Research Article

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Asymptotic normality and mean consistency of LS estimators in the errors-in-variables model with dependent errors

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Abstract: In this article, an errors-in-variables regression model in which the errors are negatively superadditive dependent (NSD) random variables is studied. First, the Marcinkiewicz-type strong law of large numbers for NSD random variables is established. Then, we use the strong law of large numbers to investigate the asymptotic normality of least square (LS) estimators for the unknown parameters. In addition, the mean consistency of LS estimators for the unknown parameters is also obtained. Some results for independent random variables and negatively associated random variables are extended and improved to the case of NSD setting. At last, two simulations are presented to verify the asymptotic normality and mean consistency of LS estimators in the model.

Keywords: errors-in-variables model, negatively superadditive dependent, asymptotic normality, mean consistency, strong law of large numbers

MSC 2020: 62F12, 60F05, 62E20, 62J05

1 Introduction

To correct the effects of sampling errors, Deaton [1] proposed the errors-in-variables (EV) regression model, which is somewhat more practical than the ordinary regression model, and hence has attracted more attention.

In this article, we consider the following linear regression model:

$$y_i = \beta_0 + \beta_1 x_i + e_i, \quad U_i = x_i + \omega_i, \quad 1 \leq i \leq n, \quad (1.1)$$

where $\beta_0, \beta_1, x_1, x_2, \dots$ are unknown parameters or constants, $(e_1, \omega_1), (e_2, \omega_2), \dots$ are random vectors and $U_i, y_i, 1 \leq i \leq n$ are observable variables. From (1.1), we have

$$y_i = \beta_0 + \beta_1 U_i + e_i - \beta_1 \omega_i, \quad 1 \leq i \leq n. \quad (1.2)$$

Consider formally (1.2) as an usual regression model of y_i on U_i with errors $e_i - \beta_1 \omega_i$, we obtain the least square (LS) estimators of β_1 and β_0 as

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$$\tilde{\beta}_{1n} = \frac{\sum_{i=1}^n (U_i - \bar{U}_n)(y_i - \bar{y}_n)}{\sum_{i=1}^n (U_i - \bar{U}_n)^2}, \quad \tilde{\beta}_{0n} = \bar{y}_n - \tilde{\beta}_{1n} \bar{U}_n, \quad (1.3)$$

where $\bar{U}_n = \frac{1}{n} \sum_{i=1}^n U_i$, $\bar{y}_n = \frac{1}{n} \sum_{i=1}^n y_i$, $\bar{\omega}_n = \frac{1}{n} \sum_{i=1}^n \omega_i$, and $\bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i$.

In the last few decades, many authors focused on investigating the EV models. In the case of independent random errors, the consistency of LS estimators in the linear EV model was established by Liu and Chen [2] and Chen et al. [3]; Miao et al. [4] obtained the central limit theorem for the LS estimators in the simple linear EV regression model; Xu and Li [5] studied the consistency of LS estimators in the linear EV regression model with replicate observations; Miao et al. [6] established some limit behaviors of estimators in the simple linear EV regression model; Miao and Yang [7] obtained the loglog law for the LS estimators in the EV regression model; Miao et al. [8] investigated the consistency and asymptotic normality of LS estimators in the simple linear EV regression model, and so on. In the case of dependent random errors, Fazekas and Kukush [9] obtained the consistency of the regression parameter of the nonlinear functional EV models under mixing conditions; Fan et al. [10] established the asymptotic properties for the LS estimators of the unknown parameters in the simple linear EV regression model with stationary-mixing errors; Miao et al. [11] derived the asymptotic normality and strong consistency of the estimators in the simple linear EV model with negatively associated (NA) errors; Miao et al. [12] obtained the weak consistency and strong consistency of LS estimators for the unknown parameters with martingale difference errors; Shen [13] studied some asymptotic properties of estimators in the EV model with martingale difference errors, and so forth. In this article, we consider model (1.1) under the assumptions that the random errors are negatively superadditive dependent (NSD) random variables whose concept was proposed by Hu [14] as follows.

