Research Article

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Jordan $\{g, h\}$ -derivations on triangular algebras

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Abstract: In this article, we give a sufficient and necessary condition for every Jordan $\{g, h\}$ -derivation to be a $\{g, h\}$ -derivation on triangular algebras. As an application, we prove that every Jordan $\{g, h\}$ -derivation on $\tau(\mathcal{N})$ is a $\{g, h\}$ -derivation if and only if $\dim \mathcal{O}_+ \neq 1$ or $\dim H^\perp \neq 1$, where $\mathcal{N}$ is a non-trivial nest on a complex separable Hilbert space H and $\tau(\mathcal{N})$ is the associated nest algebra.

Keywords: triangular algebra, $\{g, h\}$ -derivation, Jordan $\{g, h\}$ -derivation

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1 Introduction

In this article, we assume that all algebras and rings are associative and 2-torsion free. Let $\mathcal{A}$ be an algebra over $\mathcal{R}$, a commutative ring with unit. An $\mathcal{R}$ -linear map $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is said to be a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. It is said to be a Jordan derivation if $\delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y)$ for all $x, y \in \mathcal{A}$, where $x \circ y = xy + yx$. Clearly, every derivation is a Jordan derivation. In general, the converse is not true (see [1]). In which algebras or rings a Jordan derivation is necessarily a derivation is a classical area of research. Herstein [2] showed that every Jordan derivation from a prime ring into itself is a derivation. Brešar [3] proved that Herstein's result is true for semiprime rings. Zhang and Yu [4] proved that every Jordan derivation on triangular algebras is a derivation. Jordan derivations on some rings or algebras are also studied by other authors [5–7].

Let $\delta, \delta_1, \delta_2 : \mathcal{A} \rightarrow \mathcal{A}$ be $\mathcal{R}$ -linear maps. Then, δ is said to be a generalized derivation if

$$\delta(xy) = \delta_1(x)y + x\delta_2(y)$$

for all $x, y \in \mathcal{A}$. The triples $(\delta, \delta_1, \delta_2)$, where δ is a generalized derivation and δ_1, δ_2 are $\mathcal{R}$ -linear maps associated with it, are called ternary derivations. Kaygorodov [8,9] described generalized derivations of semisimple Filippov algebras and ternary Mal'tsev algebras, respectively. Jiménez-Gestal and Pérez-Izquierdo [10] investigated ternary derivations of generalized Cayley-Dickson algebras. Other results related to generalized derivations and ternary derivations have also been obtained, see for example [11–14]. It should be mentioned that Brešar [15] and Nakajima [16] introduced the other two notions of generalized derivations. Komatsu and Nakajima [17] discussed the relations of the aforementioned generalized derivations.

Let $f, g, h : \mathcal{A} \rightarrow \mathcal{A}$ be $\mathcal{R}$ -linear maps. Then, f is said to be a Jordan generalized derivation with an associated map h if

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$$f(x \circ y) = f(x) \circ y + x \circ h(y)$$

for all $x, y \in \mathcal{A}$. Li and Benkovič [18] proved that every Jordan generalized derivation is a generalized derivation on triangular algebras. f is said to be a $\{g, h\}$ -derivation if

$$f(xy) = g(x)y + xh(y) = h(x)y + xg(y)$$

for all $x, y \in \mathcal{A}$. It is clear that if f is a $\{g, h\}$ -derivation, then both (f, g, h) and (f, h, g) are ternary derivations. It is said to be a Jordan $\{g, h\}$ -derivation if

$$f(x \circ y) = g(x) \circ y + x \circ h(y)$$

for all $x, y \in \mathcal{A}$. Clearly, every $\{g, h\}$ -derivation is a Jordan $\{g, h\}$ -derivation. Brešar [19] proved that every Jordan $\{g, h\}$ -derivation of a semiprime unital algebra $\mathcal{A}$ is a $\{g, h\}$ -derivation.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a field F whose characteristic is not two, and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$ -bimodule, which is faithful as a left $\mathcal{A}$ -module and also as a right $\mathcal{B}$ -module. The algebra

$$\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations is called a triangular algebra. Let $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{Z}(\mathcal{U})$ be its center. It follows from [20] that

$$\mathcal{Z}(\mathcal{U}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : am = mb \text{ for all } m \in \mathcal{M} \right\}. \quad (1.1)$$

