = \{e_1, \dots, e_p\}$, we define the following graph classes.

- (i) Define a pair of new vertices $X_i = \{x_i^1, x_i^2\}$ for every hyperedge $e_i \in E(F)$. Let $X = \bigcup_{i \in [p]} X_i$ and let

$$V(G_1) = X \cup V(F), \quad E(G_1) = \{x_i^j v_\ell : v_\ell \in e_i, i \in [p], j \in [2], \ell \in [s]\}.$$
 The graph class $\mathcal{B}^*(F)$ contains only this graph G_1 , which is also called the double incidence graph of F .
- (ii) The class $\mathcal{B}(F)$ contains a graph G_2 if it can be obtained from the double incidence graph G_1 in the following way. We keep all vertices and edges of G_1 . For each edge $e_i \in E(F)$, we create some (maybe zero) false twins of the vertex x_i^1 . Furthermore, if $S \subseteq V(F)$ induces a complete subhypergraph in F , then we may supplement G_1 with some (maybe zero) new vertices which are adjacent to all vertices in S . We denote by Y the set of the new vertices that is $Y = V(G_2) \setminus V(G_1)$. Putting it the other way around, we say that G_1 is the γ_k -simplified graph of G_2 .
- (iii) The class $\mathcal{G}(F)$ contains G_3 if it can be obtained from a graph $G_2 \in \mathcal{B}(F)$ by supplementing it with some (maybe zero) new edges inside D and $X \cup Y$. These edges can be chosen arbitrarily, but each X_i must remain independent.

For every $k \geq 2$, the graph classes $\mathcal{B}_k^*$, $\mathcal{B}_k$, and $\mathcal{G}_k$ contain those graphs G for which there exists a k -uniform hypergraph F such that G belongs to $\mathcal{B}^*(F)$, $\mathcal{B}(F)$, and $\mathcal{G}(F)$, respectively. We say that F is the underlying hypergraph of G if $G \in \mathcal{G}(F)$.

It is clear that each member of $\mathcal{B}_k$ is bipartite and that $\mathcal{B}_k^* \subseteq \mathcal{B}_k \subseteq \mathcal{G}_k$ holds for every $k \geq 2$. In particular, for each k -uniform hypergraph F , we have exactly one graph in $\mathcal{B}_k^*$ having F as its underlying hypergraph, but there are infinitely many such graphs in $\mathcal{B}_k$. Note that in [16], where (γ, γ_2) -graphs were studied, a graph class $\mathcal{G}$ was introduced that exactly corresponds to the class $\mathcal{G}_2$ defined here. Moreover, by the results in [17,20,23], the class $\mathcal{B}_2$ is the collection of those connected graphs which satisfy $\tau(G) = \gamma(G)$ and $\delta(G) \geq 2$.

Hansberg proved the following lemma which directly implies that the class $\mathcal{G}_k$ contains all (γ, γ_k) -graphs.

Lemma 2.5. [8] *Let G be a (γ, γ_k) -graph for an integer $k \geq 2$ and suppose that D is a minimum k -dominating set of G . Then, for every k vertices $v_1, v_2, \dots, v_k$ from D , if $\bigcap_{i=1}^k N(v_i) \neq \emptyset$, then there is a nonadjacent pair $x, y \in V \setminus D$ such that $N_G(x) \cap D = N_G(y) \cap D = \{v_1, v_2, \dots, v_k\}$.*

Corollary 2.6. *If G is a connected (γ, γ_k) -graph with $\Delta(G) \geq k \geq 2$ and D is a minimum k -dominating set of G , then there exists a k -uniform underlying hypergraph F on the vertex set D such that $G \in \mathcal{G}(F)$.*

3 Bipartite (γ, γ_k) -graphs

In this section, we characterize connected bipartite graphs satisfying $\Delta(G) \geq k$ and $\gamma_k(G) = \gamma(G) + k - 2$, for every $k \geq 3$. Observe that $\gamma_k(G) \geq \gamma(G)$ holds for every graph without imposing conditions on the vertex degrees. Moreover, assuming $k \geq 3$, the equality $\gamma_k(G) = \gamma(G)$ is satisfied if and only if G consists of isolated vertices. Consequently, if $\Delta(G) \geq k \geq 3$ and if G is disconnected, then $\gamma_k(G) = \gamma(G) + k - 2$ holds if and only if G contains exactly one component which is a (γ, γ_k) -graph and all the further components are isolated vertices. Therefore, it is enough to concentrate on the connected (γ, γ_k) -graphs.

In Section 3.1, we prove that all connected bipartite (γ, γ_k) -graphs belong to $\mathcal{B}_k$ and it is enough to consider the graphs from the subclass $\mathcal{B}_k^*$ where the underlying hypergraph uniquely represents the graph G . In Section 3.2, we introduce a hypergraph invariant called “vertex-edge cover number” or shortly “TC-number” and derive the characterization of connected bipartite (γ, γ_k) -graphs via the properties of the underlying hypergraph.

3.1 Reducing the problem to γ_k -simplified graphs

Theorem 3.1. *Let $k \geq 3$ and let G be a connected bipartite graph that satisfies $\gamma_k(G) = \gamma(G) + k - 2$ and $\Delta(G) \geq k$. Then, $G \in \mathcal{B}_k$ and every minimum k -dominating set corresponds to a partite class of G .*

Proof. Let D be a minimum k -dominating set of G . The degree condition implies that $V(G) \setminus D$ is not empty and, therefore, the underlying hypergraph F contains at least one edge. Let $e_i = \{v_1, \dots, v_k\}$ be an edge of F . Since all the vertices in e_i have common neighbors in X_i , they belong to the same partite class of G . This implies that the vertices of a connected component of F are in the same partite class. If F is connected, the same is true for the entire D . Moreover, since every vertex $x \in V(G) \setminus D$ has some neighbors in D , they all must be contained in the other partite class. It finishes the proof for the case when F is connected.

If F is not connected, consider a nontrivial component F_1 and the remaining part $F_2 = F - F_1$ of the underlying hypergraph. By Corollary 2.6, no vertex of $V(G) \setminus D$ can be adjacent to a vertex from $V(F_1)$ and to a vertex from $V(F_2)$ simultaneously. By our assumption, G is connected. It is enough to consider the following two cases.