Definition 1.1. [15] A function $\phi : R^n \rightarrow R$ is called superadditive if

$$\phi(x \vee y) + \phi(x \wedge y) \geq \phi(x) + \phi(y)$$

for all $x, y \in R^n$, where $\vee$ is for component-wise maximum and $\wedge$ is for component-wise minimum.

Definition 1.2. [14] A random vector $X = (X_1, X_2, \dots, X_n)$ is said to be NSD if

$$E\phi(X_1, X_2, \dots, X_n) \leq E\phi(X_1^*, X_2^*, \dots, X_n^*), \quad (1.4)$$

where $X_1^*, X_2^*, \dots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i , and ϕ is a superadditive function such that the expectations in (1.4) exist.

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be NSD, if for all $n \geq 1$, $(X_1, X_2, \dots, X_n)$ is NSD.

Since then, a series of useful results on NSD sequences of random variables have been established. Hu [14] and Christofides and Vaggelatou [16] reported that the family of NSD sequences contains NA (in particular, independent) sequences and some more sequences of random variables which are not much deviated from being NA. Hu [14] gave an example illustrating that NSD sequences do not imply NA sequences (see ref. [17]). Moreover, Hu [14] derived some basic properties and three structural theorems of NSD. Eghbal et al. [18] provided two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables. Some Rosenthal-type inequalities for maximum partial sums of NSD sequences were established by Wang et al. [19]. The complete convergence and complete moment for arrays of rowwise NSD random variables were obtained by Meng et al. [20]. Amini et al. [21] investigated the complete convergence of moving average for NSD random variables. Yu et al. [22] studied the M-test in linear models with NSD errors. Zeng and Liu [23] established the asymptotic normality of difference-based estimator in a partially linear model with NSD errors. For more details about NSD random variables, one can refer to [24–29], and so on.

However, we have not found the studies on the asymptotic normality and mean consistency of LS estimators in the EV regression model with NSD random errors in the literature. In this article, we mainly investigate the asymptotic properties of the estimators for the unknown parameters in the simple linear EV

regression model, which was proposed by Deaton [1] to correct the effects of sampling errors and is somewhat more practical than the ordinary regression model. Many authors have obtained the asymptotic properties of the estimators for the unknown parameters in the EV regression model for independent random errors, which are not reasonable in real practice. In this article, we assume that the random errors are NSD, which include independent and NA random variables as special cases. So to study model (1.1) with NSD random errors is of considerable significance. The main novelties of this article can be outlined as follows. First, the Marcinkiewicz-type strong law of large numbers for NSD random variables is established, which extends and improves the classical Marcinkiewicz-type strong law of large numbers for independent and identically distributed random variables to NSD random variables with non-identical distribution. Second, we use the strong law of large numbers to investigate the asymptotic normality of LS estimators for the unknown parameters. In addition, the mean consistency of LS estimators for the unknown parameters is also obtained. These results include the corresponding ones of independent random errors and some dependent random errors as special cases. Finally, two simulation studies are carried out to verify the validity of the results that we have established.

Definition 1.3. [29] A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

The remainder of this article is organized as follows. Under some mild conditions, the asymptotic normality and quadratic-mean consistency of LS estimators for the unknown parameters in model (1.1) with NSD random errors are established in Section 2. We give some preliminary lemmas in Section 3. We provide the proofs of the main results in Section 4. In Section 5, two simulations are carried out to study the numerical performance of the results that we have established.

Throughout this article, let C be a positive constant whose values may vary at different places. $\xrightarrow{P}$ stands for convergence in probability, $\xrightarrow{d}$ stands for convergence in distribution, and $\xrightarrow{a.s.}$ represents almost sure convergence. $\triangleq$ means “defined as.”