Let $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ be the identities of the algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and let 1 be the identity of the triangular algebra $\mathcal{U}$. Put

$$p_1 = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}, \quad p_2 = 1 - p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}$$

and

$$\mathcal{U}_{ij} = p_i \mathcal{U} p_j, \quad 1 \leq i \leq j \leq 2.$$

It is clear that $\mathcal{U}$ can be represented as $\mathcal{U} = p_1 \mathcal{U} p_1 + p_1 \mathcal{U} p_2 + p_2 \mathcal{U} p_2$ and so for each $u \in \mathcal{U}$, $u = u_{11} + u_{12} + u_{22}$, $u_{ij} \in \mathcal{U}_{ij}$, $1 \leq i \leq j \leq 2$.

Let H be a complex separable Hilbert space and $B(H)$ denote the algebra of all bounded linear operators acting on H . A nest $\mathcal{N}$ is a totally ordered set of closed subspaces of H containing 0 and H , which is closed under the taking of arbitrary intersections and closed linear spans of its elements. For an element $N \in \mathcal{N}$, let

$$N_+ = \bigwedge \{M \in \mathcal{N} : M \supset N\} \text{ and } N_- = \bigvee \{M \in \mathcal{N} : M \subset N\}.$$

If $N_+ \neq N$ or $N_- \neq N$, then $N_+ \ominus N$ or $N \ominus N_-$ is called an atom of $\mathcal{N}$. A nest is called continuous if it has no atoms. A nest is said to be of infinite multiplicity if it has no finite dimensional atoms. It is clear that a continuous nest is of infinite multiplicity. A nest is said to be non-trivial if it contains at least one non-trivial element. The nest algebra associated with $\mathcal{N}$, denoted by $\tau(\mathcal{N})$, is the set

$$\tau(\mathcal{N}) = \{T \in B(H) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

A nest algebra $\tau(\mathcal{N})$ is called non-trivial if $\mathcal{N}$ is a non-trivial nest. We refer the reader to [21] for the basic theory of nest algebras.

It is known that the triangular algebra is not semiprime. It is natural to ask if every Jordan $\{g, h\}$ -derivation is a $\{g, h\}$ -derivation on triangular algebras. In general, it is not true (see the proof of Theorem 3.1). The purpose of this article is to give a sufficient and necessary condition for every Jordan $\{g, h\}$ -derivation to be a $\{g, h\}$ -derivation on triangular algebras. As an application, we prove that every Jordan $\{g, h\}$ -derivation on non-trivial nest algebras is a $\{g, h\}$ -derivation if and only if $\dim 0_+ \neq 1$ or $\dim H_-^\perp \neq 1$.

2 Main result

To prove the main theorem, we need some lemmas.

Lemma 2.1. *Let $\mathcal{U}$ be a triangular algebra and f be a Jordan $\{g, h\}$ -derivation on $\mathcal{U}$. Then, $f(1) = g(1) + h(1) \in \mathcal{Z}(\mathcal{U})$.*

Proof. Let f be a Jordan $\{g, h\}$ -derivation, i.e.,

$$f(x \circ y) = g(x) \circ y + x \circ h(y) \quad (2.1)$$

for all $x, y \in \mathcal{U}$. The commutativity of the Jordan product gives

$$f(x \circ y) = h(x) \circ y + x \circ g(y) \quad (2.2)$$

for all $x, y \in \mathcal{U}$. Taking $x = y = 1$ in Eq. (2.1), we obtain $f(1) = g(1) + h(1)$.

For any idempotent $p \in \mathcal{U}$, it follows from Eq. (2.1) that

$$\begin{aligned} 0 &= f(p \circ (1 - p)) \\ &= g(p) \circ (1 - p) + p \circ h(1 - p) \\ &= g(p) - g(p)p + g(p) - pg(p) + ph(1) - ph(p) + h(1)p - h(p)p. \end{aligned} \quad (2.3)$$