Case 1. There exists an edge uv in $G[D]$ such that $u \in V(F_1)$ and $v \in V(F_2)$.

Since F_1 is connected and nontrivial, there is an edge $e_i \in E(F_1)$ containing u . In this case, the k vertices from $e_i \setminus \{u\} \cup \{v\}$ cannot have a common neighbor in $V(G) \setminus D$ and consequently,

$$D' = D \setminus (e_i \setminus \{u\} \cup \{v\}) \cup \{x_i^1\},$$

where x_i^1 is from the pair X_i associated with e_i . Observe that D' is a dominating set of G . As $|D'| < |D| - k + 2$, this is a contradiction.

Case 2. There exists an edge xy in $G[V(G) \setminus D]$ such that x has neighbors from $V(F_1)$ and y has neighbors from $V(F_2)$.

Consider k vertices, namely, $v_1, \dots, v_k$ from $N(x) \cap V(F_1)$ and k vertices, namely, $u_1, \dots, u_k$ from $N(y) \cap V(F_2)$ and then define $D' = D \setminus \{v_1, \dots, v_{k-1}, u_1, \dots, u_{k-1}\} \cup \{x, y\}$. Observe that D' is a dominating set of G , since $\{v_1, \dots, v_{k-1}, u_1, \dots, u_{k-1}\} \cup \{x, y\}$ does not contain any edges of F . By our condition $k \geq 3$, it follows that $|D'| = |D| - (2k - 2) + 2 < |D| - k + 2$. We have a contradiction again which completes the proof of the theorem. $\square$

Note that Theorem 3.1 does not hold for $k = 2$. As an example, consider the graph H constructed by two vertex-disjoint copies of $K_{2,3}$ by adding exactly one edge between them such that the maximum degree remains three. Observe that this graph is bipartite and satisfies the equality $\gamma_2(H) = \gamma(H) = 4$, but neither of the partite classes corresponds to a minimum dominating set or a minimum 2-dominating set.

Theorem 3.2. *Let G be a connected bipartite graph from $\mathcal{B}_k$ and let G^* be its γ_k -simplified graph from $\mathcal{B}_k^*$, where $k \geq 3$. Then, $\gamma_k(G) = \gamma(G) + k - 2$ if and only if $\gamma_k(G^*) = \gamma(G^*) + k - 2$.*

Proof. First, we prove that for each k -uniform hypergraph F and graph $H \in \mathcal{B}(F)$, the vertex set $D = V(F)$ is a minimum k -dominating set of H . It is clear, by definition of $\mathcal{B}(F)$, that D is a k -dominating set in H . Suppose for a contradiction that D' is a minimum k -dominating set and $|D'| < |D|$ holds. If $D' \setminus D$ contains a vertex v which dominates at least $k + 1$ vertices from $D \setminus D'$, let us introduce the notations $S = N(v) \cap (D \setminus D')$ and $s = |S| \geq k + 1$. By Definition 2.4 (ii), S induces a complete k -uniform subhypergraph in F , and hence, it contains $\binom{s}{k}$ hyperedges. For each such hyperedge $e_i \subseteq S$, the vertices x_i^1 and x_i^2 from X_i have to be included in the k -dominating set D' . Let us denote by $X(S)$ the union of all such pairs, that is, $X(S) = \{X_i : e_i \in E(F) \text{ and } e_i \subseteq S\}$. Note that, since $s \geq k + 1$, we have $|X(S)| = 2 \binom{s}{k} \geq 2s$ and $X(S) \subseteq D'$. It is straightforward to check that $(D' \cup S) \setminus (X(S) \cup \{v\})$ is a k -dominating set in H which contains at most $|D'| + s - (2s + 1) < |D'|$ vertices. This contradicts the minimality of D' . We may infer that every $v \in D' \setminus D$ has at most k neighbors from $D \setminus D'$. Then, the number z of edges between $D' \setminus D$ and $D \setminus D'$ is at most $k|D' \setminus D|$. Counting the edges from the other side, as D is an independent set in H , each vertex $v \in D \setminus D'$ must be k -dominated by k different neighbors from $D' \setminus D$. As follows, $z \geq k|D \setminus D'|$ and we have $k|D' \setminus D| \geq z \geq k|D \setminus D'|$. It contradicts our assumption $|D'| < |D|$ and proves that $\gamma_k(H) = |D|$ for every $H \in \mathcal{B}(F)$. Since both G and G^* belong to $\mathcal{B}(F)$, we may conclude that $\gamma_k(G) = \gamma_k(G^*)$.

Now, we prove that $\gamma(G) = \gamma(G^*)$. Let Q' be a minimum dominating set in G^* such that $|Q' \cap D|$ is maximum. By this maximality, Q' contains at most one vertex from each pair $\{x_i^1, x_i^2\}$, since otherwise x_i^2 can be replaced by a vertex of e_i . Since x_i^1 and x_i^2 are false twins, we may suppose that the vertex x_i^2 does not belong to Q' for any i . Note that $Q' \cap D$ is a transversal in the underlying hypergraph F . We claim that Q' is a dominating set in G as well. Indeed, all vertices in $D \cup X$ are dominated. Furthermore, for any vertex $y \in Y$ we may consider k arbitrarily chosen neighbors from D , they must form a hyperedge e_j in F . The vertex x_j^2 is not in Q' but it is dominated by a vertex from $Q' \cap e_j$. This vertex dominates y as well. This proves $\gamma(G) \leq |Q'| = \gamma(G^*)$.

