Research Article

Huiwen Wang and Fang Li*

Nonlinear boundary value problems for mixed-type fractional equations and Ulam-Hyers stability

<https://doi.org/10.1515/math-2020-0051>

received August 13, 2019; accepted April 28, 2020

Abstract: In this article, we discuss the nonlinear boundary value problems involving both left Riemann-Liouville and right Caputo-type fractional derivatives. By using some new techniques and properties of the Mittag-Leffler functions, we introduce a formula of the solutions for the aforementioned problems, which can be regarded as a novelty item. Moreover, we obtain the existence result of solutions for the aforementioned problems and present the Ulam-Hyers stability of the fractional differential equation involving two different fractional derivatives. An example is given to illustrate our theoretical result.

Keywords: mixed fractional derivatives, nonlinear boundary value problem, Mittag-Leffler functions, Ulam-Hyers stability

MSC 2020: 34A08, 34A37, 34B10

1 Introduction

The fractional differential equations can model some engineering and scientific disciplines in the fields of physics, chemistry, electrodynamics of complex medium, polymer rheology, etc. In particular, the forward and backward fractional derivatives provide an excellent tool for the description of some physical phenomena such as the fractional oscillator equations and the fractional Euler-Lagrange equations [1–14]. Recently, a linear boundary value problem (BVP) involving both the right Caputo and the left Riemann-Liouville fractional derivatives has been studied by some authors [10–14].

On the other hand, Ulam's stability problem [15] has been attracted by many famous researchers (see [16,17] and references therein). Recently, studying the stability of Ulam-Hyers for fractional differential equations is gaining much importance and attention [18,19]. However, the Ulam-Hyers stabilities of differential equations involving with the forward and backward fractional derivatives have not yet been investigated.

In this article, we investigate the following BVP of the fractional differential equation with two different fractional derivatives:

$${}^c D_1^\beta ({}^L D_0^\alpha + \lambda)u(t) = f(t, u(t)), \quad t \in J := (0, 1], \quad (1.1)$$

$$\lim_{t \rightarrow 0^+} t^{1-\alpha} u(t) = u_0, \quad (I_{0^+}^{1-\alpha} u)(0) + ({}^R I_0^q u)(1) = 0, \quad (1.2)$$

where $\alpha, \beta, \alpha + \beta \in (0, 1)$, $\lambda, \rho, q > 0$, $\alpha + \rho > 1$. ${}^c D_1^\beta$ is the right Caputo fractional derivative of order β , ${}^L D_0^\alpha$ is the left Riemann-Liouville fractional derivative of order α , $I_{0^+}^{1-\alpha}$ is the Riemann-Liouville fractional integral and ${}^R I_0^q$ is the Katugampola fractional integral.

* **Corresponding author: Fang Li**, School of Mathematics, Yunnan Normal University, Kunming, 650092, People's Republic of China, e-mail: fangli860@gmail.com

Huiwen Wang: School of Mathematics, Yunnan Normal University, Kunming, 650092, People's Republic of China

The rest of this article is organized as follows. In Section 2, we collect some concepts of fractional calculus. In Section 3, we prove some properties of classical and generalized Mittag-Leffler functions. In Section 4, we present the definition of solution to (1.1) and (1.2). In Section 5, we obtain the existence and uniqueness of solutions to problem (1.1) and (1.2). In Section 6, we present the Ulam-Hyers stability result for Eq. (1.1). An example is given in Section 7 to demonstrate the application of our result.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus. Throughout this article, we denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions from J to $\mathbb{R}$, $L^p(J, \mathbb{R})$ the Banach space of all Lebesgue measurable functions $l : J \rightarrow \mathbb{R}$ with the norm $\|l\|_{L^p} = \left(\int_J |l(t)|^p dt \right)^{\frac{1}{p}} < \infty$ and $AC([a, b], \mathbb{R})$ the space of all the absolutely continuous functions defined on $[a, b]$.

Definition 2.1. [3,4] The left-sided and the right-sided fractional integrals of order γ for a function $x(t) \in L^1$ are defined as

$$\begin{aligned} (I_{a^+}^\gamma x)(t) &= \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} x(s) ds, \quad t > a, \gamma > 0, \\ (I_b^- \gamma x)(t) &= \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} x(s) ds, \quad t < b, \gamma > 0, \end{aligned}$$

respectively. Here, $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. [3,4] If $x(t) \in AC([a, b], \mathbb{R})$, then the left-sided and the right-sided Riemann-Liouville fractional derivatives ${}^L D_{a^+}^\gamma x(t)$ and ${}^L D_b^- \gamma x(t)$ of order γ exist almost everywhere on $[a, b]$ and can be written as

$$\begin{aligned} ({}^L D_{a^+}^\gamma x)(t) &= \frac{1}{\Gamma(1-\gamma)} \left(\frac{d}{dt} \right) \int_a^t (t-s)^{-\gamma} x(s) ds = \frac{d}{dt} (I_{a^+}^{1-\gamma} x)(t), \quad t > a, \quad 0 < \gamma < 1, \\ ({}^L D_b^- \gamma x)(t) &= \frac{1}{\Gamma(1-\gamma)} \left(-\frac{d}{dt} \right) \int_t^b (s-t)^{-\gamma} x(s) ds = -\frac{d}{dt} (I_b^{1-\gamma} x)(t), \quad t < b, \quad 0 < \gamma < 1, \end{aligned}$$

respectively.

Definition 2.3. [3,4] If $x(t) \in AC([a, b], \mathbb{R})$, then the right-sided Caputo derivative ${}^C D_b^\gamma x(t)$ of order γ exists almost everywhere on $[a, b]$ and can be written as

$$({}^C D_b^\gamma x)(t) = ({}^L D_b^- \gamma [x(s) - x(b)])(t), \quad t < b, \quad 0 < \gamma < 1.$$

Definition 2.4. [5] For $\rho, q > 0$, then the Katugampola fractional integral of $y(t)$ can be defined as

$$({}^\rho I_{a^+}^q y)(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_a^t (t^\rho - \tau^\rho)^{q-1} \tau^{\rho-1} y(\tau) d\tau, \quad t > a,$$

if the integral exists.

3 Properties of Mittag-Leffler functions

In this section, we prove some properties of the Mittag-Leffler functions.

Definition 3.1. [3,4] For $\mu, \nu > 0, z \in \mathbb{R}$, the classical Mittag-Leffler functions $E_\mu(z)$ and the generalized Mittag-Leffler functions $E_{\mu,\nu}(z)$ are defined by

$$E_\mu(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + 1)}, \quad E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \nu)}.$$

Clearly, $E_{\mu,1}(z) = E_\mu(z)$.

