Research Article

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A new characterization of $L_2(p^2)$
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Abstract: For a positive integer n and a prime p , let n_p denote the p -part of n . Let G be a group, $\text{cd}(G)$ the set of all irreducible character degrees of G , $\rho(G)$ the set of all prime divisors of integers in $\text{cd}(G)$, $V(G) = \{p^{e_p(G)} \mid p \in \rho(G)\}$, where $p^{e_p(G)} = \max \{\chi(1)_p \mid \chi \in \text{Irr}(G)\}$. In this article, it is proved that $G \cong L_2(p^2)$ if and only if $|G| = |L_2(p^2)|$ and $V(G) = V(L_2(p^2))$.

Keywords: irreducible characters, degree, maximal p -part, simple group, characterization

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1 Introduction

In what follows, the notations are the same as in [1]. Let G be a finite group, $|G|$ denotes the order of G , $\text{Irr}(G)$ set of all irreducible characters of G , and $\text{cd}(G)$ denotes the set of irreducible character degrees of G . For a positive integer n , and a prime p , n_p denotes the p -part of n . And we define the maximal p -part of character degrees in $\text{cd}(G)$ as $p^{e_p(G)} = \max \{\chi(1)_p \mid \chi \in \text{Irr}(G)\}$. For convenience, we set $v_p(G) = \log_p(|G|_p) - e_p(G)$, the difference between the power of p -part of $|G|$ and $e_p(G)$. Let H and G be two groups, we use $H \leq G$ to denote that H is isomorphic to a subgroup of G . Let m be an integer, n a positive integer, p a prime, $p^n \parallel m$ means that p^n divides m but p^{n+1} does not divide m , i.e., $p^n \mid m, p^{n+1} \nmid m$.

In recent years, the relationship between irreducible character degrees and structure of G was widely studied. The famous open question on this topic is the Huppert conjecture, which was proposed by B. Huppert in 2000 (see [2]).

Huppert's Conjecture: If S is a non-abelian simple group such that $\text{cd}(G) = \text{cd}(S)$, then $G \cong S \times A$, where A is an abelian group.

Although the Huppert conjecture is not completely proved, it has been shown that it is correct for many simple groups. An interesting fact is that if Huppert's conjecture has been proved, then it is natural that all finite non-abelian simple groups can be uniquely determined by their orders and the sets of irreducible character degrees. Another interesting fact is that if the condition $|G| = |S|$ is given, then it is easier to get the same conclusion and it need not consider all numbers in $\text{cd}(S)$. In other words, some simple groups may be uniquely determined by using their orders and some special degrees of complex irreducible characters. For example, in [3–5], it is shown that there are simple K_3 groups, K_4 groups, and Mathieu groups that can be uniquely determined by their orders and at most three distinct irreducible character degrees, and in [6], simple exceptional groups of lie type can be determined by their character degrees.

The research related to Huppert's conjecture and focusing on $\text{cd}(S)$ is to investigate if the character degree graph of S and its order $|S|$ can uniquely determine S .

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The character degree graph $\Gamma(G)$ is a graph whose vertices are all primes dividing the character degrees of G , the set of all vertices is denoted by $\rho(G)$, and two vertices are joined by an edge if the product of them divides some number in $\text{cd}(G)$. It has been proved that many simple groups were determined by their orders and character degree graph. For example, in [7], Khosravi et al. proved that $A_5, A_6, A_7, A_8, L_3(3), L_3(4), L_2(64), L_2(q)$, where q is an odd prime or a square of an odd prime, and $q \geq 5$, and $L_2(2^\alpha)$ (where α is a positive integer such that $2^\alpha - 1$ or $2^\alpha + 1$ is a prime) can be uniquely determined by their orders and character degree graphs.

However, not all the simple groups could be uniquely determined by their orders and character degree graphs. In fact, M_{12} and $A_4 \times M_{11}$ have the same order and character degree graph (see [8]). In order to overcome this shortcoming, Chao Qin and Guiyun Chen defined a new graph in [9], called the degree prime-power graph, whose vertices are $p^{e_p(G)}$, where $p \in \rho(G)$, and two vertices are joined by an edge if their product divides some irreducible character degree in $\text{cd}(G)$.

It is worth mentioning that M_{12} can be uniquely determined by its order and degree prime-power graph, but it cannot be uniquely determined by its order and degree graph. Moreover, it is proved that all sporadic simple groups, $L_2(p), L_2(p-1)$ (p is a Fermat prime), are determined by their orders and degree prime-power graphs by Qin et al. in [9] and his doctoral dissertation.

In this article, we continue this topic to study if $L_2(p^2)$ can be uniquely determined by its order and degree prime-power graph. In fact, we come to a more general result, which is the following theorem.

Main Theorem. *Let $p \geq 5$ be a prime, then $G \cong L_2(p^2)$ if and only if $|G| = |L_2(p^2)|$ and $V(G) = V(L_2(p^2))$.*

2 The proof of the Main Theorem

In order to prove the Main Theorem, we first present some lemmas which are useful in our proof.