Multiplying Eq. (2.3) by p from the left, we get

$$0 = pg(p) - ph(p) + ph(1)p - pg(p)p - ph(p)p.$$

Multiplying Eq. (2.3) by p from the right, we get

$$0 = g(p)p - h(p)p + h(1)p + ph(1)p - pg(p)p - ph(p)p.$$

Comparing the aforementioned two equalities, we have

$$pg(p) - ph(p) + ph(1) = g(p)p - h(p)p + h(1)p. \quad (2.4)$$

Similarly, it follows from Eq. (2.2) that

$$ph(p) - pg(p) + pg(1) = h(p)p - g(p)p + g(1)p. \quad (2.5)$$

It follows from Eqs. (2.4) and (2.5) that

$$p(g(1) + h(1)) = (g(1) + h(1))p$$

for every idempotent $p \in \mathcal{U}$. Hence for any $u_{12} \in \mathcal{U}_{12}$, we have

$$(p_1 + u_{12})(g(1) + h(1)) = (g(1) + h(1))(p_1 + u_{12})$$

and

$$p_1(g(1) + h(1)) = (g(1) + h(1))p_1.$$

It follows that

$$p_1(g(1) + h(1))p_1u_{12} = u_{12}p_2(g(1) + h(1))p_2$$

for all $u_{12} \in \mathcal{U}_{12}$ and

$$p_1(g(1) + h(1))p_2 = 0.$$

From Eq. (1.1), we have

$$\begin{aligned} g(1) + h(1) &= p_1(g(1) + h(1))p_1 + p_1(g(1) + h(1))p_2 + p_2(g(1) + h(1))p_2 \\ &= p_1(g(1) + h(1))p_1 + p_2(g(1) + h(1))p_2 \in \mathcal{Z}(\mathcal{U}). \end{aligned}$$

□

Lemma 2.2. Let $\mathcal{U}$ be a triangular algebra and f be a Jordan $\{g, h\}$ -derivation on $\mathcal{U}$. Then, $[g(1), [x, y]] = [h(1), [x, y]] = 0$ for all $x, y \in \mathcal{U}$.

Proof. It follows from the proof of [19, Lemma 4.1] that $[h(1) - g(1), [x, y]] = 0$ for all $x, y \in \mathcal{U}$. On the other hand, we have $[h(1) + g(1), [x, y]] = 0$ for all $x, y \in \mathcal{U}$ by Lemma 2.1. Hence, $[g(1), [x, y]] = [h(1), [x, y]] = 0$ for all $x, y \in \mathcal{U}$. $\square$

Lemma 2.3. Let $\mathcal{U}$ be a triangular algebra and f be a Jordan $\{g, h\}$ -derivation on $\mathcal{U}$. Then, there exist a derivation φ of $\mathcal{U}$, $\omega \in \mathcal{Z}(\mathcal{U})$ and $\omega_1, \omega_2 \in \mathcal{U}$ such that

$$f(x) = \varphi(x) + \omega \circ x, \quad g(x) = \varphi(x) + \omega_1 \circ x, \quad h(x) = \varphi(x) + \omega_2 \circ x$$

for all $x \in \mathcal{U}$.

Proof. Let $\omega = \frac{1}{2}f(1)$, $\omega_1 = \frac{1}{2}g(1)$, $\omega_2 = \frac{1}{2}h(1)$. It follows from Lemma 2.1 that $\omega \in \mathcal{Z}(\mathcal{U})$. Taking $y = 1$ in Eq. (2.1), we obtain $g(x) = f(x) - x\omega_2 - \omega_2x$. Taking $x = 1$ in Eq. (2.1), we obtain $h(y) = f(y) - \omega_1y - y\omega_1$. It follows from Lemma 2.2 that for all $x, y \in \mathcal{U}$, we have

$$\begin{aligned} f(x \circ y) &= g(x) \circ y + x \circ h(y) \\ &= (f(x) - x\omega_2 - \omega_2x) \circ y + x \circ (f(y) - \omega_1y - y\omega_1) \\ &= f(x) \circ y + x \circ f(y) - x\omega y - y\omega x - \omega_2xy - \omega_1yx - xy\omega_1 - yx\omega_2 \\ &= f(x) \circ y + x \circ f(y) - \omega xy - \omega yx - (\omega - \omega_1)xy - \omega_1yx - xy\omega_1 - yx(\omega - \omega_1) \\ &= f(x) \circ y + x \circ f(y) - 2\omega(x \circ y) + \omega_1[x, y] - [x, y]\omega_1 \\ &= f(x) \circ y + x \circ f(y) - 2\omega(x \circ y) + [\omega_1, [x, y]] \\ &= f(x) \circ y + x \circ f(y) - 2\omega(x \circ y). \end{aligned}$$