Lemma 3.2. [4,9] Let $\alpha \in (0, 1), \theta > \alpha$ be arbitrary. The functions $E_\alpha, E_{\alpha,\alpha}$ and $E_{\alpha,\theta}$ are nonnegative and have the following properties:

- (i) For any $t \in J$, $E_\alpha(-\lambda t^\alpha) \leq 1$, $E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\alpha)}$, $E_{\alpha,\theta}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\theta)}$.
- (ii) For any $t_1, t_2 \in J$,

$$\begin{aligned} |E_\alpha(-\lambda t_2^\alpha) - E_\alpha(-\lambda t_1^\alpha)| &= O(|t_2 - t_1|^\alpha), \quad \text{as } t_2 \rightarrow t_1, \\ |E_{\alpha,\alpha}(-\lambda t_2^\alpha) - E_{\alpha,\alpha}(-\lambda t_1^\alpha)| &= O(|t_2 - t_1|^\alpha), \quad \text{as } t_2 \rightarrow t_1, \\ |E_{\alpha,\alpha+1}(-\lambda t_2^\alpha) - E_{\alpha,\alpha+1}(-\lambda t_1^\alpha)| &= O(|t_2 - t_1|^\alpha), \quad \text{as } t_2 \rightarrow t_1. \end{aligned}$$

Lemma 3.3. [4,9] For $\gamma, \mu, \nu, \lambda > 0, t > 0, 0 < \alpha, \beta < 1$, the usual derivatives of $E_{\mu,\nu}$ and the fractional integrals and derivatives of $E_{\mu,\nu}$ are expressed by

- (1) $\frac{d}{dt} E_\mu(-\lambda t^\mu) = -\lambda t^{\mu-1} E_{\mu,\mu}(-\lambda t^\mu)$;
- (2) $I_{0+}^\gamma (s^{\nu-1} E_{\mu,\nu}(-\lambda s^\mu))(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} s^{\nu-1} E_{\mu,\nu}(-\lambda s^\mu) ds = t^{\gamma+\nu-1} E_{\mu,\gamma+\nu}(-\lambda t^\mu)$;
- (3) $[{}^L D_{a+}^\nu (s-a)^{\beta-1} E_{\alpha,\beta}(-\lambda(s-a)^\alpha)](t) = (t-a)^{\beta-\nu-1} E_{\alpha,\beta-\nu}(-\lambda(t-a)^\alpha)$.

Lemma 3.4. For $\nu, \lambda > 0, t > 0, 0 < \alpha, \beta < 1$, the fractional integrals and derivatives of $E_{\mu,\nu}$ are expressed by

- (1) $[{}^L D_{a+}^\alpha (s-a)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(s-a)^\alpha)](t) = -\lambda(t-a)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-a)^\alpha)$;
- (2) $[{}^C D_{1-}^\beta {}^L D_{0+}^\alpha s^\alpha E_{\alpha,\alpha+1}(-\lambda s^\alpha)](t) + \lambda [{}^C D_{1-}^\beta s^\alpha E_{\alpha,\alpha+1}(-\lambda s^\alpha)](t) = 0$;
- (3) $[{}^C D_{1-}^\beta {}^L D_{0+}^\alpha s^{\alpha-1} E_{\alpha,\alpha}(-\lambda s^\alpha)](t) + \lambda [{}^C D_{1-}^\beta s^{\alpha-1} E_{\alpha,\alpha}(-\lambda s^\alpha)](t) = 0$;
- (4) $[{}^\rho I_{0+}^q s^{\nu-1} E_{\alpha,\nu}(-\lambda s^\alpha)](t) = \frac{t^{q\rho+\nu-1}}{\rho^q \Gamma(q)} \int_0^1 s^{\frac{\nu-1}{\rho}} (1-s)^{q-1} E_{\alpha,\nu}(-\lambda t^\alpha s^{\frac{\alpha}{\rho}}) ds, \nu \geq \alpha$.

Proof.

- (1) By using Lemma 3.3, we find

$$\begin{aligned} [{}^L D_{a+}^\alpha (s-a)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(s-a)^\alpha)](t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} (s-a)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(s-a)^\alpha) ds \\ &= \frac{d}{dt} [E_\alpha(-\lambda(t-a)^\alpha)] = -\lambda(t-a)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-a)^\alpha). \end{aligned}$$

Similarly, we have

$$[{}^L D_{a+}^\alpha (s-a)^\alpha E_{\alpha,\alpha+1}(-\lambda(s-a)^\alpha)](t) = E_\alpha(-\lambda(t-a)^\alpha). \quad (3.1)$$

- (2) Using (3.1) and $({}^C D_{1-}^\beta 1)(t) = 0$, we arrive at

$$\begin{aligned} [{}^C D_{1-}^\beta {}^L D_{0+}^\alpha s^\alpha E_{\alpha,\alpha+1}(-\lambda s^\alpha)](t) + \lambda [{}^C D_{1-}^\beta s^\alpha E_{\alpha,\alpha+1}(-\lambda s^\alpha)](t) &= [{}^C D_{1-}^\beta (E_\alpha(-\lambda s^\alpha) + \lambda s^\alpha E_{\alpha,\alpha+1}(-\lambda s^\alpha))](t) \\ &= ({}^C D_{1-}^\beta 1)(t) = 0. \end{aligned}$$

(3) The proof of (3) is similar to that of (2).

(4) Clearly, for $\nu \geq \alpha$, the integral $\int_0^t (1-\tau)^{q-1} \tau^{\frac{\nu-1}{\rho}} E_{\alpha,\nu}(-\lambda t^\alpha \tau^{\frac{\alpha}{\rho}}) d\tau$ exists, then we get

$$\begin{aligned} [{}^\rho I_0^q s^{\nu-1} E_{\alpha,\nu}(-\lambda s^\alpha)](t) &= \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t (t^\rho - s^\rho)^{q-1} s^{\rho+\nu-2} E_{\alpha,\nu}(-\lambda s^\alpha) ds \\ &= \frac{\rho^{1-q} t^{\rho q + \nu - 1}}{\rho \Gamma(q)} \int_0^1 (1-\tau)^{q-1} \tau^{\frac{\nu-1}{\rho}} E_{\alpha,\nu}(-\lambda t^\alpha \tau^{\frac{\alpha}{\rho}}) d\tau \\ &= \frac{t^{\rho q + \nu - 1}}{\rho^q \Gamma(q)} \int_0^1 (1-\tau)^{q-1} \tau^{\frac{\nu-1}{\rho}} E_{\alpha,\nu}(-\lambda t^\alpha \tau^{\frac{\alpha}{\rho}}) d\tau. \quad \square \end{aligned}$$

4 Solutions for BVP

In this section, we present the formulas of solutions to problem (1.1) and (1.2).