Lemma 2.1. [9, Lemma 2.2] *Suppose that for every $p \in \pi(G)$, $v_p(G) = 0$, that is, for every $p \in \pi(G)$, there always exists an irreducible character χ such that $\chi(1)_p = |G|_p$. Then, G is non-solvable, moreover every minimal subnormal subgroup of G is a non-abelian simple group.*

Lemma 2.2. [3, Lemma 1] *Let G be a non-solvable group, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $G/K \leq \text{Out}(K/H)$.*

Lemma 2.3. *Let a, m, n be three positive integers, then the greatest common divisor $(a^m - 1, a^n - 1) = a^{(m,n)} - 1$.*

Proof. This follows by the Euclid algorithm. □

Lemma 2.4. *Let $p \geq 5$ be a prime, n a positive integer, and q a power of a prime different from p , then the following hold.*

- (1) *Let $h = \prod_{i=1}^n (q^i - 1)$ such that $p^2 \nmid h$, then $p^2 \leq q^n - 1$.*
- (2) *Let $h = \prod_{i=1}^n (q^{2i} - 1)$ such that $p^2 \nmid h$, then $p^2 \leq q^{2n} - 1$.*

Proof. (1) On the contrary, $p^2 > q^n - 1$. By $p^2 \nmid h$, there exists $1 \leq i < j \leq n$ such that $p \nmid q^i - 1$, $p \nmid q^j - 1$, and we assume that i is the smallest positive integer such that $p \nmid q^i - 1$. Hence, $p \nmid q^{(i,j)} - 1$ by Lemma 2.3, it follows that $j = 2i$ and $2i \leq n < 3i$. Therefore, $q^n - 1 < p^2 \leq (q^i - 1)^2 \leq (q^{\frac{n}{2}} - 1)^2$, a contradiction, so (1) follows. In the same manner, we can show that (2) follows. □

Lemma 2.5. *Let p, r be two different primes, $p \geq 5$, q is a power of r , then there is no positive integer s to meet all the following requirements:*

- (1) $s = q^b \cdot u_1 \cdot u_2 \cdots u_l$ such that $(r, u_i) = 1$ and (u_i, u_j) is a power of 2, where $1 \leq i \neq j \leq l$;
- (2) for every $1 \leq i \leq l$, $u_i < q^b$; and
- (3) $p^2 \nmid s$ and $s \mid \frac{p^2(p^b - 1)}{2}$.

Proof. Since $p^2 \parallel s$ and $p \neq r$, $p \geq 5$, we can conclude from (1) that $p^2 \parallel u_i$ for some $1 \leq i \leq l$. According to (2), we have $p^2 < q^b$. On the other hand, $s \mid \frac{p^2(p^4-1)}{2}$ implies that $q^b \mid \frac{p^2+1}{2}$ or $q^b \mid p^2 - 1$, in either case, we get that $q^b < p^2$, a contradiction to $p^2 < q^b$. $\square$

Lemma 2.6. Let p, r be two different primes, $p \geq 5$, q is a power of r , then there is no positive integer s to meet all the following requirements:

- (1) $s = q^b \cdot u_1 \cdot u_2 \cdots u_l$ such that $(r, u_i) = 1$ and (u_i, u_j) is a power of 2, where $1 \leq i \neq j \leq l$;
- (2) for every $1 \leq i \leq l$, $u_i < q^b$; and
- (3) $p \parallel s$ and $s^2 \mid \frac{p^2(p^4-1)}{2}$.

Proof. With the same argument as Lemma 2.5, we can prove the lemma holds. $\square$

Lemma 2.7. Let $p \geq 5$ be a prime, then there is no non-abelian simple group S such that $p \parallel |S|$, $|S| \mid p(p^4 - 1)$.

Proof. This follows from [10, Theorem 2.7, Step 2, (A)]. $\square$

Now we start to prove the Main Theorem.

Proof of the Main Theorem. The necessity is obvious. It is enough to prove the sufficiency.

Because $\text{cd}(L_2(p^2)) = \left\{1, p^2, p^2 - 1, \frac{p^2+1}{2}, p^2 + 1\right\}$ and $|G| = |L_2(p^2)| = \frac{1}{2}p^2(p^2 - 1)(p^2 + 1)$, we have that for every $r \in \pi(G)$, $v_r(G) = 0$. By Lemma 2.1, we get that G is non-solvable, and the minimal subnormal subgroup of G is a non-abelian simple group. Furthermore, it follows from Lemma 2.2 that there exists some non-abelian simple group S , such that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, $K/H = \underbrace{S \times S \times \cdots \times S}_m$, and $G/K \leq \text{Out}(K/H)$.

Step 1 To prove that $p \nmid |G/K|$.

On the contrary, $p \mid |G/K|$. Since $|G/K| \mid |\text{Out}(K/H)|$, we have $p \mid |\text{Out}(K/H)|$. Note that $\text{Out}(K/H) \cong \text{Out}(S) \wr S_m$, we deduce that $p \mid |S_m|$ or $p \mid |\text{Out}(S)|$. If $p \mid |S_m|$, then $p \leq m$, further,

$$|G| = \frac{1}{2}p^2(p^2 - 1)(p^2 + 1) \geq |K/H| = |S|^m \geq 60^p,$$

which is impossible. Therefore, $p \nmid |\text{Out}(S)|$. By [11] and $p \geq 5$, we know that S can only be a simple group of Lie type over $GF(q)$, where $q = r^f$, r is a prime. Thus, $p \mid |\text{Out}(S)| = dfg$, where d, f , and g are the orders of diagonal, field, and graph automorphisms of S , respectively. On the other hand, observing the order of a simple group of Lie type over $GF(q)$, we get that $\frac{1}{2}q(q^2 - 1) \leq |S|$. If $p \mid f$, then $\frac{1}{2} \cdot 2^p(4^p - 1) \leq S \leq \frac{p^2(p^4-1)}{2}$, $2^p(4^p - 1) \leq p^2(p^4 - 1)$, which contradicts $p \geq 5$. Now by $g \leq 3$, we get that $p \mid d$. By [11], we conclude that $d \leq 4$ for all the simple groups of Lie type except $A_n(q)$ and ${}^2A_n(q)$.