Therefore,

$$f(x \circ y) - 2\omega(x \circ y) = (f(x) - 2\omega x) \circ y + x \circ (f(y) - 2\omega y). \quad (2.6)$$

We define a map $\varphi : \mathcal{U} \rightarrow \mathcal{U}$ by

$$\varphi(x) = f(x) - 2\omega x$$

for all $x \in \mathcal{U}$. Then, $\varphi(x \circ y) = \varphi(x) \circ y + x \circ \varphi(y)$ for all $x, y \in \mathcal{U}$ by Eq. (2.6), and so φ is a Jordan derivation on $\mathcal{U}$. Therefore, φ is a derivation by [4, Theorem 2.1] and

$$f(x) = \varphi(x) + 2\omega x = \varphi(x) + \omega \circ x. \quad (2.7)$$

It follows from Eq. (2.7) that

$$f(xy) = f(x)y + x\varphi(y) = \varphi(x)y + xf(y),$$

which implies that f is a $\{f, \varphi\}$ -derivation. Hence, f is a Jordan $\{f, \varphi\}$ -derivation, i.e.,

$$f(x \circ y) = f(x) \circ y + x \circ \varphi(y).$$

On the other hand,

$$f(x \circ y) = g(x) \circ y + x \circ h(y).$$

Comparing the aforementioned two equalities, we have

$$(g(x) - f(x)) \circ y = x \circ (\varphi(y) - h(y)). \quad (2.8)$$

Taking $y = 1$ in Eq. (2.8), we obtain $g(x) = f(x) - \omega_2 \circ x = \varphi(x) + \omega_1 \circ x$. Similarly, taking $x = 1$ in Eq. (2.8), we obtain $h(y) = \varphi(y) + \omega_2 \circ y$. $\square$

The main result of this article is as follows.

Theorem 2.1. Let $\mathcal{U}$ be a triangular algebra. Then, every Jordan $\{g, h\}$ -derivation f on $\mathcal{U}$ is a $\{g, h\}$ -derivation if and only if $g(1) \in \mathcal{Z}(\mathcal{U})$ or $h(1) \in \mathcal{Z}(\mathcal{U})$.

Proof. We assume $h(1) \in \mathcal{L}(\mathcal{U})$ without loss of generality by Lemma 2.1. It follows from Lemma 2.3 that

$$\begin{aligned} f(xy) &= \varphi(xy) + \omega \circ (xy) \\ &= \varphi(x)y + x\varphi(y) + \omega \circ (xy) \\ &= (g(x) - \omega_1 \circ x)y + x(h(y) - \omega_2 \circ y) + \omega \circ (xy) \\ &= g(x)y + xh(y) + [\omega_2, xy] \\ &= g(x)y + xh(y). \end{aligned}$$

Similarly,

$$\begin{aligned} f(xy) &= \varphi(xy) + \omega \circ (xy) \\ &= \varphi(x)y + x\varphi(y) + \omega \circ (xy) \\ &= (h(x) - \omega_2 \circ x)y + x(g(y) - \omega_1 \circ y) + \omega \circ (xy) \\ &= h(x)y + xg(y) + [xy, \omega_2] \\ &= h(x)y + xg(y). \end{aligned}$$

Therefore, f is a $\{g, h\}$ -derivation.

Conversely, it follows from the definition of $\{g, h\}$ -derivations that $g(1) \in \mathcal{L}(\mathcal{U})$ and $h(1) \in \mathcal{L}(\mathcal{U})$. $\square$

3 An application

As a notational convenience, if N is a closed subspace of H , let $P(N)$ denote the orthogonal projection from H onto N .

Theorem 3.1. *Let $\mathcal{N}$ be a non-trivial nest on a complex separable Hilbert space H . Then, every Jordan $\{g, h\}$ -derivation on $\tau(\mathcal{N})$ is a $\{g, h\}$ -derivation if and only if $\dim \mathcal{O}_+ \neq 1$ or $\dim H_-^\perp \neq 1$.*

Proof. Let f be a Jordan $\{g, h\}$ -derivation. It follows from Lemma 2.2 that

$$[h(1), [X, Y]] = 0 \quad (3.1)$$

for all $X, Y \in \tau(\mathcal{N})$. Next, we will prove $h(1) \in \mathcal{L}(\tau(\mathcal{N}))$, and so f is a $\{g, h\}$ -derivation by Theorem 2.1.