Lemma 4.1. [4] For $\theta > 0$, a general solution of the fractional differential equation ${}^c D_1^\theta u(t) = 0$ is given by

$$u(t) = \sum_{i=0}^{n-1} C_i (1-t)^i,$$

where $C_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$ ($n = [\theta] + 1$), and $[\theta]$ denotes the integer part of the real number θ .

Lemma 4.2. For $\alpha, \beta \in (0, 1)$, ${}^c D_1^\beta ({}^L D_0^\alpha + \lambda)u(t) = h(t)$, $t \in J$, then

$$u(t) = C_0 t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) + C_1 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^1 K(t, \tau) h(\tau) d\tau, \quad t \in J,$$

if the integral exists. Here,

$$K(t, \tau) = \frac{1}{\Gamma(\beta)} \begin{cases} \int_0^\tau (t-s)^{\alpha-1} (\tau-s)^{\beta-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) ds, & 0 < \tau < t \leq 1, \\ \int_0^t (t-s)^{\alpha-1} (\tau-s)^{\beta-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) ds, & 0 < t < \tau \leq 1. \end{cases}$$

Formally, by Lemma 4.1, for $C_0 \in \mathbb{R}$, we obtain $({}^L D_0^\alpha + \lambda)u(t) = C_0 + (I_1^\beta h)(t)$. Based on the arguments of [4], we derive

$$\begin{aligned} u(t) &= C_1 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) (C_0 + (I_1^\beta h)(s)) ds \\ &= C_0 t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) + C_1 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \left(\int_s^1 (\tau-s)^{\beta-1} h(\tau) d\tau \right) ds \end{aligned}$$

$$\begin{aligned}
&= C_0 t^\alpha E_{\alpha, \alpha+1}(-\lambda t^\alpha) + C_1 t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^\alpha) \\
&\quad + \frac{1}{\Gamma(\beta)} \left[\int_0^t h(\tau) d\tau \int_0^\tau (t-s)^{\alpha-1} (\tau-s)^{\beta-1} E_{\alpha, \alpha}(-\lambda(t-s)^\alpha) ds \right. \\
&\quad \left. + \int_t^1 h(\tau) d\tau \int_0^t (t-s)^{\alpha-1} (\tau-s)^{\beta-1} E_{\alpha, \alpha}(-\lambda(t-s)^\alpha) ds \right] \\
&= C_0 t^\alpha E_{\alpha, \alpha+1}(-\lambda t^\alpha) + C_1 t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^\alpha) + \int_0^1 K(t, \tau) h(\tau) d\tau.
\end{aligned}$$

We define $C_{1-\alpha}([0, 1], \mathbb{R}) = \{u \in C(J, \mathbb{R}) : t^{1-\alpha}u(t) \in C([0, 1], \mathbb{R})\}$ with the norm $\|u\|_{1-\alpha} = \max_{t \in [0, 1]} t^{1-\alpha}|u(t)|$ and we abbreviate $C_{1-\alpha}([0, 1], \mathbb{R})$ to $C_{1-\alpha}$.

To prove our results, we make the following assumptions.

(H1) Let $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(\cdot, u) : J \rightarrow \mathbb{R}$ is measurable for all $u \in \mathbb{R}$ and $f(t, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous for a.e. $t \in J$, and there exists a function $\varphi \in L^{\frac{1}{p}}(J, \mathbb{R}^+)$ ($0 < p < \min\{\alpha, \beta\}$) such that

$$|f(t, u(t)) - f(t, v(t))| \leq \varphi(t) \|u - v\|_{1-\alpha}.$$

(H2) $\sup_{t \in J} |f(t, 0)| < \infty$.

For convenience of the following presentation, we set

$$\begin{aligned}
A(\theta, t) &= \frac{t^{q\rho+\theta}}{\rho^q \Gamma(q)} \int_0^1 s^{\frac{\theta}{\rho}} (1-s)^{q-1} E_{\alpha, \theta+1}(-\lambda t^\alpha s^{\frac{\theta}{\rho}}) ds; \\
N_\theta &= \frac{B\left(\frac{\theta}{\rho} + 1, q\right)}{\rho^q \Gamma(q)} \frac{1}{\Gamma(\theta + 1)}; \\
M_1 &= \frac{1}{\rho^q \Gamma(q+1) \Gamma(\alpha+1) A(\alpha, 1)} + 1; \\
M_2 &= \left[\frac{1}{\beta} \left(\frac{1-p}{\alpha-p} \right)^{1-p} + \frac{1}{\alpha} \left(\frac{1-p}{\beta-p} \right)^{1-p} \right] \frac{\|\varphi\|_{L^{\frac{1}{p}}}}{\Gamma(\alpha) \Gamma(\beta)}; \\
M_3 &= \frac{2}{\Gamma(\alpha+1) \Gamma(\beta+1)},
\end{aligned}$$

where $B(\cdot, \cdot)$ is the Beta function.

For $t \in J$, $y > p$, applying the Hölder inequality, there hold the following estimates:

$$\int_a^t (t-s)^{y-1} \varphi(s) ds \leq \left(\int_a^t (t-s)^{\frac{y-1}{1-p}} ds \right)^{1-p} \|\varphi\|_{L^{\frac{1}{p}}} = \left(\frac{1-p}{y-p} \right)^{1-p} (t-a)^{y-p} \|\varphi\|_{L^{\frac{1}{p}}}, \quad t > a, \quad (4.1)$$

$$\int_t^b (s-t)^{y-1} \varphi(s) ds \leq \left(\int_t^b (s-t)^{\frac{y-1}{1-p}} ds \right)^{1-p} \|\varphi\|_{L^{\frac{1}{p}}} = \left(\frac{1-p}{y-p} \right)^{1-p} (b-t)^{y-p} \|\varphi\|_{L^{\frac{1}{p}}}, \quad t < b. \quad (4.2)$$

Lemma 4.3.

$$\left| \int_0^\tau [(t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_1 - s)^\alpha) - (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_2 - s)^\alpha)] (\tau - s)^{\beta-1} ds \right| \leq O((t_2 - t_1)^\alpha), \quad (4.3)$$

$$0 < \tau < t_1 < t_2 \leq 1,$$

$$\left| \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] (\tau - s)^{\beta-1} ds \right| \leq (\tau - t_2)^{\beta-1} O((t_2 - t_1)^\alpha), \quad 0 < t_1 < t_2 < \tau \leq 1. \quad (4.4)$$