If $S \cong A_n(q)$, then $d = (n + 1, q - 1)$, so $p \mid q - 1$. Since $p \geq 5$, we have $n \geq 4$, which follows that $(q - 1)^4 \mid |A_n(q)|$. Therefore, $p^4 \mid |S|$, a contradiction to $p^2 \parallel |G|$.

If $S \cong {}^2A_n(q)$, then $d = (n + 1, q + 1)$, thus $p \mid q + 1$. Similarly, we get that $n \geq 4$, $(q + 1)^4 \mid |A_n(q)|$, $p^4 \mid |S|$, a contradiction. Consequently, $p \nmid |G/K|$. Therefore, $p \mid |H|$ or $p \mid |K/H|$.

Step 2 To prove that K has a composition factor $N \cong S$ or $N \cong S \times S$, such that either $N \cong S$, $p^2 \mid |S|$, $|S| \mid \frac{p^2(p^4-1)}{2}$ or $N \cong S \times S$, $p \parallel |S|$, $|S|^2 \mid \frac{p^2(p^4-1)}{2}$, where S is a non-abelian simple group.

By Step 1, one has that either $p \mid |H|$ or $p \nmid |H|$.

If $p \mid |H|$, then there are two possibilities: $p \parallel |H|$ and $p^2 \parallel |H|$. If $p \parallel |H|$, then $p \parallel |K/H|$. Hence, K/H is a non-abelian simple group, and $|K/H| \mid \frac{p(p^4-1)}{2}$. But this is impossible by Lemma 2.7. Now we have $p^2 \parallel |H|$.

Since for every $r \in \pi(G)$, $v_r(G) = 0$, it follows that $v_r(H) = 0$, therefore, H is non-solvable. Again by Lemma 2.2, H has a normal series $1 \trianglelefteq H_2 \trianglelefteq H_1 \trianglelefteq H$ such that H_1/H_2 is a direct product of t copies of a non-abelian simple group and $H/H_1 \leq \text{Out}(H_1/H_2)$. Now with the same argument above, we obtain that $p^2 \parallel |H_2|$. If we repeat the methods above, then we come to that G has a minimal subnormal subgroup N such that $p^2 \parallel |N|$, $|N| \mid \frac{p^2(p^4-1)}{2}$ and N is the direct product of isomorphic non-abelian simple groups. Therefore, $N \cong S$ or $N \cong S \times S$, where S is a non-abelian simple group. That is, either $N \cong S$, $p^2 \parallel |S|$, $|S| \mid \frac{p^2(p^4-1)}{2}$ or $N \cong S \times S$, $p \parallel |S|$, $|S|^2 \mid \frac{p^2(p^4-1)}{2}$.

If $p \nmid |H|$, then $p^2 \parallel |K/H|$. But $K/H = \underbrace{S \times S \times \cdots \times S}_m$, which concludes $K/H \cong S$ or $S \times S$, where S is a non-abelian simple group, Step 2 holds.

Step 3 Suppose that $N = S$, $p^2 \parallel |S|$ and $|S| \mid \frac{p^2(p^4-1)}{2}$, then $G \cong L_2(p^2)$.

Using the classification of finite simple groups, we consider all simple groups such that

$$p^2 \parallel |S|, |S| \mid \frac{p^2(p^4-1)}{2}. \quad (*)$$

If S is a sporadic simple group, comparing the orders of sporadic simple groups with $|G|$, we get that there are no sporadic simple groups satisfying $(*)$.

If $S \cong A_n$, then $p^2 \parallel \frac{n!}{2}$ and $n! \mid p^2(p^4-1)$, which implies that $2p \leq n$, so that $(2p)! \mid p^2(p^4-1)$, hence, $p^p < (2p)! \leq p^2(p^4-1) < p^6$. By $p \geq 5$, we get that $p = 5$, a contradiction to $(2p)! \mid p^2(p^4-1)$.

Now suppose that S is a simple group of Lie type over a field of order q , where $q = r^f$. If $r = p$, then comparing the orders of simple groups of Lie type with $|G|$, we easily get a contradiction unless $S \cong A_1(p^2)$. If $S \cong A_1(p^2)$, then $|S| = |G| = \frac{p^2(p^4-1)}{2}$, moreover, $G \cong L_2(p^2)$.

Now we assume that S is one of the simple groups of Lie type over a field of order q , where $q = r^f$ and $p \neq r$. In the following, we shall eliminate the possibilities of S one by one.

(1) To prove that $S \not\cong A_n(q)$ ($n \geq 1$).