On the other hand, consider a minimum dominating set Q in G such that $|Q \cap D|$ is maximum. By this maximality, we have $|Q \cap X_i| \leq 1$ for every i . If $Q \cap X_i$ contains a vertex different from x_i^1 , we may replace it with x_i^1 in the dominating set. Again, $Q \cap D$ must be a transversal in F . Now assume that Q contains a vertex y from Y . Let $z_1, \dots, z_\ell$ be those vertices from D which are privately dominated by y (they have no further neighbors in the dominating set). Note that all of $z_1, \dots, z_\ell$ are outside of Q . By Corollary 2.6 and the definition of F , if $\ell \geq k$, then the set $\{z_1, \dots, z_\ell\}$ must contain at least one edge from F . This contradicts the fact that Q is a transversal in F . If $\ell \leq k - 1$, then there is an edge $e_i \in E(F)$ containing $z_1, \dots, z_\ell$. As x_i^1 can dominate all these vertices $z_1, \dots, z_\ell$ and the vertex which dominates x_i^2 in Q also dominates y , the set $Q \setminus \{y\} \cup \{x_i^1\}$ is a dominating set of G .

Perform these changes while there is a vertex from Y in the dominating set. At the end, we have a minimum dominating set $Q^* \subseteq (D \cup X)$ in G . Clearly, Q^* is a dominating set of G^* , and we have $\gamma(G^*) \leq |Q^*| = \gamma(G)$. This implies $\gamma(G) = \gamma(G^*)$, which completes the proof. $\square$

3.2 Characterization via underlying hypergraphs

Definition 3.3. For a hypergraph H , a set $S \subseteq V(H) \cup E(H)$ is a vertex-edge cover or shortly a TC-set of H if $S \cap V(H)$ is a transversal (vertex cover) in H and the edges in $S \cap E(H)$ together cover all vertices outside $S \cap V(H)$. The smallest cardinality of a TC-set in H is called the TC-number of H and denoted by $tc(H)$.

Proposition 3.4. For every hypergraph H of order n , the following statements hold and the upper bounds given in (i) and (ii) are sharp.

- (i) If r is the smallest size of an edge in H , then $tc(H) \leq n - r + 2$.
- (ii) If H is k -uniform, then $tc(H) \leq n - k + 2$.
- (iii) If H is 2-uniform, that is a simple graph, then $tc(H) = n$.

Proof. Let H be a hypergraph and $e \in E(H)$ an edge of minimum cardinality r . If v is an arbitrary vertex from e , the $r - 1$ vertices in $e \setminus \{v\}$ do not contain any edges of H . Hence, $T = (V(H) \setminus e) \cup \{v\}$ is a transversal and $T \cup \{e\}$ is a TC-set in H . Then, we have $tc(H) \leq |T| + 1 = n - (r - 1) + 1 = n - r + 2$ that proves (i). From (i), statement (ii) can be obtained as a direct consequence. Observe further that every complete k -uniform hypergraph gives a sharp example as every transversal of $\mathcal{K}_n^k$ contains at least $n - k + 1$ vertices and, if it is not the entire vertex set, we also have to put an edge into the TC-set. Thus, $tc(\mathcal{K}_n^k) = n - k + 2$ holds for every $n \geq k \geq 2$.

Now, let H be an arbitrary graph and let T be its transversal set. Since every edge of H intersects $V(H) \setminus T$ in at most one vertex, we cannot cover $V(H) \setminus T$ with less than $|V(H) \setminus T| = n - |T|$ edges. This gives $tc(H) \geq n$ and it is clear, or concluded from part (ii), that $tc(H) \leq n$. This proves (iii). $\square$

Concerning the extremal cases in part (ii) of Proposition 3.4, we prove the following.

Theorem 3.5. A k -uniform hypergraph H of order n satisfies $tc(H) = n - k + 2$, if and only if, for every $\ell \leq k - 1$ and for every ℓ edges $e_{i_1}, \dots, e_{i_\ell}$ of H , the union $L = \bigcup_{j=1}^{\ell} e_{i_j}$ contains at most $\ell + k - 2$ weakly independent vertices.

Proof. We prove the equivalence for the negations. First suppose that there exist ℓ edges, say $e_1, \dots, e_\ell$, in H such that $L = \bigcup_{j=1}^{\ell} e_j$ contains $s = \ell + k - 1$ (weakly) independent vertices. Let $X = \{v_1, \dots, v_s\}$ be such an independent subset of L . Then, the set $T = V(H) \setminus X$ is a transversal of cardinality $n - s$ and the remaining vertices in X can be covered by using the ℓ edges $e_1, \dots, e_\ell$. This TC-set contains $n - s + \ell = n - (\ell + k - 1) + \ell = n - k + 1$ elements and, as follows, we have $tc(H) \leq n - k + 1 < n - k + 2$. This proves the first direction of the equivalence.

Now we assume that $tc(H) \leq n - k + 1$ and show the existence of $\ell \leq k - 1$ edges such that the union L contains more than $\ell + k - 2$ independent vertices. Consider a TC-set S with $|S| = n - k + 1$ and, renaming the edges and vertices if necessary, let $S \cap E(H) = \{e_1, \dots, e_\ell\}$ and $X = V(H) \setminus (S \cap V(H)) = \{v_1, \dots, v_s\}$, where $s = n - |S \cap V(H)| = n - (n - k + 1 - \ell) = k + \ell - 1$. By the definition of a TC-set, X is an independent set in H and all vertices in X can be covered by the ℓ edges $e_1, \dots, e_\ell$. Therefore, the union L of these ℓ edges contains a set X of $s > \ell + k - 2$ independent vertices.

Now it suffices to prove that we can ensure $\ell \leq k - 1$. Suppose that $\ell \geq k$ and the union L of $e_1, \dots, e_\ell$ contains a set X of $s = k + \ell - 1$ independent vertices. Then, $\ell = s - k + 1 \geq s - \ell + 1$ holds and we may derive that $2\ell - 1 \geq s$. By the pigeonhole principle, there exists an edge e_q , where $q \in [\ell]$, which contains at most one “private” vertex from X , that is,

$$|(e_q \setminus \bigcup_{j \in [\ell], j \neq q} e_j) \cap X| \leq 1.$$

Removing this edge e_q from the list, we have $\ell - 1$ edges such that their union contains at least $(\ell - 1) + k - 1$ independent vertices. If $\ell - 1$ is still greater than $k - 1$, we may repeat the procedure. At the end, we have $\ell' \leq k - 1$ edges such that their union contains $k + \ell' - 1$ independent vertices. Consequently, it is enough to check the property for at most $k - 1$ edges. $\square$

Motivated by Theorem 3.5, we define the following classes of hypergraphs.