Case 1: If $\dim \mathcal{O}_+ = 0$, then $\mathcal{O}_+ = 0$ and so there exists a decreasing sequence N_n of closed subspaces in $\mathcal{N} \setminus \{0, H\}$ such that $P_n = P(N_n)$ converges strongly to 0. For any $T \in B(H)$, since $P_1 T P_1^\perp \in \tau(\mathcal{N})$, taking $X = P_1, Y = P_1 T P_1^\perp$ in Eq. (3.1), we obtain

$$[h(1), P_1 T P_1^\perp] = 0.$$

Hence, $P_1 h(1) P_1 + P_1^\perp h(1) P_1^\perp \in \mathcal{L}(\tau(\mathcal{N}))$. Since $P_n T (P_1 - P_n) \in \tau(\mathcal{N})$, taking $X = P_n, Y = P_n T (P_1 - P_n)$ ($n \geq 2$) in Eq. (3.1), we obtain

$$0 = [h(1), P_n T (P_1 - P_n)] = [P_1 h(1) P_1^\perp, P_n T (P_1 - P_n)] = -P_n T (P_1 - P_n) P_1 h(1) P_1^\perp.$$

It follows that $P_n B(H) (P_1 - P_n) P_1 h(1) P_1^\perp = \{0\}$ and so $(P_1 - P_n) P_1 h(1) P_1^\perp = 0$. Letting $n \rightarrow \infty$, we get $P_1 h(1) P_1^\perp = 0$. Hence, $h(1) \in \mathcal{L}(\tau(\mathcal{N}))$.

Case 2: If $\dim H_-^\perp = 0$, then $H_-^\perp = H$ and so there exists an increasing sequence M_n of closed subspaces in $\mathcal{N} \setminus \{0, H\}$ such that $Q_n = P(M_n)$ converges strongly to I . For any $T \in B(H)$, taking $X = Q_1, Y = Q_1 T Q_1^\perp$ in Eq. (3.1), we have $Q_1 h(1) Q_1 + Q_1^\perp h(1) Q_1^\perp \in \mathcal{L}(\tau(\mathcal{N}))$. Since $(Q_n - Q_1) T Q_n^\perp \in \tau(\mathcal{N})$, taking $X = Q_n, Y = (Q_n - Q_1) T Q_n^\perp$ ($n \geq 2$) in Eq. (3.1), we obtain

$$0 = [h(1), (Q_n - Q_1) T Q_n^\perp] = [Q_1 h(1) Q_1^\perp, (Q_n - Q_1) T Q_n^\perp] = Q_1 h(1) Q_1^\perp (Q_n - Q_1) T Q_n^\perp.$$

It follows that $Q_1 h(1) Q_1^\perp (Q_n - Q_1) B(H) Q_n^\perp = \{0\}$ and so $Q_1 h(1) Q_1^\perp (Q_n - Q_1) = 0$. Letting $n \rightarrow \infty$, we get $Q_1 h(1) Q_1^\perp = 0$. Hence, $h(1) \in \mathcal{L}(\tau(\mathcal{N}))$.

Case 3: If $\dim O_+ > 1$, let $P = P(O_+)$. For any $T \in B(H)$, taking $X = P$, $Y = PTP^\perp$ in Eq. (3.1), we have $Ph(1)P + P^\perp h(1)P^\perp \in \mathcal{L}(\tau(\mathcal{N}))$. Since $PB(H) \subset \tau(\mathcal{N})$, we have from Eq. (3.1) that

$$0 = [h(1), [PTP, PSP]] = [Ph(1)P^\perp, [PTP, PSP]] = -[PTP, PSP]Ph(1)P^\perp$$

for any $T, S \in B(H)$. Hence,

$$[PB(H)P, PB(H)P]Ph(1)P^\perp = \{0\}. \quad (3.2)$$

If $\dim O_+ = n$, then $PB(H)P$ is isomorphic to $M_n(\mathbb{C})$. Let E_{ij} ($i, j = 1, 2, \dots, n$) be the matrix unit of order n whose (i, j) position is 1 and all other positions are 0. If n is odd, let

$$S_1 = E_{11} - E_{\frac{n+1}{2}, \frac{n+1}{2}} + \sum_{(i, n-i+1) \neq (\frac{n+1}{2}, \frac{n+1}{2})} E_{in-i+1}.$$