Assume $S \cong A_n(q)$, let $k = \prod_{i=1}^n (q^{i+1} - 1)$. Since $|S| = |A_n(q)| = q^{\frac{n(n+1)}{2}} k d^{-1}$, $d = (n+1, q-1)$, $p^2 \parallel |S|$, and $p \neq r$, we have $p^2 \nmid k$. By Lemma 2.4(1), $p^2 \leq q^{n+1} - 1$. And because $q^{i-1} < q^i - 1$, $d < q$, we get

$$q^{-1} \cdot q^{\frac{n(n+1)}{2}} \cdot q^{1+2+\cdots+n} < |S| < \frac{p^6}{2} \leq \frac{(q^{n+1}-1)^3}{2} < q^{3(n+1)}.$$

Consequently, $n \leq 3$.

a. If $n = 1$, we have $|S| = q(q+1)(q-1)d^{-1}$. By $p^2 \parallel |S|$ and $p \neq r$, we come to that $p^2 \mid q+1$ or $p^2 \mid q-1$. According to $|S| \mid |G|$, we get that $q \mid p^2 - 1$ or $q \mid \frac{p^2+1}{2}$, but in each case, we obtain that $q < p^2$. Therefore, $p^2 \mid q-1$ is impossible, which means $p^2 \mid q+1$.

Suppose that $q \mid p^2 - 1$ and $p^2 \mid q+1$ hold, then we may set $p^2 = qt + 1$, $q = p^2m - 1$. Hence, $p^2 = qt + 1 = (p^2m - 1)t + 1$, from which it follows that $p^2 \mid t - 1$. If $t > 1$, then $p^2 + 1 \leq t$. Therefore, $p^2 = (p^2m - 1)t + 1 \geq (p^2 - 1)(p^2 + 1) + 1 = p^4$, a contradiction. If $t = 1$, then $p^2 = q + 1$. Consequently, $|S| = (p^2 - 1)p^2(p^2 - 2)$, by $|S| \mid |G|$, we get that $p^2 - 2 \mid \frac{p^2+1}{2}$, which implies $p^2 - 2 \leq \frac{p^2+1}{2}$, as a result $p^2 \leq 5$, a contradiction to $p \geq 5$.

Suppose that $q \mid \frac{p^2+1}{2}$ and $p^2 \mid q+1$, then $2q - 1 \leq p^2 \leq q + 1$, which forces $q = 2$, contradicting $q \mid \frac{p^2+1}{2}$ since $\frac{p^2+1}{2}$ is an odd number.

- b. If $n = 2$, we have $|S| = q^3(q-1)^2(q+1)(q^2+q+1)d^{-1}$. By $p^2 \mid |S|$, we obtain that: $p^2 \mid (q-1)^2$ or $p^2 \mid q+1$ or $p^2 \mid 1+q+q^2$. Since for each case above we have $p^2 < q^3$, by Lemma 2.5 we get a contradiction. In the same way, we can get $n = 3$ is impossible.

(2) To prove that $S \neq B_n(q) (n \geq 2)$, $C_n(q) (n \geq 3)$.

If $S \cong B_n(q)$, let $k = \prod_{i=1}^n (q^{2i} - 1)$, then $|S| = |B_n(q)| = q^{n^2}kd^{-1}$, $d = (2, q-1)$. We conclude that $p^2 \mid k$ since $p^2 \nmid |S|$. By Lemma 2.4(2), $p^2 \leq q^{2n} - 1$. Note that $d \leq 2$, we have

$$\frac{1}{2}q^{n^2} \cdot q^{2(1+2+\dots+n)} < |S| < \frac{p^6}{2} \leq \frac{(q^{2n-1})^3}{2} < \frac{q^{6n}}{2},$$

which implies $n = 2$ and that $p^2 \leq q^4 - 1 < q^4$. But $|S| \mid |G|$ implies that $q^4 \mid \frac{p^2+1}{2}$ or $q^4 \mid p^2 - 1$, in either case, we have $q^4 < p^2$, contradicting $p^2 < q^4$. We can get that $S \neq C_n(q)$ since $|C_n(q)| = |B_n(q)|$.

(3) To prove that $S \neq D_n(q) (n \geq 4)$.

If $S \cong D_n(q)$, we set $k = \prod_{i=1}^{n-1} (q^{2i} - 1)$, then $|S| = |D_n(q)| = q^{n(n-1)}(q^n - 1)kd^{-1}$, $d = (4, q^n - 1)$. One has that $p^2 \mid (q^n - 1)k$. By Lemma 2.4(2), $p^2 \leq q^{2n} - 1$. Note that $d \leq q^n - 1$, therefore,

$$q^{n(n-1)} \cdot (q^n - 1) \cdot (q^n - 1)^{-1} \cdot q^{(n-1)^2} < |S| < \frac{p^6}{2} \leq \frac{(q^{2n-1})^3}{2} < q^{6n},$$

which implies $n = 4$. At this moment, $|S| = q^{12}(q-1)^4(q+1)^4(q^2+1)^2(q^2-q+1)(q^2+q+1)d^{-1}$, we get that p^2 divides one of $(q^2+1)^2$, q^2-q+1 , and q^2+q+1 . Each case above implies that $p^2 + p^{12}$. By Lemma 2.5, we easily get a contradiction.

(4) To prove that $S \neq G_2(q)$.

If $S \cong G_2(q)$, then $|S| = |G_2(q)| = q^6(q^6-1)(q^2-1)$, moreover, $p^2 \mid (q^6-1)(q^2-1)$, it follows that p^2 divides one of $(q+1)^2$, $(q-1)^2$, q^2+q+1 , and q^2-q+1 . Since for each case above we have $p^2 < q^6$, by Lemma 2.5 we get a contradiction.

(5) To prove that $S \neq F_4(q)$.