Proof. For $0 < t_1 < t_2 \leq 1$, it follows from Lemma 3.2 and the mean value theorem that

$$|t_1^{\alpha-1} E_{\alpha,\alpha}(-\lambda t_1^\alpha) - t_2^{\alpha-1} E_{\alpha,\alpha}(-\lambda t_2^\alpha)| \leq |t_1^{\alpha-1} - t_2^{\alpha-1}| E_{\alpha,\alpha}(-\lambda t_1^\alpha) + |E_{\alpha,\alpha}(-\lambda t_1^\alpha) - E_{\alpha,\alpha}(-\lambda t_2^\alpha)| t_2^{\alpha-1} := O((t_2 - t_1)^\alpha),$$

which yields

$$\begin{aligned} & \left| \int_0^\tau [(t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_1 - s)^\alpha) - (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_2 - s)^\alpha)] (\tau - s)^{\beta-1} ds \right| \\ & \leq \int_0^\tau (\tau - s)^{\beta-1} ds \cdot O((t_2 - t_1)^\alpha) := O((t_2 - t_1)^\alpha), \quad \text{for } 0 < \tau < t_1 < t_2 \leq 1. \end{aligned}$$

For $0 < t_1 < t_2 < \tau \leq 1$, by the mean value theorem, one can see that

$$\begin{aligned} \left| \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] (\tau - s)^{\beta-1} ds \right| & \leq \left| \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] ds \right| \cdot (\tau - t_1)^{\beta-1} \\ & \leq \frac{1}{\alpha} [t_2^\alpha - t_1^\alpha + (t_2 - t_1)^\alpha] \cdot (\tau - t_1)^{\beta-1} \\ & \leq (\tau - t_2)^{\beta-1} \cdot O((t_2 - t_1)^\alpha). \end{aligned}$$

□

Let us define

$$\begin{aligned} k_1(t, \tau) &= \frac{1}{\Gamma(\beta)} \int_0^\tau (t - s)^{\alpha-1} (\tau - s)^{\beta-1} E_{\alpha,\alpha}(-\lambda(t - s)^\alpha) ds, \quad 0 < \tau < t \leq 1, \\ k_2(t, \tau) &= \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\alpha-1} (\tau - s)^{\beta-1} E_{\alpha,\alpha}(-\lambda(t - s)^\alpha) ds, \quad 0 < t < \tau \leq 1, \end{aligned}$$

clearly,

$$\int_0^1 K(t, \tau) d\tau = \int_0^t k_1(t, \tau) d\tau + \int_t^1 k_2(t, \tau) d\tau,$$

one can obtain the following estimates.

Lemma 4.4.

$$|k_1(t, \tau)| \leq \frac{(t - \tau)^{\alpha-1}}{\Gamma(\beta + 1)\Gamma(\alpha)}, \quad 0 < \tau < t \leq 1, \quad (4.5)$$

$$|k_2(t, \tau)| \leq \frac{(\tau - t)^{\beta-1}}{\Gamma(\alpha + 1)\Gamma(\beta)}, \quad 0 < t < \tau \leq 1, \quad (4.6)$$

$$\int_0^1 |K(t_2, \tau) - K(t_1, \tau)| \varphi(\tau) d\tau = O((t_2 - t_1)^{\alpha-p}) + O((t_2 - t_1)^{\beta-p}), \quad (4.7)$$

$$\int_0^1 |K(t_2, \tau) - K(t_1, \tau)| d\tau = O((t_2 - t_1)^\alpha) + O((t_2 - t_1)^\beta), \quad (4.8)$$

$$\int_0^1 K(t, \tau) \varphi(\tau) d\tau \leq M_2, \quad (4.9)$$

$$\int_0^1 K(t, \tau) d\tau \leq M_3. \quad (4.10)$$

Proof. Since

$$\begin{aligned} \int_0^\tau (t-s)^{\alpha-1}(\tau-s)^{\beta-1} ds &\leq \int_0^\tau (t-\tau)^{\alpha-1}(\tau-s)^{\beta-1} ds = \frac{\tau^\beta(t-\tau)^{\alpha-1}}{\beta}, \quad 0 < \tau < t \leq 1, \\ \int_0^t (t-s)^{\alpha-1}(\tau-s)^{\beta-1} ds &\leq \int_0^t (t-s)^{\alpha-1}(\tau-t)^{\beta-1} ds = \frac{t^\alpha(\tau-t)^{\beta-1}}{\alpha}, \quad 0 < t < \tau \leq 1, \end{aligned}$$

(4.5) and (4.6) hold. Moreover, (4.1) and (4.2) imply

$$\int_{t_1}^{t_2} k_1(t_2, s) \varphi(s) ds \leq \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1} \varphi(s)}{\Gamma(\beta+1)\Gamma(\alpha)} ds \leq \left(\frac{1-p}{\alpha-p} \right)^{1-p} \frac{(t_2-t_1)^{\alpha-p}}{\Gamma(\beta+1)\Gamma(\alpha)} \|\varphi\|_{L^{\frac{1}{p}}}, \quad (4.11)$$

$$\int_{t_1}^{t_2} k_2(t_1, s) \varphi(s) ds \leq \int_{t_1}^{t_2} \frac{(s-t_1)^{\beta-1} \varphi(s)}{\Gamma(\alpha+1)\Gamma(\beta)} ds \leq \left(\frac{1-p}{\beta-p} \right)^{1-p} \frac{(t_2-t_1)^{\beta-p}}{\Gamma(\alpha+1)\Gamma(\beta)} \|\varphi\|_{L^{\frac{1}{p}}}. \quad (4.12)$$

For $0 < \tau < t_1 < t_2 \leq 1$, by (4.3), we get

$$\begin{aligned} |k_1(t_2, \tau) - k_1(t_1, \tau)| &\leq \frac{1}{\Gamma(\beta)} \left| \int_0^\tau [(t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_2-s)^\alpha) - (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_1-s)^\alpha)] (\tau-s)^{\beta-1} ds \right| \\ &= O((t_2 - t_1)^\alpha), \end{aligned}$$

furthermore,

$$\int_0^{t_1} |k_1(t_2, \tau) - k_1(t_1, \tau)| \varphi(\tau) d\tau \leq \|\varphi\|_{L^1} \cdot O((t_2 - t_1)^\alpha) =: O((t_2 - t_1)^\alpha), \quad (4.13)$$

$$\int_0^{t_1} |k_1(t_2, \tau) - k_1(t_1, \tau)| d\tau =: O((t_2 - t_1)^\alpha). \quad (4.14)$$