Definition 3.6. For every $k \geq 3$, a k -uniform hypergraph H belongs to the class $\mathcal{H}_k$, if and only if, $\rho(H) \leq k - 1$ and $\alpha_w(H) \geq \rho(H) + k - 1$.

The condition $\rho(H) \leq k - 1$ implies that for each k and every $H \in \mathcal{H}_k$, the order of H is at most $k(k - 1)$ and, consequently, $\mathcal{H}_k$ is a finite set of hypergraphs. Having the definition of $\mathcal{H}_k$ in hand, we can formulate the following corollary.

Corollary 3.7. For each $k \geq 3$, a k -uniform hypergraph F satisfies $tc(F) = |V(F)| - k + 2$ if and only if F is $\mathcal{H}_k$ -free.

The next theorem and its consequences give a characterization for connected bipartite (γ, γ_k) -graphs. In fact, these characterizations refer to the properties of the underlying hypergraph but, since we know that every bipartite (γ, γ_k) -graph G belongs to a class $\mathcal{B}_k(F)$, these results also characterize the structure of G .

Theorem 3.8. For each $k \geq 3$, a connected bipartite graph G satisfies $\gamma_k(G) = \gamma(G) + k - 2$, if and only if, $G \in \mathcal{B}_k$ and the underlying hypergraph F of G satisfies $tc(F) = |V(F)| - k + 2$.

Proof. If G is a connected bipartite (γ, γ_k) -graph, then, by Theorem 3.1, we have $G \in \mathcal{B}_k$. Furthermore, by Theorem 3.2, G is a (γ, γ_k) -graph if and only if its γ_k -simplified graph G^* is a (γ, γ_k) -graph as well. Note that G and G^* admit the same underlying hypergraph F where $|V(F)| = \gamma_k(G) = \gamma_k(G^*)$ by Corollary 2.6. Hence, it suffices to prove that $\gamma(G^*) = tc(F)$.

Now consider a minimum dominating set Q of G^* such that $|Q \cap V(F)|$ is maximum under this condition. By this maximality, Q does not contain two false twins from $V(G) \setminus V(F)$. Indeed, $x_i^1 \in Q$ dominates itself and all vertices from the associated edge e_i of F , and then, x_i^2 can be replaced by any vertex of e_i in the dominating set. We may suppose, without loss of generality, that $x_i^2 \notin Q$ holds for each $e_i \in E(F)$. Since $N_{G^*}(x_i^2) = e_i$ and $|N_{G^*}(x_i^2) \cap Q| \geq 1$, we infer that Q contains at least one vertex from each $e_i \in E(F)$. Thus, $Q \cap V(F)$ is a transversal of F . Furthermore, if a vertex $v_j \in V(F)$ does not belong to Q , it is dominated by a vertex $x_s^1 \in Q$ where the edge e_s of the underlying hypergraph is incident with v_j . It means that the edge set

$$R = \{e_s : e_s \in E(F) \text{ and } x_s^1 \in Q\}$$

covers all vertices of $V(F)$ which are outside Q . Then, $(Q \cap V(F)) \cup R$ is a TC-set in F and since $|R|$ equals the number of vertices in $Q \setminus V(F)$, the cardinality of this TC-set is $|Q| = \gamma(G^*)$. Therefore, $tc(F) \leq |Q| = \gamma(G^*)$.

To prove the other direction, suppose that $S' = T' \cup R'$ is a minimum TC-set of F where $T' = S' \cap V(F)$ and $R' = S' \cap E(F)$. Now, determine the set $Q^* \subseteq V(G^*)$ as $Q^* = T' \cup R^*$ where R^* consists of those vertices x_i^1 which represent the edges $e_i \in R'$, that is,

$$R^* = \{x_i^1 : N_{G^*}(x_i^1) \in R'\}.$$

Hence, we have $|Q^*| = |S'| = tc(F)$. Observe that, by the definition of the underlying hypergraph, the transversal T' of F dominates all vertices in $V(G) \setminus V(F)$ in the graph G^* . Moreover, each vertex u of $V(F)$ which is outside T' is covered by an edge $e_i \in R'$ in the underlying hypergraph and, therefore, u is dominated by the vertex $x_i^1 \in R^*$ in G . We conclude that Q^* is a dominating set of G^* and we have $\gamma(G^*) \leq |Q^*| = tc(F)$. This finishes the proof of the theorem. $\square$

Theorem 3.8, Theorem 3.5, and Corollary 3.7 immediately imply other formulations of the characterization.

Corollary 3.9. For each $k \geq 3$, a connected bipartite graph G is a (γ, γ_k) -graph if and only if $G \in \mathcal{B}_k$ and the underlying hypergraph F of G satisfies the following property: for every $\ell \leq k - 1$ and for every ℓ edges $e_{i_1}, \dots, e_{i_\ell}$ of F , the union $L = \bigcup_{j=1}^{\ell} e_{i_j}$ contains at most $\ell + k - 2$ weakly independent vertices.

Corollary 3.10. For each $k \geq 3$, a connected bipartite graph G is a (γ, γ_k) -graph if and only if $G \in \mathcal{B}_k$ and the underlying hypergraph of G is $\mathcal{H}_k$ -free.

For the case of $k = 3$, Corollary 3.9 can be reformulated as follows.

Corollary 3.11. A connected bipartite graph G is a (γ, γ_3) -graph if and only if $G \in \mathcal{B}_3$ and the underlying hypergraph F satisfies the following property:

(*) Every four different vertices that can be split into two pairs of adjacent vertices induce at least one hyperedge in F .

Proposition 3.12. If G is a connected bipartite (γ, γ_3) -graph and D is a minimum 3-dominating set of G , then $G^2[D]$ is a threshold graph.