2 Main results

Model (1.1) to be studied can be exactly described as follows:

$$\begin{cases} y_i = \beta_0 + \beta_1 x_i + e_i, U_i = x_i + \omega_i, 1 \leq i \leq n; \\ Ee_i = E\omega_i = 0, 1 \leq i \leq n, \end{cases} \quad (2.1)$$

where (U_i, y_i) , $1 \leq i \leq n$, are observable vectors, while x_i , $1 \leq i \leq n$, are constants, β_0 and β_1 are unknown parameters. Denote $T_n = \sum_{i=1}^n (x_i - \bar{x}_n)^2$ for all $n \geq 1$.

Based on the notations above and by simple calculation, we have

$$\tilde{\beta}_{1n} - \beta_1 = \frac{\sum_{i=1}^n (\omega_i - \bar{\omega}_n) e_i + \sum_{i=1}^n (x_i - \bar{x}_n) (e_i - \beta_1 \omega_i) - \beta_1 \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2}{\sum_{i=1}^n (U_i - \bar{U}_n)^2} \quad (2.2)$$

and

$$\tilde{\beta}_{0n} - \beta_0 = (\beta_1 - \tilde{\beta}_{1n}) \bar{x}_n + (\beta_1 - \tilde{\beta}_{1n}) \bar{\omega}_n + \bar{e}_n - \beta_1 \bar{\omega}_n. \quad (2.3)$$

To obtain our results, the following conditions are sufficient.

- (C₁) $0 < Ee_i^2 = h_1 < \infty$ and $0 < E\omega_i^2 = h_2 < \infty$;
- (C₂) $\lim_{n \rightarrow \infty} \frac{n}{\sqrt{T_n}} = 0$;
- (C₃) $r_n = O(n^{-p})$ for some $p > 1/2$, where $r_n = \max_{1 \leq i \leq n} \frac{|x_i - \bar{x}_n|}{\sqrt{T_n}}$;

- (C₄) $\liminf_{n \rightarrow \infty} \sigma_{1n} \geq c_1 > 0$, where $\sigma_{1n}^2 = \text{Var}\left[\frac{1}{\sqrt{T_n}} \sum_{i=1}^n (x_i - \bar{x}_n)(e_i - \beta_1 \omega_i)\right]$;
- (C₅) $\liminf_{n \rightarrow \infty} \sigma_{0n} \geq c_2 > 0$, where $\sigma_{0n}^2 = \text{Var}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^n (e_i - \beta_1 \omega_i)\right]$;
- (C₆) $\frac{T_n}{n\bar{x}_n^2} \rightarrow \infty$ as $n \rightarrow \infty$;
- (C₇) $\lim_{n \rightarrow \infty} \frac{n}{T_n} = 0$;
- (C₈) $|\omega_i| \leq M$ for some $M \geq 0$ and all $1 \leq i \leq n$.

2.1 Asymptotic normality

In this subsection, we state the asymptotic normality of LS estimators $\tilde{\beta}_{1n}$ and $\tilde{\beta}_{0n}$ for the unknown parameters β_1 and β_0 .

Theorem 2.1. *In model (2.1), let $\{e_i, i \geq 1\}$ and $\{\omega_i, i \geq 1\}$ be both stationary NSD sequences of random variables, and they are independent with each other. Suppose that conditions (C₁)–(C₄) hold. Then,*

$$\frac{\sqrt{T_n}}{\sigma_{1n}}(\tilde{\beta}_{1n} - \beta_1) \xrightarrow{d} N(0, 1), \quad (2.4)$$

where $N(0, 1)$ represents the standard normal distribution.

Theorem 2.2. *In model (2.1), let $\{e_i, i \geq 1\}$ and $\{\omega_i, i \geq 1\}$ be both strictly stationary NSD sequences of random variables, and they are independent with each other. Suppose that conditions (C₁)–(C₃) and (C₅)–(C₆) hold. Then,*

$$\frac{\sqrt{n}}{\sigma_{0n}}(\tilde{\beta}_{0n} - \beta_0) \xrightarrow{d} N(0, 1). \quad (2.5)$$

Remark 2.1. Since the family of NSD sequences of random variables includes independent and NA sequences, the results of Theorems 2.1 and 2.2 also hold for independent random errors and NA random errors.