For $0 < t_1 < t_2 < \tau \leq 1$, by (4.4) and Lemma 3.2, we find

$$\begin{aligned}
|k_2(t_2, \tau) - k_2(t_1, \tau)| &\leq \frac{1}{\Gamma(\beta)} \left[\frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] (\tau - s)^{\beta-1} ds \right| \right. \\
&\quad + \int_0^{t_1} (t_2 - s)^{\alpha-1} (\tau - s)^{\beta-1} |E_{\alpha, \alpha}(-\lambda(t_1 - s)^\alpha) - E_{\alpha, \alpha}(-\lambda(t_2 - s)^\alpha)| ds \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} (\tau - s)^{\beta-1} ds \right] \\
&\leq (\tau - t_2)^{\beta-1} \cdot O((t_2 - t_1)^\alpha) + \frac{(\tau - t_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
&:= (\tau - t_2)^{\beta-1} \cdot O((t_2 - t_1)^\alpha),
\end{aligned}$$

then by (4.2), we arrive at

$$\int_{t_2}^1 |k_2(t_2, \tau) - k_2(t_1, \tau)| \varphi(\tau) d\tau \leq \int_{t_2}^1 (\tau - t_2)^{\beta-1} \varphi(\tau) d\tau \cdot O((t_2 - t_1)^\alpha) := O((t_2 - t_1)^\alpha), \quad (4.15)$$

$$\int_{t_2}^1 |k_2(t_2, \tau) - k_2(t_1, \tau)| d\tau := O((t_2 - t_1)^\alpha). \quad (4.16)$$

From (4.13), (4.15), (4.11) and (4.12), it follows that

$$\begin{aligned}
\int_0^1 |K(t_2, \tau) - K(t_1, \tau)| \varphi(\tau) d\tau &\leq \int_0^{t_1} |k_1(t_2, s) - k_1(t_1, s)| \varphi(s) ds + \int_{t_1}^{t_2} k_1(t_2, s) \varphi(s) ds \\
&\quad + \int_{t_2}^1 |k_2(t_2, s) - k_2(t_1, s)| \varphi(s) ds + \int_{t_1}^{t_2} k_2(t_1, s) \varphi(s) ds \\
&= O((t_2 - t_1)^{\alpha-p}) + O((t_2 - t_1)^{\beta-p}).
\end{aligned}$$

Similarly, one can conclude from (4.14), (4.16), (4.5) and (4.6) that

$$\int_0^1 |K(t_2, \tau) - K(t_1, \tau)| d\tau := O((t_2 - t_1)^\alpha) + O((t_2 - t_1)^\beta).$$

By (4.5), (4.6), (4.1) and (4.2), we observe that

$$\begin{aligned}
\int_0^1 K(t, \tau) \varphi(\tau) d\tau &= \int_0^t k_1(t, \tau) \varphi(\tau) d\tau + \int_t^1 k_2(t, \tau) \varphi(\tau) d\tau \leq M_2, \\
\int_0^1 K(t, \tau) d\tau &\leq M_3, \\
\int_0^1 K(t, \tau) |f(\tau, 0)| d\tau &\leq \int_0^1 K(t, \tau) d\tau \sup_{t \in J} |f(t, 0)| \leq M_3 \sup_{t \in J} |f(t, 0)|.
\end{aligned}$$

□

For the sake of convenience, we adopt the following notation:

$$(Fu)(t) = \int_0^1 K(t, \tau) f(\tau, u(\tau)) d\tau = \int_0^t k_1(t, \tau) f(\tau, u(\tau)) d\tau + \int_t^1 k_2(t, \tau) f(\tau, u(\tau)) d\tau.$$

Since

$$|f(t, u(t))| \leq |f(t, u(t)) - f(t, 0)| + |f(t, 0)| \leq \varphi(t) \|u\|_{1-\alpha} + |f(t, 0)|,$$

then by (4.9) and (4.10),

$$|(Fu)(t)| \leq \int_0^1 K(t, \tau) \varphi(\tau) d\tau \|u\|_{1-\alpha} + \int_0^1 K(t, \tau) |f(\tau, 0)| d\tau \leq M_2 \|u\|_{1-\alpha} + M_3 \sup_{t \in J} |f(t, 0)|. \quad (4.17)$$

Lemma 4.5. Assume that (H1) and (H2) hold. For $u \in C_{1-\alpha}$, $t \in J$, $(Fu)(t)$ satisfies the following relations:

- (1) $(Fu)(t) \in AC(J, \mathbb{R})$;
- (2) $[{}^L D_{0^+}^\alpha (Fu)](t) = -\lambda(Fu)(t) + (I_{1-}^\beta f)(t)$;
- (3) $[{}^c D_{1-}^\beta {}^L D_{0^+}^\alpha (Fu)](t) + \lambda[{}^c D_{1-}^\beta (Fu)](t) = f(t, u(t))$;
- (4) $[{}^\rho I_{0^+}^q (Fu)](t) = \frac{t^{q\rho}}{\rho^q \Gamma(q)} \int_0^1 (1-s)^{q-1} (Fu)(ts^{\frac{1}{\rho}}) ds$.

Proof. For every finite collection $\{(a_j, b_j)\}_{1 \leq j \leq n}$ on J with $\sum_{j=1}^n (b_j - a_j) \rightarrow 0$, noting that (4.7) and (4.8), we derive

$$\begin{aligned} \sum_{j=1}^n |(Fu)(b_j) - (Fu)(a_j)| &= \sum_{j=1}^n \left| \int_0^1 K(b_j, \tau) f(\tau, u(\tau)) d\tau - \int_0^1 K(a_j, \tau) f(\tau, u(\tau)) d\tau \right| \\ &\leq \sum_{j=1}^n \int_0^1 |K(b_j, \tau) - K(a_j, \tau)| \varphi(\tau) d\tau \|u\|_{1-\alpha} \\ &\quad + \sum_{j=1}^n \int_0^1 |K(b_j, \tau) - K(a_j, \tau)| d\tau \sup_{t \in J} |f(t, 0)| \rightarrow 0. \end{aligned}$$

Hence, $(Fu)(t)$ is absolutely continuous on J . Furthermore, for almost all $t \in J$, $[{}^L D_{0^+}^\alpha (Fu)(s)](t)$ exists and from the fact $(Fu)(s) = \int_0^s (s-\tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda(s-\tau)^\alpha) (I_{1-}^\beta f)(\tau, u(\tau)) d\tau$ and Lemma 3.3, we get

$$\begin{aligned} [{}^L D_{0^+}^\alpha (Fu)(s)](t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \int_0^s (t-s)^{-\alpha} (s-\tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda(s-\tau)^\alpha) (I_{1-}^\beta f)(\tau) d\tau ds \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (I_{1-}^\beta f)(\tau) \int_\tau^t (t-s)^{-\alpha} (s-\tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda(s-\tau)^\alpha) ds d\tau \\ &= \frac{d}{dt} \int_0^t (I_{1-}^\beta f)(\tau) E_\alpha(-\lambda(t-\tau)^\alpha) d\tau = -\lambda(Fu)(t) + (I_{1-}^\beta f)(t). \end{aligned}$$