Proof. Suppose that G and D satisfy the conditions of the proposition and consider the underlying hypergraph F with $V(F) = D$. Since $G \in \mathcal{B}_3$, any two vertices $u, v \in D$ having distance 2 in G belong to a common hyperedge of F . In other words, if $uv \in E(G^2[D])$, then uv is an edge in the 2-section graph H of F . Similar argumentation shows that the other direction is valid too. Thus, $H = G^2[D]$. By Corollary 3.11, the underlying hypergraph F satisfies (*). This implies that for every two vertex disjoint edges vv' and uu' of H , the vertex set $\{v, v', u, u'\}$ contains at least one hyperedge of F . We infer that there exist three vertices in $\{v, v', u, u'\}$, which induces a triangle in H . Checking all possible extensions of a $2K_2$ on four vertices, we obtain that $2K_2$, P_4 , and C_4 are the forbidden induced subgraphs. Since $(2K_2, P_4, C_4)$ -free graphs are exactly the threshold graphs, the statement follows. $\square$

4 Characterization of bipartite (γ, γ_3) -perfect graphs

In this section, we prove a characterization for bipartite (γ, γ_3) -perfect graphs. First, we define the graph classes S_k and prove that all members of S_k are (γ, γ_k) -perfect for every $k \geq 2$. Then, the main theorem of this section states that S_3 is the set of all bipartite (γ, γ_3) -perfect graphs.

Definition 4.1. Let $k, r, i_1, \dots, i_r$ be integers which satisfy $k \geq 2, r \geq 1$, and $i_j \geq k$ for every $j \in [r]$. We define $S_k(i_1, \dots, i_r)$ as the graph that is obtained from the star $K_{1,r}$ in the following way. First replace the edges $e_1, \dots, e_r$ of the star with $i_1, \dots, i_r$ parallel edges, respectively, and subdivide each edge exactly once. Finally, supplement the graph by $k - 2$ new vertices which are false twins with the center. For a fixed integer $k \geq 2$, let S_k be the graph class that contains all $S_k(i_1, \dots, i_r)$ with $r \geq 1$ and $i_j \geq k$ for every $j \in [r]$.

With this notation, $S_2(\ell)$ corresponds to $K_{2,\ell}$ whenever $\ell \geq 2$. In general, $S_k(\ell)$ is just $K_{k,\ell}$ if $\ell \geq k \geq 2$. For another example that is not a complete bipartite graph see Figure 1.

Proposition 4.2. If $G \in S_k$, then G is (γ, γ_k) -perfect.

Proof. Consider a graph $G = S_k(i_1, \dots, i_r)$ from the graph class S_k . It is straightforward to check that the minimum vertex degree is k and that $\gamma_k(G) = r + k - 1$ and $\gamma(G) = r + 1$ hold. Alternatively, the equality $\gamma_k(G) = \gamma(G) + k - 2$ can be verified directly by Theorem 3.8. Thus, G is a (γ, γ_k) -graph. Note that every subdivision vertex has degree k . Now, consider an induced subgraph H of G that satisfies $\delta(H) \geq k$. Suppose that a vertex $u \in V(G)$ does not belong to $V(H)$. If u is the center or its false twin, then, since

$\delta(H) \geq k$, no subdivision vertices can be present in H . This contradicts the degree condition. If u is a leaf in the original star, then, by the degree condition again, no neighboring subdivision vertices belong to H . If u is a subdivision vertex, then consider the neighbor of u which was a leaf in the star, say u' , and denote the degree $d_G(u')$ by i_j . When u does not belong to H , we have two cases: either u' and at least k of its neighbors are still present in H ; or the entire $N_G[u']$ is omitted from H . We infer that every induced subgraph H of G with $\delta(H) \geq k$ is isomorphic to a graph $S_k(i'_1, \dots, i'_{r'})$, where $i'_j \geq k$ for all $j' \in [r']$. Therefore, H is a (γ, γ_k) -graph again. This proves the (γ, γ_k) -perfectness of G . $\square$

As proved in [16], S_2 is the set of all (γ, γ_2) -perfect graphs. Since each S_k contains only bipartite graphs, it is equivalent to saying that S_2 is the set of all bipartite (γ, γ_2) -perfect graphs. Here, we prove an analogous statement for S_3 .

Theorem 4.3. *G is a bipartite (γ, γ_3) -perfect graph if and only if $G \in S_3$.*

Proof. By Definition 4.1 and Proposition 4.2, every member of S_3 is (γ, γ_3) -perfect and bipartite. To prove the other direction, suppose that G is a bipartite (γ, γ_3) -perfect graph, D is a minimum 3-dominating set, and F is the 3-uniform underlying hypergraph of G (with respect to D). Under these conditions, we first prove a couple of claims on the structure of F and G .

Claim A. *Every edge of F contains a vertex of degree 1.*

Proof of Claim A. Suppose, to the contrary, that $e_1 = \{u, u', u''\}$ is an edge in F such that $d_F(u) \geq d_F(u') \geq d_F(u'') \geq 2$. Let Y_4 be the set of vertices of degree at least 4 in $V(G) \setminus D$ and let $Z = (N_G(u) \cap N_G(u') \cap N_G(u'')) \setminus \{x_1^1\}$.

Consider the induced subgraph $H = G[V(G) \setminus (Y_4 \cup Z)]$ of G . Since G is bipartite and no vertex from D was deleted, every vertex $y \in V(H) \setminus D$ remains of degree 3 in H . If v is a vertex from D such that $d_F(v) = 1$, then v cannot be contained in a clique of order at least 4 in F and, by Corollary 2.6, v cannot be adjacent to any vertex from Y_4 . By our assumption, e_1 does not contain the degree-1 vertex v , and therefore, v is adjacent to none of the vertices in Z . Thus, $d_F(v) = 1$ implies $d_H(v) = d_G(v) \geq 3$. Now, consider a vertex $v \in D$ that satisfies $d_F(v) \geq 2$. This vertex v is incident with at least two different edges in F , say v is incident with e_i and e_j . Observe that $d_G(v) \geq 4$ as v is adjacent to all vertices from $X_i \cup X_j$. Since $Y_4 \cup Z$ contains at most one vertex, namely x_1^2 , from $X_i \cup X_j$, the degree of v is still at least 3 in H . We conclude $\delta(H) = 3$.