2.2 Mean consistency

In this subsection, we state the quadratic-mean consistency of LS estimators $\tilde{\beta}_{1n}$ and $\tilde{\beta}_{0n}$ for the unknown parameters β_1 and β_0 .

Theorem 2.3. *In model (2.1), let $\{e_i, i \geq 1\}$ and $\{\omega_i, i \geq 1\}$ be both stationary NSD sequences of random variables, and they are independent of each other. Suppose that conditions (C₁), (C₇), and (C₈) hold. Then,*

$$\lim_{n \rightarrow \infty} E|\tilde{\beta}_{1n} - \beta_1|^2 = 0. \quad (2.6)$$

Remark 2.2. As independent random variables are special NSD random variables, Theorem 2.3 generalizes and improves the corresponding result of Liu and Chen [2] for independent and identically distributed random errors in the case of NSD setting.

Theorem 2.4. *Suppose that the conditions of Theorem 2.3 are satisfied. Then,*

$$\lim_{n \rightarrow \infty} E|\tilde{\beta}_{0n} - \beta_0|^2 = 0. \quad (2.7)$$

Remark 2.3. As independent and NA random variables are special NSD random variables, the result of Theorem 2.4 also holds for NA random errors.

3 Preliminary lemmas

In this section, we present some important lemmas which will be used to prove the main results of the article.

Lemma 3.1. [14] Suppose that $(X_1, X_2, \dots, X_n)$ is NSD.

- (i) $(-X_1, -X_2, \dots, -X_n)$ is also NSD.
- (ii) If $g_1, g_2, \dots, g_n$ are all non-decreasing or non-increasing functions, then $((g_1(X_1), g_2(X_2), \dots, g_n(X_n)))$ is NSD.

Lemma 3.2. [19, Rosenthal-type inequality] Let $p > 1$ and $\{X_n, n \geq 1\}$ be a sequence of NSD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$. Then, there exists a positive constant D_p depending only on p such that for all $n \geq 1$,

$$E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i \right|^p \right) \leq D_p \sum_{i=1}^n E|X_i|^p$$

for $1 < p \leq 2$ and

$$E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i \right|^p \right) \leq D_p \left(\sum_{i=1}^n E|X_i|^p + \left(\sum_{i=1}^n EX_i^2 \right)^{p/2} \right)$$

for $p > 2$.

From Lemmas 3.1 and 3.2, we can easily derive the following corollary.

Corollary 3.1. (Khinchine-Kolmogorov-type convergence theorem) Let $\{X_n, n \geq 1\}$ be an NSD sequence of random variables with $\sum_{i=1}^{\infty} \text{Var}(X_i) < \infty$, then

$$\sum_{i=1}^{\infty} (X_i - EX_i)$$

converges a.s.

Lemma 3.3. [14] Suppose that $X = (X_1, X_2, \dots, X_n)$ and $Z = (Z_1, Z_2, \dots, Z_n)$ are independent random vectors. If X and Z are both NSD, then $(X_1 + Z_1, X_2 + Z_2, \dots, X_n + Z_n)$ is NSD.

Lemma 3.4. [22,23] Let $\{X_n, n \geq 1\}$ be an NSD sequence of random variables with $EX_n = 0$ and $\sup_{j \geq 1} \sum_{i: |i-j| \geq n} |\text{Cov}(X_i, X_j)| \rightarrow 0$ as $n \rightarrow \infty$. Assume that $\{a_{ni}, 1 \leq i \leq n\}$ is an array of real numbers with $\sum_{i=1}^n a_{ni}^2 = O(1)$ and $\max_{1 \leq i \leq n} |a_{ni}| \rightarrow 0$ as $n \rightarrow \infty$. If $\{X_n, n \geq 1\}$ is uniformly integral in L_2 , then

$$\sigma_n^{-1} \sum_{i=1}^n a_{ni} X_i \xrightarrow{d} N(0, 1),$$

where $\sigma_n^2 = \text{Var}(\sum_{i=1}^n a_{ni} X_i)$.