Thus,

$$[{}^c D_{1-}^\beta {}^L D_{0^+}^\alpha (Fu)(s)](t) + \lambda[{}^c D_{1-}^\beta (Fu)(s)](t) = [{}^c D_{1-}^\beta (-\lambda(Fu)(s) + (I_{1-}^\beta f)(s))](t) + \lambda[{}^c D_{1-}^\beta (Fu)(s)](t) = f(t, u(t)).$$

It follows from (4.17) that $\int_0^1 (1-s)^{q-1} (Fu)(ts^{\frac{1}{\rho}}) ds$ exists and

$$[{}^\rho I_{0^+}^q (Fu)](t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t (t^\rho - s^\rho)^{q-1} s^{\rho-1} (Fu)(s) ds = \frac{t^{q\rho}}{\rho^q \Gamma(q)} \int_0^1 (1-s)^{q-1} (Fu)(ts^{\frac{1}{\rho}}) ds. \quad \square$$

Lemma 4.6. Assume that (H1) and (H2) hold. A function u is a solution of the following fractional integral equation:

$$u(t) = (Pu)(t) + (Qu)(t) \quad (4.18)$$

if and only if u is a solution of problem (1.1) and (1.2), where

$$\begin{aligned} (Pu)(t) &= -\frac{u_0\Gamma(\alpha)(1+A(\alpha-1,1))}{A(\alpha,1)}t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) + u_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha), \\ (Qu)(t) &= -\frac{\frac{1}{\rho^q\Gamma(q)}\int_0^1(1-\tau)^{q-1}(Fu)(\tau^{\frac{1}{\rho}})d\tau}{A(\alpha,1)}t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) + (Fu)(t). \end{aligned}$$

Proof. (Sufficiency) Let u be the solution of (1.1) and (1.2), Lemmas 3.3, 3.4, 4.2 and 4.5 imply

$$\begin{aligned} u(t) &= at^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) + bt^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + (Fu)(t), \quad t \in J, \\ (I_0^{1-\alpha}u)(t) &= atE_{\alpha,2}(-\lambda t^\alpha) + bE_\alpha(-\lambda t^\alpha) + [I_0^{1-\alpha}(Fu)](t), \\ [{}^\rho I_0^q u](t) &= aA(\alpha, t) + bA(\alpha-1, t) + \frac{t^{q\rho}}{\rho^q\Gamma(q)}\int_0^1(1-s)^{q-1}(Fu)(ts^{\frac{1}{\rho}})ds, \end{aligned}$$

where a, b are constants. Using the boundary value condition (1.2), we derive that $b = u_0\Gamma(\alpha)$ and

$$aA(\alpha, 1) + u_0\Gamma(\alpha)(1+A(\alpha-1, 1)) + \frac{1}{\rho^q\Gamma(q)}\int_0^1(1-\tau)^{q-1}(Fu)(\tau^{\frac{1}{\rho}})d\tau = 0,$$

which means

$$a = -\frac{u_0\Gamma(\alpha)(1+A(\alpha-1, 1)) + \frac{1}{\rho^q\Gamma(q)}\int_0^1(1-\tau)^{q-1}(Fu)(\tau^{\frac{1}{\rho}})d\tau}{A(\alpha, 1)}.$$

Now we can see that (4.18) holds.

(Necessity) Let u satisfy (4.18). According to Lemma 3.4(2), (3) and Lemma 4.5(3), $[{}^c D_1^{\beta L} D_0^\alpha u](t)$ exists and ${}^c D_1^\beta ({}^L D_0^\alpha + \lambda)u(t) = f(t, u(t))$ for $t \in J$. Clearly, the boundary value condition (1.2) holds and hence the necessity is proved. $\square$

5 Existence and uniqueness result

In this section, we deal with the existence and uniqueness of solutions to problem (1.1) and (1.2).

Theorem 5.1. Assume that (H1) and (H2) are satisfied, then problem (1.1) and (1.2) has a unique solution $u \in C_{1-\alpha}$ if $M_1 M_2 < 1$.

Proof. We consider an operator $\mathcal{F} : C_{1-\alpha} \rightarrow C_{1-\alpha}$ defined by

$$(\mathcal{F}u)(t) = (Pu)(t) + (Qu)(t).$$

Clearly, $\mathcal{F}$ is well defined and the fixed point of $\mathcal{F}$ is the solution of problem (1.1) and (1.2).

Let $B_r = \{u \in C_{1-\alpha} : \|u\|_{1-\alpha} \leq r\}$ be a bounded set in $C_{1-\alpha}$, where

$$r \geq \frac{\frac{|u_0|}{\alpha} \left[\frac{1+N_{\alpha-1}}{A(\alpha, 1)} + \alpha \right] + M_1 M_3 \sup_{t \in J} |f(t, 0)|}{1 - M_1 M_2}.$$

For $u \in B_r$, it follows from Lemmas 3.2(i) and (4.17) that

$$\begin{aligned}\|Pu\|_{1-\alpha} &\leq \frac{|u_0|}{\alpha} \left[\frac{1 + N_{\alpha-1}}{A(\alpha, 1)} + \alpha \right], \\ \|Qu\|_{1-\alpha} &\leq M_1 [M_2 r + M_3 \sup_{t \in J} |f(t, 0)|]\end{aligned}$$

and hence

$$\|(\mathcal{F}u)\|_{1-\alpha} \leq \|Pu\|_{1-\alpha} + \|Qu\|_{1-\alpha} \leq \frac{|u_0|}{\alpha} \left[\frac{1 + N_{\alpha-1}}{A(\alpha, 1)} + \alpha \right] + M_1 [M_2 r + M_3 \sup_{t \in J} |f(t, 0)|] \leq r.$$

Then for $t \in J$, $u \in B_r$, $\mathcal{F}u \in B_r$.