It is straightforward to show that, under our assumptions, $(D \cup \{x_1^1\}) \setminus \{u, u', u''\}$ is a dominating set of H and hence we have $\gamma(H) \leq |D| - 2$. We now prove that $\gamma_3(H) \geq |D|$ that gives the desired contradiction. Consider an arbitrary 3-dominating set Q in H . Denote $|D \setminus Q|$ by s and $|Q \cap (V(H) \setminus D)|$ by ℓ . To 3-dominate all vertices in $D \setminus Q$, we need at least $3s$ edges between the vertex sets $D \setminus Q$ and $Q \cap (V(H) \setminus D)$. On the other

Figure 1: An illustration for the graph $S_3(3, 3, 4)$. The thick red box represents two false twin vertices.

hand, we may have at most 3ℓ edges between the two sets because every vertex in the second set is of degree 3. Consequently, $\ell \geq s$ holds and this implies

$$|Q| = |Q \cap D| + |Q \cap (V(H) \setminus D)| = |D| - s + \ell \geq |D|$$

for every 3-dominating set Q of H . Therefore, we have $\gamma_3(H) \geq |D| > \gamma(H) + 1$ for an induced subgraph of minimum degree 3 that contradicts the (γ, γ_3) -perfectness of G . This contradiction finishes the proof of Claim A.

Claim B. Every vertex from $V(G) \setminus D$ is of degree 3 in G .

Proof of Claim B. By Claim A, there is no complete subhypergraph in F on four (or more) vertices. Together with Corollary 2.6 these imply that there exist no vertices of degree greater than 3 in $V(G) \setminus D$. Since $\delta(G) \geq 3$, this results in $d_G(y) = 3$ for every $y \in V(G) \setminus D$ as stated.

Claim C. If three vertices form an edge in the underlying hypergraph F , then they have at least three common neighbors in G .

Proof of Claim C. By Claim A, every edge $e = \{u, u', u''\}$ of F contains a vertex of degree 1. We may assume that u is such a vertex in e . By the definition of the underlying hypergraph and by Claim B, all neighbors of the degree-1 vertex u are associated with the edge e , that is,

$$N_G(u) = \{y \in V(G) \setminus D : N_G(y) = \{u, u', u''\}\}.$$

Since $|N_G(u)| = d_G(u) \geq 3$, there exist at least three common neighbors of u, u' , and u'' .

Claim D. F does not have two edges that share exactly one vertex.

Proof of Claim D. First suppose that two edges, namely e_i and e_j , of F share exactly one vertex w . Let $e_i = \{u, u', w\}$ and $e_j = \{v, v', w\}$. By Claim C u, u' , and w have at least three common neighbors, denote them by x_i^1, x_i^2 , and x_i^3 . Similarly, let the common neighbors of v, v' , and w be x_j^1, x_j^2 , and x_j^3 . Consider the subgraph H of G induced by these 11 vertices. Observe that H has minimum degree 3, $\gamma_3(H) = 5$ and, as w, x_i^1, x_j^1 form a dominating set, $\gamma(H) \leq 3$. Thus, we have $\gamma_3(H) > \gamma(H) + 1$ for a connected induced subgraph of G with minimum degree 3. This contradicts our condition on the (γ, γ_3) -perfectness of G . Consequently, F cannot contain two edges intersecting in exactly one vertex.

Concerning the statement and proof of Claim D, remark that even if $e_i \cup e_j$ contains further edges in F , the considered subgraph H is an induced subgraph of G .

To complete the proof of Theorem 4.3 we make the following observations. If F contains only one edge, then G is a $K_{3,\ell}$ that is isomorphic to $S_3(\ell)$ for an integer $\ell \geq 3$. So, we may assume that $|D| \geq 4$ and F contains at least two edges. Since G is a connected bipartite graph and D is one of the partite classes, the underlying hypergraph F must be connected as well. Thus, F has two intersecting edges e_i and e_j . By Claim D, e_i and e_j share two vertices, say v_0^1 and v_0^2 . Claim A implies that the third vertex of e_i is of degree 1 and the same is true for e_j . The connectivity of F then requires that each edge of F is incident with v_0^1 and v_0^2 and contains a further private vertex. Therefore, the underlying hypergraph F of a (γ, γ_3) -perfect G can always be obtained (up to isomorphism) in the following way:

$$V(F) = \{v_0^1, v_0^2, v_1, \dots, v_r\};$$

$$E(F) = \{e_1, \dots, e_r\} \quad \text{where} \quad e_i = \{v_0^1, v_0^2, v_i\} \text{ for every } i \in [r].$$

Then, G can be constructed by assigning at least three, say i_j , new vertices to each edge e_j of F and making adjacent the vertices in e_j to the new associated vertices. This clearly results in the graph $S_3(i_1, \dots, i_r)$ with $i_j \geq 3$ for all $j \in [r]$. Note that the condition $\delta(G) \geq 3$ implies that F is not edgeless and consequently, $r \geq 1$ must hold. We conclude that every (γ, γ_3) -perfect graph belongs to S_3 . $\square$

We show that the statement analogous to Theorem 4.3 holds neither for $k = 4$ nor for any even integer $k > 4$. First, consider the lexicographic product $C_6 \circ \overline{K_2}$, that is, every vertex of the cycle C_6 is replaced with

two independent vertices. Then, $\gamma_4(C_6 \circ \overline{K}_2) = 6 = \gamma(C_6 \circ \overline{K}_2) + 2$ and, since the graph is 4-regular, every proper subgraph has a vertex of degree smaller than 4. Thus, $C_6 \circ \overline{K}_2$ is (γ, γ_4) -perfect and bipartite but does not belong to S_4 . As an infinite class of similar examples we propose the following construction.

Example. For each even integer $k = 2\ell \geq 4$ consider the bipartite graph $G_k \in \mathcal{G}_k^*$ obtained as the double incidence graph of the following underlying hypergraph F_k . The vertex set of F_k is $V = W \cup U$ where $W = \{w_1, \dots, w_\ell\}$ and $U = \{u_1, \dots, u_{\ell+1}\}$; the edge set is $E = \{e_1, \dots, e_{\ell+1}\}$ where $e_i = V \setminus \{u_i\}$ for all $i \in [\ell + 1]$. One can check that F_k satisfies the condition given in Corollary 3.9 and, therefore, G_k is a (γ, γ_k) -graph. On the other hand, one can check that every proper subgraph of G_k has minimum degree less than k . We may conclude that G_k is a bipartite (γ, γ_k) -perfect graph but does not belong to S_k .