Remark 3.1. [11] If $\{X_n, n \geq 1\}$ is a stationary sequence with $EX_n^2 < \infty$, then $\{X_n, n \geq 1\}$ is uniformly integral in L_2 . In addition, the assumption $EX_1^2 + 2 \sum_{j=2}^{\infty} EX_1 X_j > 0$ implies $\sup_{j \geq 1} \sum_{i: |i-j| \geq n} |\text{Cov}(X_i, X_j)| \rightarrow 0$.

Lemma 3.5. [29] Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X . For any $a > 0$ and $\beta > 0$, the following two statements hold:

$$\begin{aligned} E|X_n|^\beta I(|X_n| \leq a) &\leq C_1[E|X|^\beta I(|X| \leq a) + a^\beta P(|X| > a)], \\ E|X_n|^\beta I(|X_n| > a) &\leq C_2 E|X|^\beta I(|X| > a), \end{aligned}$$

where C_1 and C_2 are positive constants. Thus,

$$E|X_n|^\beta \leq CE|X|^\beta,$$

where C is a positive constant.

Lemma 3.6. (Marcinkiewicz-type strong law of large numbers) Let $\{X_n, n \geq 1\}$ be an NSD sequence of random variables which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for $0 < p < 2$. Assume that $EX_n = 0$ if $1 \leq p < 2$. Then,

$$n^{-1/p} \sum_{i=1}^n X_i \xrightarrow{a.s.} 0. \quad (3.1)$$

Proof. Denote $X'_n = -n^{1/p}I(X_n \leq -n^{1/p}) + X_n I(|X_n| < n^{1/p}) + n^{1/p}I(X_n \geq n^{1/p})$, then we obtain by $E|X|^p < \infty$ that

$$\sum_{n=1}^{\infty} P(X_n \neq X'_n) = \sum_{n=1}^{\infty} P(|X_n| \geq n^{1/p}) \leq C \sum_{n=1}^{\infty} P(|X| \geq n^{1/p}) \leq CE|X|^p < \infty.$$

Hence, it follows by the Borel-Cantelli lemma that

$$X_n = X'_n \text{ a.s.} \quad (3.2)$$

Thus, to prove (3.1), it suffices to show that

$$n^{-1/p} \sum_{i=1}^n X'_i \xrightarrow{a.s.} 0. \quad (3.3)$$

Hence, to prove (3.3), it suffices to show that

$$n^{-1/p} \sum_{i=1}^n EX'_i \xrightarrow{a.s.} 0 \quad (3.4)$$

and

$$n^{-1/p} \sum_{i=1}^n (X'_i - EX'_i) \xrightarrow{a.s.} 0. \quad (3.5)$$

First, we will prove (3.4). It will be divided into the following two cases:

(i) If $p = 1$, it follows from $EX_n = 0$ and $E|X|^p < \infty$ that

$$|EX_n I(|X_n| \leq n)| \leq CE|X|I(|X| > n) \rightarrow 0$$

as $n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} nP(|X_n| > n) \leq \lim_{n \rightarrow \infty} E|X_n|I(|X_n| > n) \leq C \lim_{n \rightarrow \infty} E|X|I(|X| > n) = 0.$$

Hence,

$$|EX'_n| \leq nP(|X_n| > n) + |EX_n I(|X_n| \leq n)| \rightarrow 0$$

as $n \rightarrow \infty$. By the Toeplitz lemma, (3.4) holds.