If $S \cong F_4(q)$, then $|S| = |F_4(q)| = q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$ and $p^2 \mid (q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$. Similarly, $p^2 \leq q^{12} - 1$. Hence, $q^{24} \cdot q^{11} \cdot q^7 \cdot q^5 \cdot q < |S| < \frac{p^6}{2} \leq \frac{(q^{12}-1)^3}{2} < q^{36}$, consequently, $q^{48} < |S| < q^{36}$, a contradiction.

(6) Let S be one of the groups: $E_6(q)$, $E_7(q)$, and $E_8(q)$, we can get a contradiction in the same way as in (5).

(7) To prove that $S \neq {}^2A_n(q) (n \geq 2)$.

If $S \cong {}^2A_n(q)$, put $k = \prod_{i=1}^n (q^{i+1} - (-1)^{i+1})$, then $|S| = |{}^2A_n(q)| = q^{\frac{n(n+1)}{2}}kd^{-1}$, $d = (n+1, q+1)$ and $p^2 \mid k$. Clearly, $p \leq q^{n+1} + 1$, $d \leq q+1$. Thus, $q^{\frac{n(n+1)}{2}}q^2+\dots+n < \frac{kq^{\frac{n(n+1)}{2}}}{q+1} \leq |S| < \frac{p^6}{2} < \frac{(q^{n+1}+1)^6}{2} < q^{6(n+2)}$. Consequently, $n \leq 6$.

- a. If $n = 2$, then $|S| = q^3(q^2-1)(q^3+1) = q^3(q-1)(q+1)^2(q^2-q+1)d^{-1}$. Since $p^2 \nmid |S|$, it follows that $p^2 \nmid q-1$ or $p^2 \nmid (q+1)^2$ or $p^2 \nmid q^2-q+1$. Both $p^2 \nmid q-1$ and $p^2 \nmid q^2-q+1$ imply that $p^2 < q^3$, by Lemma 2.5 we get a contradiction. If $p^2 \nmid (q+1)^2$, then $p \nmid q+1$, $p-1 \leq q$. Since $|S| \mid |G|$, we get that $q^3 \mid \frac{p^2-1}{2}$ or $q^3 \mid p^2-1$, but in either case, we always have $q^3 < p^2$. Consequently, we get a contradiction by $(p-1)^3 \leq q^3 < p^2$.
- b. If $n = 3$, then $|S| = q^6(q^2-1)(q^3+1)(q^4-1)d^{-1}$. Because $p^2 \nmid |S|$, it follows that $p^2 \nmid (q^2-1)(q^3+1)(q^4-1)$. Consequently, one of the cases $p^2 \nmid (q-1)^2$, $p^2 \nmid q^2-q+1$, and $p^2 \nmid q^2+1$ holds. Since for each case above

we have $p^2 < q^6$, by Lemma 2.5 we get a contradiction. In the same way as above, we can get a contradiction if $n = 4, 5, 6$.

(8) To prove that $S \not\equiv {}^2B_2(q)$, where $q = 2^{2f+1}$.

If $S \equiv {}^2B_2(q)$, then $|S| = |{}^2B_2(q)| = q^2(q^2 + 1)(q - 1)$, $p^2 | (q^2 + 1)(q - 1)$. Since $p \neq r$ and $|S| \nmid |G|$, we get that $q^2 | \frac{p^4 - 1}{2} < p^4$, hence, $q < p^2$. Noting that $(q^2 + 1, q - 1) = 1$, we conclude that $p^2 | q^2 + 1$. Moreover, $q^2 | \frac{p^2 + 1}{2}$ or $q^2 | p^2 - 1$.

If $q^2 | \frac{p^2 + 1}{2}$, then $2q^2 \leq p^2 + 1 \leq q^2 + 2$, which implies that $q^2 \leq 2$, a contradiction.

If $q^2 | p^2 - 1$, then we may set $p^2 - 1 = q^2 m$, also $q^2 = p^2 t - 1$, where m, t are two positive integers. Now we have that $p^2 - 1 = (p^2 t - 1)m$, then $p^2 | m - 1$, which yields $p^2 + 1 \leq m$ if $m > 1$, and $p^2 - 1 = q^2 m \geq q^2(p^2 + 1)$, a contradiction. If $m = 1$, then $t = 1$ and $p^2 = q^2 + 1$. By $q = 2^{2f+1}$, we get $5 | q^2 + 1$, which forces $p = 5$, $q^2 = 24$, a contradiction.

(9) To prove that $S \not\equiv {}^2D_n(q)$ ($n \geq 4$).

If $S \equiv {}^2D_n(q)$, let $k = \prod_{i=1}^{n-1} (q^{2i} - 1)$, then $|S| = |{}^2D_n(q)| = q^{n(n-1)}(q^n + 1)kd^{-1}$, $d = (4, q^n + 1)$ and $p^2 | (q^n + 1)k$. By Lemma 2.4(2), $p^2 \leq q^{2n} - 1$. Noting that $d \leq q^n + 1$, we get

$$q^{n(n-1)}q^{(n-1)^2} < q^{n(n-1)} \cdot (q^n + 1)k(q^n + 1)^{-1} \leq |S| < \frac{p^6}{2} < \frac{(q^{2n} - 1)^3}{2} < q^{6n}.$$

As a result, $n = 4$, $k = (q^4 + 1)(q^2 - 1)(q^4 - 1)(q^6 - 1)$, moreover,

$$k = (q^4 + 1)(q - 1)^3(q + 1)^3(q^2 + 1)(q^2 + q + 1)(q^2 - q + 1).$$

By $p^2 \nmid |S|$, it follows that p^2 divides one of $q^4 + 1$, $(q - 1)^3$, $(q + 1)^3$, $q^2 + 1$, $q^2 + q + 1$, and $q^2 - q + 1$. Since for each case above we have $p^2 < q^5$, by Lemma 2.5 we get a contradiction.