5 Complexity results

Proposition 5.1. *Let k be a fixed integer with $k \geq 3$ and let G be a bipartite graph with $\Delta(G) \geq k$. It can be decided in polynomial time whether the graph G satisfies the equality $\gamma_k(G) = \gamma(G) + k - 2$.*

Proof. If G is a connected bipartite (γ, γ_k) -graph with $\Delta(G) \geq k$, then, by Theorem 3.1, every minimum k -dominating set is a partite class of G and $G \in \mathcal{B}_k$. This can be checked efficiently and, assuming that the conditions are satisfied, the k -uniform underlying hypergraph F can be determined in polynomial time as well. Then, by Corollary 3.10, it is enough to decide whether F is $\mathcal{H}_k$ -free. Recall that $\mathcal{H}_k$ is a finite set of k -uniform hypergraphs each of which is of order at most $k(k - 1)$. Thus, this step can also be performed in polynomial time.

Finally, note that if G is disconnected, it satisfies $\gamma_k(G) = \gamma(G) + k - 2$, if and only if, G contains one component which is a (γ, γ_k) -graph and the further components are isolated vertices. Hence, the statement remains true over the class of disconnected bipartite graphs. $\square$

Theorem 5.2. *For each $k \geq 2$, it is NP-hard to decide whether the equality $\gamma_k(G) = \gamma(G) + k - 2$ holds for G over the class $\mathcal{G}_k$.*

Proof. In order to prove that the corresponding decision problem is NP-hard, we construct a polynomial time reduction from 3-SAT problem, a classical NP-complete problem [24]. Note that we adopt the approach we used in [16] to prove NP-hardness. Since the construction we need here is more general, we only present the construction explicitly and give a sketch of the rest of the proof, for the brevity.

Let C be a 3-SAT instance with clauses $C_1, C_2, \dots, C_\ell$ over the Boolean variables $X = \{x_1, x_2, \dots, x_s\}$. We may assume $s \geq 4$, $\ell \geq 2$ and that for every three variables $x_{i_1}, x_{i_2}, x_{i_3}$ there exists a clause C_j , where $j \in [\ell]$, such that C_j does not contain any of the variables $x_{i_1}, x_{i_2}, x_{i_3}$ (neither in the positive form nor in the negative form). Otherwise, the problem could be reduced to at most eight (separated) 2-SAT problems, which are solvable in polynomial time.

We construct a graph $G \in \mathcal{G}_k$, such that the given instance C of 3-SAT problem is satisfiable if and only if $\gamma(G) < \gamma_k(G) - k + 2$.

For every variable x_i , we create three vertices $\{x_i^t, x_i^f, v_i\}$ and then we add the edges $x_i^t v_i$ and $x_i^f v_i$. For every clause $C_j \in C$, we create a vertex c_j , and if x_i is a literal in C_j , then $x_i^t c_j \in E(G)$; if $\neg x_i$ is a literal in C_j , then $x_i^f c_j \in E(G)$. Moreover, we add a vertex c^* and the edges $c^* x_i^t$ and $c^* x_i^f$ for every $i \in [s]$. We also add a vertex v_{s+1} and the edge set $\{c_i v_{s+1} : 1 \leq i \leq \ell\} \cup \{c^* v_{s+1}\}$. Finally, we add $k - 1$ new vertices $v_0^1, \dots, v_0^{k-1}$, such that each vertex v_0^r is adjacent to every vertex in $V(G) \setminus \{v_1, v_2, \dots, v_{s+1}\}$ for $r \in [k - 1]$ (for an illustration of the construction, see Figure 2). The order of G is obviously $3s + \ell + k + 1$ and this construction can be done in polynomial time. Note that $G \in \mathcal{G}(F)$ and any two hyperedges in F share $k - 1$ vertices, namely $v_0^1, \dots, v_0^{k-1}$.

Figure 2: An illustration of the construction for 3-SAT reduction: The clauses C_1 and C_ℓ corresponding to the vertices c_1 and c_ℓ , resp., are $C_1 = (x_1 \vee \neg x_3 \vee \neg x_s)$ and $C_\ell = (x_1 \vee \neg x_2 \vee x_s)$. The thick red box represents $k - 1$ independent vertices with the same open neighborhood.

It is straightforward to prove that $\{v_1, \dots, v_{s+1}, v_0^1, \dots, v_0^{k-1}\}$ is a minimum k -dominating set in G and, therefore, $\gamma_k(G) = k + s$. We refer the reader to [16] for a more detailed proof for $k = 2$.

In order to finish the proof, it suffices to show that C is satisfiable if and only if $\gamma(G) < \gamma_k(G) - k + 2$. First, assume that C is satisfiable and let $\varphi : X \rightarrow \{t, f\}$ be the corresponding truth assignment. The set $D' = D_1 \cup D_2 \cup \{c^*\}$ is a dominating set, where $D_1 = \bigcup_{i \in [s]} \{x_i^t : \varphi(x_i) = t\}$ and let $D_2 = \bigcup_{i \in [s]} \{x_i^f : \varphi(x_i) = f\}$. Since $|D'| = s + 1$, we have $\gamma(G) < \gamma_k(G) - k + 2 = (k + s) - k + 2$ for $k \geq 2$.

For the other direction, assume that $\gamma(G) < \gamma_k(G) - k + 2$ and consider a dominating set D' such that $|D'| \leq s + 1$. In order to dominate v_i , the set D' contains at least one vertex from the set $\{x_i^t, x_i^f, v_i\}$, for each $i \in [s]$. Similarly, to dominate v_{s+1} , the set D' contains at least one vertex from the set $\{c_1, c_2, \dots, c_\ell, c^*, v_{s+1}\}$. Since $|D'| \leq s + 1$, we have $|D' \cap \{x_i^t, x_i^f, v_i\}| = 1$ for every $i \in [s]$. Moreover, $|D' \cap \{c_1, c_2, \dots, c_\ell, c^*, v_{s+1}\}| = 1$ and $\{v_0^1, \dots, v_0^{k-1}\} \cap D' = \emptyset$. There are three cases to consider and for a detailed discussion, we refer the reader to [16].