For any $u, v \in B_r$, $t \in J$, by (4.9),

$$|(Fu)(t) - (Fv)(t)| \leq \int_0^1 K(t, \tau) |f(\tau, u(\tau)) - f(\tau, v(\tau))| d\tau \leq \int_0^1 K(t, \tau) \varphi(\tau) d\tau \|u - v\|_{1-\alpha} \leq M_2 \|u - v\|_{1-\alpha},$$

and thus

$$|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \leq \frac{\int_0^1 (1 - \tau)^{q-1} \left| (Fu)(\tau^{\frac{1}{p}}) - (Fv)(\tau^{\frac{1}{p}}) \right| d\tau}{A(\alpha, 1) \Gamma(\alpha + 1) \rho^q \Gamma(q)} + |(Fu)(t) - (Fv)(t)| \leq M_1 M_2 \|u - v\|_{1-\alpha}.$$

Furthermore, $\|\mathcal{F}u - \mathcal{F}v\|_{1-\alpha} < \|u - v\|_{1-\alpha}$. The proof now can be finished by using the Banach contraction mapping principle. $\square$

6 Ulam-Hyers stability

Let $\tilde{\epsilon}$ be a positive real number. We consider Eq. (1.1) with inequality

$$|^c D_1^\beta ({}^L D_0^\alpha + \lambda) v(t) - f(t, v(t))| \leq \tilde{\epsilon}, \quad t \in J. \quad (6.1)$$

Definition 6.1. Eq. (1.1) is Ulam-Hyers stable if there exists a real number $c > 0$ such that for each solution $v(t)$ of inequality (6.1) there exists a solution u of Eq. (1.1) with

$$|v(t) - u(t)| \leq c\tilde{\epsilon}, \quad t \in J.$$

Remark 6.2. A function $v \in C_{1-\alpha}$ is a solution of inequality (6.1) if and only if there exists a function $g \in C_{1-\alpha}$ such that (i) $|g(t)| \leq \tilde{\epsilon}$; (ii) ${}^c D_1^\beta ({}^L D_0^\alpha + \lambda) v(t) = f(t, v(t)) + g(t)$.

Let

$$\begin{aligned}\tilde{v}(t) &= - \frac{u_0 \Gamma(\alpha)(1 + A(\alpha - 1, 1))}{A(\alpha, 1)} t^\alpha E_{\alpha, \alpha+1}(-\lambda t^\alpha) + u_0 \Gamma(\alpha) t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^\alpha) \\ &\quad - \frac{1}{\rho^q \Gamma(q)} \int_0^1 (1 - \tau)^{q-1} (Fv)(\tau^{\frac{1}{p}}) d\tau \\ &\quad - \frac{1}{A(\alpha, 1)} t^\alpha E_{\alpha, \alpha+1}(-\lambda t^\alpha) + (Fv)(t),\end{aligned}$$

where

$$(Fv)(t) = \int_0^1 K(t, \tau) f(\tau, v(\tau)) d\tau.$$

Remark 6.3. Let $v \in C_{1-\alpha}$ be a solution of inequality (6.1) with $\lim_{t \rightarrow 0^+} t^{1-\alpha} v(t) = u_0$, $(I_{0^+}^{1-\alpha} v)(0) + ({}^\rho I_{0^+}^q v)(1) = 0$. Then, v is a solution of the inequality $|v(t) - \tilde{v}(t)| \leq M_1 M_3 \tilde{\epsilon}$.

Indeed, by Remark 6.2, we have

$$\begin{aligned} {}^c D_1^\beta ({}^L D_{0^+}^\alpha + \lambda) v(t) &= f(t, v(t)) + g(t), \\ \lim_{t \rightarrow 0^+} t^{1-\alpha} v(t) &= u_0, \quad (I_{0^+}^{1-\alpha} v)(0) + ({}^\rho I_{0^+}^q v)(1) = 0. \end{aligned}$$

Applying the same arguments as in the proof of Lemma 4.6, we obtain

$$\begin{aligned} v(t) &= -\frac{u_0 \Gamma(\alpha)(1 + A(\alpha - 1, 1))}{A(\alpha, 1)} t^\alpha E_{\alpha, \alpha+1}(-\lambda t^\alpha) + u_0 \Gamma(\alpha) t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^\alpha) \\ &\quad - \frac{\frac{1}{\rho^q \Gamma(q)} \int_0^1 (1-\tau)^{q-1} (Gv)(\tau^{\frac{1}{\rho}}) d\tau}{A(\alpha, 1)} t^\alpha E_{\alpha, \alpha+1}(-\lambda t^\alpha) + (Gv)(t), \end{aligned}$$

where

$$(Gv)(t) = \int_0^1 K(t, \tau) [f(\tau, v(\tau)) + g(\tau)] d\tau.$$

Therefore, by (4.10), we conclude that $|v(t) - \tilde{v}(t)| \leq M_1 M_3 \tilde{\epsilon}$.

Theorem 6.4. Assume that (H1) and (H2) are satisfied, then Eq. (1.1) is Ulam-Hyers stable if $M_1 M_2 < 1$.

Proof. Let $v \in C_{1-\alpha}$ be a solution of inequality (6.1) with $\lim_{t \rightarrow 0^+} t^{1-\alpha} v(t) = u_0$, $(I_{0^+}^{1-\alpha} v)(0) + ({}^\rho I_{0^+}^q v)(1) = 0$. u denotes the unique solution of the following problem:

$$\begin{cases} {}^c D_1^\beta ({}^L D_{0^+}^\alpha + \lambda) u(t) = f(t, u(t)), & t \in J, \\ \lim_{t \rightarrow 0^+} t^{1-\alpha} u(t) = u_0, & (I_{0^+}^{1-\alpha} u)(0) + ({}^\rho I_{0^+}^q u)(1) = 0. \end{cases}$$

It is easy to check that

$$\begin{aligned} |v(t) - u(t)| &\leq |v(t) - \tilde{v}(t)| + |\tilde{v}(t) - u(t)| \\ &\leq M_1 M_3 \tilde{\epsilon} + \frac{\int_0^1 (1-\tau)^{q-1} |(Fv)(\tau^{\frac{1}{\rho}}) - (Fu)(\tau^{\frac{1}{\rho}})| d\tau}{A(\alpha, 1) \Gamma(\alpha + 1) \rho^q \Gamma(q)} + |(Fv)(t) - (Fu)(t)| \\ &\leq M_1 M_3 \tilde{\epsilon} + M_1 M_2 \|v - u\|_{1-\alpha}, \end{aligned}$$

then

$$\|v - u\|_{1-\alpha} \leq \frac{M_1 M_3 \tilde{\epsilon}}{1 - M_1 M_2},$$

that is, Eq. (1.1) is Ulam-Hyers stable. $\square$

7 Example

As an example, we consider here the following BVP of the fractional differential equation with two different fractional derivatives:

$${}^c D_1^{\frac{1}{2}} \left({}^L D_{0^+}^{\frac{3}{2}} + 3 \right) u(t) = \frac{1}{30\sqrt[10]{t}} \sin(t^{\frac{2}{5}} u(t) + t^{\frac{3}{4}}), \quad t \in J := (0, 1], \quad (7.1)$$

$$\lim_{t \rightarrow 0^+} t^{\frac{2}{5}} u(t) = u_0, \quad \left(I_{0^+}^{\frac{2}{5}} u \right)(0) + ({}^2 I_{0^+}^3 u)(1) = 0. \quad (7.2)$$

Corresponding to (1.1) and (1.2), we recognize that $\alpha = \frac{3}{5}$, $\beta = \frac{1}{5}$, $\lambda = 3$, $\rho = 2$, $q = 3$ and

$$f(t, u(t)) = \frac{1}{30\sqrt[10]{t}} \sin(t^{\frac{2}{5}} u(t) + t^{\frac{3}{4}}).$$

The space $C_{1-\alpha} = \{u \in C(J, \mathbb{R}) : t^{\frac{2}{5}} u(t) \in C([0, 1], \mathbb{R})\}$ with the norm $\|u\|_{\frac{5}{2}} = \max_{t \in [0, 1]} |t^{\frac{2}{5}} u(t)|$.