(10) To prove that $S \not\equiv {}^3D_4(q)$, ${}^2E_6(q)$.

If $S \equiv {}^3D_4(q)$, let $k = (q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1) = (q + 1)^2(q - 1)^2(q^2 + q + 1)^2(q^2 - q + 1)^2(q^4 - q^2 + 1)$, then $|S| = |{}^3D_4(q)| = q^{12}k$. Since $p^2 \nmid |S|$, it follows that one of the following holds:

$$p^2 | (q + 1)^2, p^2 | (q - 1)^2, p^2 | (q^2 + q + 1)^2, p^2 | (q^2 - q + 1)^2, p^2 | q^4 - q^2 + 1.$$

Consequently, $p^2 < q^6$. But $q^{12} | \frac{p^4 - 1}{2} < p^4$, hence, $q^6 < p^2$, a contradiction. In the same way as above, we can get that $S \not\equiv {}^2E_6(q)$.

(11) To prove that $S \not\equiv {}^2G_2(q)$, where $q = 3^{2m+1}$.

If $S \equiv {}^2G_2(q)$, then $|S| = |{}^2G_2(q)| = q^3(q^3 + 1)(q - 1)$ and $p^2 | (q + 1)(q - 1)(q^2 - q + 1)$, furthermore, there are three possibilities: $p^2 | q + 1$, $p^2 | q - 1$, and $p^2 | q^2 - q + 1$. Since for each case above we have $p^2 < q^3$, by Lemma 2.5 we get a contradiction.

(12) To prove that $S \not\equiv {}^2F_4(q)$, where $q = 2^{2m+1}$.

If $S \equiv {}^2F_4(q)$, let $k = (q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1) = (q^2 + 1)^2(q - 1)^2(q + 1)^2(q^2 - q + 1)(q^4 - q^2 + 1)$, then $|S| = |{}^2F_4(q)| = q^{12}k$. So we can get a contradiction by the same argument as in (11).

(13) To prove $S \not\equiv {}^2F_4(2)'$.

If $S \equiv {}^2F_4(2)'$, then $|S| = |{}^2F_4(2)'| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$, we can easily obtain a contradiction.

Till now we have proved Step 3.

Step 4 Suppose that $N = S \times S$, $p \parallel |S|$, and $|S|^2 \mid \frac{p^2(p^4-1)}{2}$, then we come to a contradiction.

Now using the classification of finite simple groups and similar to the above argument, we consider all the simple groups such that

$$p \parallel |S|, |S|^2 \mid \frac{p^2(p^4-1)}{2}. \quad (**)$$

If S is a sporadic simple group, we take possible prime $p \parallel |S|$ one by one to check if S satisfies $|S|^2 \mid \frac{p^2(p^4-1)}{2}$, and we see that no sporadic simple group satisfying (**).

If $S \cong A_n (n \geq 5)$, then $p \parallel \frac{n!}{2}$, thus $p \parallel n!$, moreover, $\left(\frac{n!}{2}\right)^2 = |S|^2 \leq \frac{p^2(p^4-1)}{2} < p^6$. Therefore, $p \leq n$, $n! < 2p^3$. Hence, $p! \leq n! < 2p^3$. If $p \geq 7$, we conclude that $(p-3)(p-2)(p-1) < 3 \times 4 \times \cdots (p-1) < p^2$, which yields $p^3 - 7p^2 + 11p < 6 < p$, consequently, $2 < p < 5$, a contradiction. If $p = 5$, by $n! < 2p^3$, we have $n = 5$. Consequently, $|S|^2 = 3600$, $|G| = 7800$, a contradiction to $|S|^2 \mid |G|$.

Suppose that S is a simple group of Lie type over a field of order q , where $q = r^f$. If $r = p$, then by $p \parallel |S|$, we get the unique possibility $S = PSL_2(p)$. Assume that $S = PSL_2(p)$, then $|S| = \frac{1}{2}p(p^2-1)$, but $|G| = \frac{1}{2}p^2(p^4-1)$, $|S|^2 \nmid |G|$, a contradiction.

We assume that S is a simple group of Lie type over a field of order q , where $q = r^f$ and $p \neq r$. In the following, we shall eliminate the possibilities of S one by one.

Now we state a fact that is frequently used in the following proofs. If S is a simple group of Lie type over a field of order q , then $|S| = q^l \cdot u \cdot d^{-1}$, where l, u are two positive integers, d is the order of diagonal automorphisms of S and $(r, u) = 1$. Therefore, by $p \parallel |S|$, $|S|^2 \mid \frac{p^2(p^4-1)}{2}$, and $p \neq r$, we conclude that $p \parallel u$, $q^{2l} \mid \frac{p^4-1}{2}$. Moreover, $q^{2l} \mid \frac{p^2+1}{2}$ or $q^{2l} \mid p^2-1$, which implies that $q^{2l} < p^2$.

(1) To prove $S \not\cong A_n(q) (n \geq 1)$.