Then, S_1 is invertible in $M_n(\mathbb{C})$ and

$$S_1 = \left[E_{1\frac{n+1}{2}}, E_{\frac{n+1}{2}1} \right] + \sum_{(i, n-i+1) \neq (\frac{n+1}{2}, \frac{n+1}{2})} [E_{ii}, E_{in-i+1}].$$

If n is even, let

$$S_2 = E_{11} - E_{22} + \sum_{i=1}^n E_{in-i+1}.$$

Then, S_2 is invertible in $M_n(\mathbb{C})$ and

$$S_2 = [E_{12}, E_{21}] + \sum_{i=1}^n [E_{ii}, E_{in-i+1}].$$

Therefore, there exist $A_i, B_i \in M_n(\mathbb{C})$ ($i = 1, 2, \dots, l$) such that $\sum_{i=1}^l [A_i, B_i] = S$ is invertible in $M_n(\mathbb{C})$. It follows from Eq. (3.2) that $Ph(1)P^\perp = 0$ and so $h(1) \in \mathcal{L}(\tau(\mathcal{N}))$.

If $\dim O_+ = +\infty$, since $[PB(H)P, PB(H)P] = PB(H)P$, it follows from Eq. (3.2) that $Ph(1)P^\perp = 0$ and so $h(1) \in \mathcal{L}(\tau(\mathcal{N}))$.

Case 4: If $\dim H_-^\perp > 1$, let $Q = P(H_-)$. For any $T \in B(H)$, taking $X = Q$, $Y = QTQ^\perp$ in Eq. (3.1), we have $Qh(1)Q + Q^\perp h(1)Q^\perp \in \mathcal{L}(\tau(\mathcal{N}))$. Since $B(H)Q^\perp \subset \tau(\mathcal{N})$, we have from Eq. (3.1)

$$0 = [h(1), [Q^\perp TQ^\perp, Q^\perp S Q^\perp]] = [Qh(1)Q^\perp, [Q^\perp TQ^\perp, Q^\perp S Q^\perp]] = Qh(1)Q^\perp [Q^\perp TQ^\perp, Q^\perp S Q^\perp]$$

for any $T, S \in B(H)$. Hence, $Qh(1)Q^\perp [Q^\perp B(H)Q^\perp, Q^\perp B(H)Q^\perp] = \{0\}$. A similar proof to Case 3 gives $h(1) \in \mathcal{L}(\tau(\mathcal{N}))$.

Conversely, if $\dim O_+ = 1$ and $\dim H_-^\perp = 1$, then $O_+ = H_-^\perp = \mathbb{C}$. Let $H_0 = H \ominus (\mathbb{C} \oplus \mathbb{C})$ and so for every $X \in \tau(\mathcal{N})$, X has the following form:

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{pmatrix},$$

where $X_{11}, X_{13}, X_{33} \in \mathbb{C}$, $X_{12} \in B(H_0, \mathbb{C})$, $X_{23} \in B(\mathbb{C}, H_0)$ and $X_{22} \in B(H_0)$. For each $X = (X_{ij}) \in \tau(\mathcal{N})$, we define

$$f(X) = \begin{pmatrix} 2X_{11} & X_{12} & \frac{3}{2}(X_{11} - X_{33}) \\ 0 & 2X_{22} & X_{23} \\ 0 & 0 & 2X_{33} \end{pmatrix},$$

$$g(X) = \begin{pmatrix} X_{11} & 0 & X_{11} - 2X_{33} - X_{13} \\ 0 & X_{22} & 0 \\ 0 & 0 & X_{33} \end{pmatrix}$$

and

$$h(X) = \begin{pmatrix} X_{11} & 0 & 2X_{11} - X_{33} - X_{13} \\ 0 & X_{22} & 0 \\ 0 & 0 & X_{33} \end{pmatrix}.$$

It is easy to check that f is a Jordan $\{g, h\}$ -derivation. However, f is not a $\{g, h\}$ -derivation. If it was, then

$$\begin{aligned} \begin{pmatrix} 2 & 0 & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \\ &= g \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} h \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

which is a contradiction. □

As an immediate result by Theorem 3.1, we have the following corollary.

Corollary 3.1. Let $\mathcal{N}$ be a non-trivial nest of infinite multiplicity. Then, every Jordan $\{g, h\}$ -derivation on $\tau(\mathcal{N})$ is a $\{g, h\}$ -derivation.

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