It is not difficult to obtain

$$|f(t, u(t)) - f(t, v(t))| \leq \varphi(t) \|u - v\|_{\frac{5}{2}},$$

where $\varphi(t) = \frac{1}{30\sqrt[10]{t}} \in L^{\frac{1}{p}}[0, 1]$ $\left(p = \frac{3}{20}\right)$ and $\|\varphi\|_{L^{\frac{1}{p}}} = \frac{3^{\frac{3}{20}}}{30}$. Moreover,

$$|f(t, 0)| = \frac{1}{30\sqrt[10]{t}} |\sin t^{\frac{3}{4}}| \leq \frac{1}{30},$$

and thus $\sup_{t \in J} |f(t, 0)| \leq \frac{1}{30}$.

By direct computation, we find

$$\begin{aligned} A\left(\frac{3}{5}, 1\right) &= \frac{1}{8\Gamma(3)} \int_0^1 s^{\frac{3}{10}} (1-s)^2 E_{\frac{3}{5}, \frac{8}{5}} \left(-3s^{\frac{3}{10}}\right) ds \approx 0.005, \\ M_1 &= \frac{1}{\rho^q \Gamma(q+1) \Gamma(\alpha+1) A(\alpha, 1)} + 1 = \frac{1}{8\Gamma(4) \Gamma\left(\frac{8}{5}\right) A\left(\frac{3}{5}, 1\right)} + 1 \approx 5.6; \\ M_2 &= \left[\frac{1}{\beta} \left(\frac{1-p}{\alpha-p}\right)^{1-p} + \frac{1}{\alpha} \left(\frac{1-p}{\beta-p}\right)^{1-p} \right] \frac{\|\varphi\|_{L^{\frac{1}{p}}}}{\Gamma(\alpha) \Gamma(\beta)} = \left[5 \left(\frac{17}{9}\right)^{\frac{17}{20}} + \frac{5}{3} (17)^{\frac{17}{20}} \right] \frac{3^{\frac{3}{20}}}{30\Gamma\left(\frac{3}{5}\right) \Gamma\left(\frac{1}{5}\right)} \approx 0.16, \end{aligned}$$

consequently, $M_1 M_2 \approx 0.9 < 1$, by Theorems 5.1 and 6.4, problem (7.1) and (7.2) has a unique solution and Eq. (7.1) is Ulam-Hyers stable.

References

- [1] Kenneth S. Miller and Bertram Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
- [2] Stefan G. Samko, Anatoly A. Kilbas, and Oleg I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, 1993.
- [3] Igor Podlubny, *Fractional Differential Equations*, vol. 198, *Mathematics in Science and Engineering*, Academic Press, San Diego, 1999.
- [4] Anatoly A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*, *North-Holland Mathematics Studies 204*, Elsevier, Amsterdam, 2006.
- [5] Udit N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl. **6** (2014), no. 4, 1–15.
- [6] Fang Li, Jin Liang, and Hong-Kun Xu, *Existence of mild solutions for fractional integrodifferential equations of Sobolev type with nonlocal conditions*, J. Math. Anal. Appl. **391** (2012), no. 2, 510–525, DOI: 10.2478/s12175-012-0055-4.
- [7] Jin Liang, Yunyi Mu, and Ti-Jun Xiao, *Solutions to fractional Sobolev-type integro-differential equations in Banach spaces with operator pairs and impulsive conditions*, Banach J. Math. Anal. **13** (2019), no. 4, 745–768, DOI: 10.1215/17358787-2019-0017.
- [8] Jin Liang, James H. Liu, and Ti-Jun Xiao, *Condensing operators and periodic solutions of infinite delay impulsive evolution equations*, Discrete Contin. Dyn. Syst. Ser. S **10** (2017), no. 3, 475–485, DOI: 10.3934/dcdss.2017023.
- [9] Yunsong Miao and Fang Li, *Boundary value problems of the nonlinear multiple base points impulsive fractional differential equations with constant coefficients*, Adv. Difference Equ. **2017** (2017), 190, DOI: 10.1186/s13662-017-1249-4.
- [10] Om P. Agrawal, *Fractional variational calculus and transversality condition*, J. Phys. A: Math. Gen. **39** (2006), no. 33, 10375–10384, DOI: 10.1088/0305-4470/39/33/008.

- [11] Teodor M. Atanackovic and Bogoljub Stankovic, *On a differential equation with left and right fractional derivatives*, Fract. Calc. Appl. Anal. **10** (2007), no. 2, 139–150.
- [12] Tomasz Blaszczyk and Mariusz Ciesielski, *Fractional oscillator equation – transformation into integral equation and numerical solution*, Appl. Math. Comput. **257** (2015), 428–435, DOI: 10.1016/j.amc.2014.12.122.
- [13] Assia Guezane-Lakoud, Rabah Khaldi, and Adem Kiliçman, *Solvability of a boundary value problem at resonance*, SpringerPlus **5** (2016), 1504, DOI: 10.1186/s40064-016-3172-7.
- [14] Rabah Khaldi and Assia Guezane-Lakoud, *Higher order fractional boundary value problems for mixed type derivatives*, J. Nonlinear Funct. Anal. **2017** (2017), 30, DOI: 10.23952/jnfa.2017.30.
- [15] Stanisław M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1960.
- [16] Donald H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. **27** (1941), no. 4, 222–224, DOI: 10.1073/pnas.27.4.222.
- [17] Donald H. Hyers, George Isac, and Themistocles M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
- [18] Wei Wei, Xuezhu Li, and Xia Li, *New stability results for fractional integral equation*, Comput. Math. Appl. **64** (2012), no. 10, 3468–3476, DOI: 10.1016/j.camwa.2012.02.057.
- [19] JinRong Wang and Xuezhu Li, *Ulam-Hyers stability of fractional Langevin equations*, Appl. Math. Comput. **258** (2015), 72–83, DOI: 10.1016/j.amc.2015.01.111.