If $S \cong A_n(q)$, then $|S| = |A_n(q)| = q^{\frac{n(n+1)}{2}} \prod_{i=1}^n (q^{i+1}-1)d^{-1}$, $d = (n+1, q-1)$ and $p \parallel \prod_{i=1}^n (q^{i+1}-1)$, which concludes that there exists some $1 \leq i \leq n$ such that $p \mid q^{i+1}-1$, thus $p \leq q^{n+1}-1$. By $q^{i-1} < q^i-1$ and $d \leq q-1 < q$, we come to that $|S| > q^{\frac{n(n+1)}{2}} \cdot q^{1+2+\cdots+n} \cdot q^{-1} = q^{n^2+n-1}$. Hence, $q^{2(n^2+n-1)} < |S|^2 < p^6 \leq q^{6(n+1)}$, which implies that $n \leq 3$.

If $n = 1$, then $|S| = q(q+1)(q-1)d^{-1}$. Therefore, $p \mid q+1$ or $p \mid q-1$. Moreover, $q^2 \mid \frac{p^2+1}{2}$ or $q^2 \mid p^2-1$. Suppose that $q^2 \mid \frac{p^2+1}{2}$, then by $p \mid q+1$ and $p \mid q-1$, we get that $p-1 \leq q$. Thus, $2(p-1)^2-1 \leq 2q^2-1 \leq p^2$, which implies that $p < 4$, contradicting $p \geq 5$. Suppose that $q^2 \mid p^2-1$ and $p \mid q+1$, then we may set $p^2 = q^2t+1$, $q = pm-1$. Therefore, $p^2 = q^2t+1 = (pm-1)^2t+1$, which implies that $p \mid t+1$, thus, $p-1 \leq t$. Hence, $p^2 = (pm-1)^2t+1 \geq (p-1)^3+1$, a contradiction to $p \geq 5$. In the same way as above, we can get a contradiction if $q^2 \mid p^2-1$ and $p \mid q-1$.

If $n = 2$, then $|S| = q^3(q^2-1)(q^3-1)d^{-1} = q^3(q-1)^2(q+1)(q^2+q+1)d^{-1}$. By $p \parallel |S|$, we have either $p \mid q+1$ or $p \mid q^2+q+1$. Since $p < q^3$ in each case above, by Lemma 2.6 we get a contradiction. With the same argument, we can get a contradiction if $n = 3$.

(2) To prove $S \not\cong B_n(q) (n \geq 2)$, $C_n(q) (n \geq 3)$.

If $S \cong B_n(q)$, then $|S| = |B_n(q)| = q^{n^2} \prod_{i=1}^n (q^{2i}-1)d^{-1}$, $d = (2, q-1)$, and $p \parallel \prod_{i=1}^n (q^{2i}-1)$. Hence, there exists some $1 \leq i \leq n$ such that $p \mid q^{2i}-1$, which implies that $p \leq q^n+1$. Similarly, by $d < q$, we get that $q^{n^2} \cdot q^{1+3+\cdots+(2n-1)} \cdot \frac{1}{q} < |S|$, thus, $q^{2(2n^2-1)} < |S|^2 < p^6 < q^{6(n+1)}$, consequently, that $n = 2$. Therefore, $|S| = q^4(q^2-1)(q^4-1)d^{-1} = q^4(q-1)^2(q+1)^2(q^2+1)d^{-1}$. Since $p \parallel |S|$, we have $p \mid (q^2+1)$. Because $p \leq q^2+1 < q^4$, by Lemma 2.6 we get a contradiction. We can get that $S \not\cong C_n(q)$ since $|B_n(q)| = |C_n(q)|$.

(3) To prove that $S \neq D_n(q) (n \geq 4)$.

If $S \cong D_n(q)$, then $|S| = |D_n(q)| = q^{n(n-1)}(q^n - 1) \prod_{i=1}^{n-1} (q^{2i} - 1)d^{-1}$, $d = (4, q^n - 1)$. We get that $p \mid q^n - 1$ or $p \mid \prod_{i=1}^{n-1} (q^{2i} - 1)$. Therefore, $p \leq q^n$. Note that $d \leq q^n - 1$, thus $|S| > q^{n(n-1)} \cdot q^{1+3+\dots+(2n-3)} = q^{2n^2-3n+1}$. As a result, $q^{2(2n^2-3n+1)} < |S|^2 < p^6 \leq q^{6n}$, which implies that $n < 3$, a contradiction to $n \geq 4$.

(4) To prove that $S \neq G_2(q)$.

If $S \cong G_2(q)$, then $|S| = |G_2(q)| = q^6(q^6 - 1)(q^2 - 1)$ and $p \mid (q^6 - 1)(q^2 - 1)$, so $p \leq q^6 - 1 < q^6$. On the other hand, either $q^{12} \mid \frac{p^2+1}{2}$ or $q^{12} \mid p^2 - 1$, in each case we obtain that $q^{12} < p^2$, therefore, we get a contradiction by $p^2 < q^{12} < p^2$.

(5) To prove that $S \neq F_4(q)$, $E_6(q)$, $E_7(q)$, and $E_8(q)$.

If $S \cong F_4(q)$, then $|S| = |F_4(q)| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$ and $p \mid (q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$, so $p \leq q^{12} - 1 < q^{12}$. Note that $q^{48} \mid \frac{p^4-1}{2}, \frac{p^4-1}{2} < p^4$, we get a contradiction by $p^4 < q^{48} < p^4$. In the same way, we can get a contradiction if S is one of Lie type: $E_6(q)$, $E_7(q)$ and $E_8(q)$.

(6) To prove that $S \neq {}^2A_n(q) (n \geq 2)$.