- (i) If $v_{s+1} \in D'$, then, to dominate x_i^t and x_i^f for each $i \in [s]$, every v_i must be contained in D' . Hence, the vertices $\{v_0^1, \dots, v_0^{k-1}\}$ are not dominated by a vertex from D' , a contradiction.
- (ii) If $c_j \in D'$ for some $j \in [\ell]$, then for all but at most three $i \in [s]$ we have $v_i \in D'$. Then, there is a vertex c_q , such that $q \in [\ell]$ and c_q is not dominated by a vertex from D' , a contradiction.
- (iii) If $c^* \in D'$, then c_j must be dominated by a vertex x_i^t or x_i^f for every $j \in [\ell]$. Now, consider the truth assignment $\varphi : X \rightarrow \{t, f\}$ where $\varphi(x_i) = t$ if and only if $x_i^t \in D'$ and observe that φ satisfies the 3-SAT instance C .

This finishes the proof. □

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References

- [1] John Frederick Fink and Michael S. Jacobson, *n*-domination in graphs, in: Graph Theory with Application to Algorithms and Computer Science, 1985, pp. 283–300.
- [2] John Frederick Fink and Michael S. Jacobson, *On n-domination, n-dependence and forbidden subgraphs*, in: Graph Theory Applications to Algorithms Computer Science, 1985, pp. 301–311.
- [3] Csilla Bujtás and Szilárd Jaskó, *Bounds on the 2-domination number*, Discrete Appl. Math. **242** (2018), 4–15, DOI: 10.1016/j.dam.2017.05.014.
- [4] Yair Caro, *On the k-domination and k-transversal numbers of graphs and hypergraphs*, Ars Combin. **29** (1990), 49–55.
- [5] Yair Caro and Yehuda Roditty, *A note on the k-domination number of a graph*, Int. J. Math. Math. Sci. **13** (1990), no. 1, 205–206, DOI: 10.1155/S016117129000031X.
- [6] Odile Favaron, *k-domination and k-independence in graphs*, Ars Combin. **25C** (1988), 159–167.
- [7] Odile Favaron, Adriana Hansberg, and Lutz Volkmann, *On k-domination and minimum degree in graphs*, J. Graph Theory **57** (2008), no. 1, 33–40, DOI: 10.1002/jgt.20279.
- [8] Adriana Hansberg, *On the k-domination number, the domination number and the cycle of length four*, Util. Math. **98** (2015), 65–76.
- [9] Adriana Hansberg and Ryan Pepper, *On k-domination and j-independence in graphs*, Discrete Appl. Math. **161** (2013), no. 10, 1472–1480, DOI: 10.1016/j.dam.2013.02.008.
- [10] Ramy S. Shaheen, *Bounds for the 2-domination number of toroidal grid graphs*, Int. J. Comput. Math. **86** (2009), no. 4, 584–588, DOI: 10.1080/00207160701690284.
- [11] Teresa W. Haynes, Stephen Hedetniemi, and Peter Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1997.
- [12] Teresa W. Haynes, Stephen Hedetniemi, and Peter Slater, *Fundamentals of Domination in Graphs*, CRC Press, Boca Raton, 1998.
- [13] Mustapha Chellali, Odile Favaron, Adriana Hansberg, and Lutz Volkmann, *k-domination and k-independence in graphs: a survey*, Graphs Combin. **28** (2012), no. 1, 1–55, DOI: 10.1007/s00373-011-1040-3.
- [14] Adriana Hansberg, Bert Randerath, and Lutz Volkmann, *Claw-free graphs with equal 2-domination and domination numbers*, Filomat **30** (2016), no. 10, 2795–2801, DOI: 10.2298/FIL1610795H.
- [15] Adriana Hansberg and Lutz Volkmann, *On graphs with equal domination and 2-domination numbers*, Discrete Math. **308** (2008), no. 11, 2277–2281, DOI: 10.1016/j.disc.2007.04.057.
- [16] Gülnaz Boruzanlı Ekinci and Csilla Bujtás, *On the equality of domination number and 2-domination number*, arXiv preprint, arXiv:1907.07866, 2019.
- [17] S. Arumugam, Bibin K. Jose, Csilla Bujtás, and Zsolt Tuza, *Equality of domination and transversal numbers in hypergraphs*, Discrete Appl. Math. **161** (2013), no. 13, 1859–1867, DOI: 10.1016/j.dam.2013.02.009.
- [18] Mostafa Blidia, Mustapha Chellali, and Teresa W. Haynes, *Characterizations of trees with equal paired and double domination numbers*, Discrete Math. **306** (2006), no. 16, 1840–1845, DOI: 10.1016/j.disc.2006.03.061.
- [19] Magda Dettlaff, Magdalena Lemańska, Gabriel Semanišin, and Rita Zuazua, *Some variations of perfect graphs*, Discuss. Math. Graph Theory **36** (2016), no. 3, 661–668, DOI: 10.7151/dmgt.1880.
- [20] Bert Hartnell and Douglas F. Rall, *A characterization of graphs in which some minimum dominating set covers all the edges*, Czechoslovak Math. J. **45** (1995), no. 2, 221–230.
- [21] Marcin Krzywkowski, Michael A. Henning, and Christoph Brause, *A characterization of trees with equal 2-domination and 2-independence numbers*, Discrete Math. Theor. Comput. Sci. **19** (2017), DOI: 10.23638/DMTCS-19-1-1.
- [22] Bert Randerath and Lutz Volkmann, *Characterization of graphs with equal domination and covering number*, Discrete Math. **191** (1998), no. 1–3, 159–169, DOI: 10.1016/S0012-365X(98)00103-4.
- [23] Andrzej Lingas, Mateusz Miotk, Jerzy Topp, and Paweł Żyliński, *Graphs with equal domination and covering numbers*, J. Comb. Optim. **39** (2020), no. 1, 55–71, DOI: 10.1007/s10878-019-00454-6.
- [24] Michael R. Garey and David S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, WH Freeman and Co., New York, 1979.