If $S \cong {}^2A_n(q)$, then $|S| = |{}^2A_n(q)| = q^{\frac{n(n+1)}{2}} \prod_{i=1}^n (q^{i+1} - (-1)^{i+1})d^{-1}$, $d = (n+1, q+1)$. So $p \mid \prod_{i=1}^n (q^{i+1} - (-1)^{i+1})$, which implies that there exists some $1 \leq i \leq n$ such that $p \mid q^{i+1} - (-1)^{i+1}$, thus $p \leq q^{n+1} + 1$. By $d \leq q+1$, we come to that $|S| > q^{\frac{n(n+1)}{2}} \cdot q^{2+\dots+n} = q^{n^2+n-1}$. Hence, $q^{2(n^2+n-1)} < |S|^2 < p^6 \leq q^{6(n+2)}$, which implies that $n \leq 3$.

If $n = 2$, then $|S| = q^3(q^2 - 1)(q^3 + 1)d^{-1} = q^3(q - 1)(q + 1)^2(q^2 - q + 1)d^{-1}$. By $p \mid |S|$, we have $p \mid q - 1$ or $p \mid q^2 - q + 1$. Since in either case we get $p < q^3$, by Lemma 2.6 we get a contradiction. In the same approach, we can get a contradiction if $n = 3$.

(7) To prove that $S \neq {}^2B_2(q)$, $q = 2^{2f+1}$.

If $S \cong {}^2B_2(q)$, then $|S| = |{}^2B_2(q)| = q^2(q^2 + 1)(q - 1)$. Hence, $p \mid (q^2 + 1)(q - 1)$, so $p \mid (q^2 + 1)$ or $p \mid q - 1$. Note that either $q^4 \mid \frac{p^2+1}{2}$ or $q^4 \mid p^2 - 1$. Since $q^4 < p^2$, $q^2 < p$, it follows that $p \mid q - 1$ is impossible. Suppose that $p \mid (q^2 + 1)$ and $q^4 \mid \frac{p^2+1}{2}$, then we get that $2(p - 1)^2 - 1 \leq 2q^4 - 1 \leq p^2$, which implies that $p < 4$, contradicting $p \geq 5$. If $p \mid q^2 + 1$ and $q^4 \mid p^2 - 1$, then we may set $q^2 = pm - 1$, $p^2 - 1 = q^4t$, hence $p^2 = q^4t + 1 = (pm - 1)^2t + 1$, it follows that $p \mid t + 1$, thus, $p - 1 \leq t$. Therefore, $p^2 = (pm - 1)^2t + 1 \geq (p - 1)^3 + 1$, which implies that $0 < p < 1$, a contradiction.

(8) To prove that $S \neq {}^2D_n(q) (n \geq 4)$.

If $S \cong {}^2D_n(q)$, then $|S| = |{}^2D_n(q)| = q^{n(n-1)}(q^n + 1) \prod_{i=1}^{n-1} (q^{2i} - 1)d^{-1}$, $d = (4, q^n + 1)$. We have $p \mid q^n + 1$ or $p \mid \prod_{i=1}^{n-1} (q^{2i} - 1)$. Similarly, we get that $p \leq q^n + 1$. By $d \leq q^n + 1$, we come to that $|S| \geq q^{n(n-1)} \cdot q^{1+3+\dots+(2n-3)} = q^{2n^2-3n+1}$. Hence, $q^{2(2n^2-3n+1)} < |S|^2 < p^6 \leq q^{6(n+1)}$, which implies that $n \leq 3$, a contradiction to $n \geq 4$.

(9) To prove that $S \neq {}^3D_4(q)$.

If $S \cong {}^3D_4(q)$, then $|S| = |{}^3D_4(q)| = q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$. Thus, $p \mid (q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$, so $p \leq q^8 + q^4 + 1 < q^9$. On the other hand, either $q^{24} \mid \frac{p^2+1}{2}$ or $q^{24} \mid p^2 - 1$, in each case, we obtain that $q^{24} < p^2$; therefore, we get a contradiction by $q^{24} < p^2 < q^{18}$.

(10) To prove that $S \neq {}^2G_2(q)$, $q = 3^{2f+1}$.

If $S \cong {}^2G_2(q)$, then $|S| = |{}^2G_2(q)| = q^3(q^3 + 1)(q - 1)$. Hence, $p \mid (q + 1)(q - 1)(q^2 - q + 1)$, thus $p < q^3$. By Lemma 2.6, we get a contradiction.

(11) To prove that $S \not\cong {}^2F_4(q), {}^2E_6(q)$.

If $S \cong {}^2F_4(q)$, then $|S| = |{}^2F_4(q)| = q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)$. So $p | (q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)$, we get that $p < q^6 + 1 < p^7$. On the other hand, either $q^{24} | \frac{p^2 + 1}{2}$ or $q^{24} | p^2 - 1$, it always follows that $q^{24} < p^2$; therefore, we get a contradiction by $q^{24} < p^2 < q^{14}$. In the same way as above, we can get that $S \not\cong {}^2E_6(q)$.

(12) To prove that $S \not\cong {}^2F_4(2)'$.

If $S \cong {}^2F_4(2)'$, then $|S| = |{}^2F_4(2)'| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$. By $p || |S|$, we come to that $p = 13$, but at this moment $|S|^2 \nmid \frac{13^6 - 1}{2}$, a contradiction. So (12) follows.

This is the end of the proof of Step 4. And the Main Theorem follows from Steps 1 to 4.

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