(ii) If $p \neq 1$, by the Kronecker lemma, to prove (3.4), it suffices to show that

$$\sum_{n=1}^{\infty} \frac{|EX'_n|}{n^{1/p}} < \infty. \quad (3.6)$$

For $0 < p < 1$, it follows from $E|X|^p < \infty$ and Lemma 3.5 that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|EX'_n|}{n^{1/p}} &\leq C \sum_{n=1}^{\infty} \frac{n^{1/p}P(|X| \geq n^{1/p}) + E|X|I(|X| < n^{1/p})}{n^{1/p}} \leq C + C \sum_{n=1}^{\infty} \sum_{j=1}^n n^{-1/p} E|X|I(j-1 \leq |X|^p < j) \\ &= C + C \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} n^{-1/p} E|X|I(j-1 \leq |X|^p < j) \leq C + C \sum_{j=1}^{\infty} j^{-1/p+1} E|X|^p j^{(1-p)/p} I(j-1 \leq |X|^p < j) \\ &\leq C + C \sum_{j=1}^{\infty} j^{-1/p+1} E|X|^p j^{(1-p)/p} I(j-1 \leq |X|^p < j). \end{aligned}$$

For $1 < p < 2$, it follows from $EX_n = 0$, $E|X|^p < \infty$ and Lemma 3.5 that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|EX'_n|}{n^{1/p}} &\leq C \sum_{n=1}^{\infty} \frac{n^{1/p}P(|X| \geq n^{1/p}) + |EX|I(|X| < n^{1/p})}{n^{1/p}} \leq C \sum_{n=1}^{\infty} \frac{n^{1/p}P(|X| \geq n^{1/p}) + |EX|I(|X| < n^{1/p})}{n^{1/p}} \\ &\leq C + C \sum_{n=1}^{\infty} n^{-1/p} E|X|I(|X| \geq n^{1/p}) = C + C \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} n^{-1/p} E|X|I(j \leq |X|^p < j+1) \\ &= C + C \sum_{j=1}^{\infty} \sum_{n=1}^j n^{-1/p} E|X|I(j \leq |X|^p < j+1) \leq C + C \sum_{j=1}^{\infty} j^{-1/p+1} E|X|^p j^{(1-p)/p} I(j \leq |X|^p < j+1) \\ &= C + C \sum_{j=1}^{\infty} E|X|^p I(j \leq |X|^p < j+1) < \infty. \end{aligned}$$

Hence, (3.6) holds.

Next, we will prove (3.5).

By Lemma 3.1, we know that $\{X'_n - EX'_n, n \geq 1\}$ is still an NSD sequence of random variables. Hence, by Corollary 3.1 and Kronecker lemma, to prove (3.5), we only need to show that

$$\sum_{n=1}^{\infty} \text{Var}(X'_n/n^{1/p}) < \infty.$$

By the Markov inequality, Lemma 3.5 and $E|X|^p < \infty$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \text{Var}\left(\frac{X'_n}{n^{1/p}}\right) &\leq \sum_{n=1}^{\infty} \frac{E(X'_n)^2}{n^{2/p}} \leq C \sum_{n=1}^{\infty} \frac{n^{2/p}P(|X| \geq n^{1/p}) + EX^2I(|X| < n^{1/p})}{n^{2/p}} \\ &\leq C + C \sum_{n=1}^{\infty} n^{-2/p} \sum_{k=1}^n EX^2I(k-1 \leq |X|^p < k) = C + C \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} n^{-2/p} EX^2I(k-1 \leq |X|^p < k) \\ &\leq C + C \sum_{k=1}^{\infty} k^{-2/p+1} E|X|^p k^{(2-p)/p} I(k-1 \leq |X|^p < k) \\ &= C + C \sum_{k=1}^{\infty} E|X|^p I(k-1 \leq |X|^p < k) < \infty. \end{aligned}$$

This completes the proof of Lemma 3.6. $\square$

Remark 3.2. Lemma 3.5 is the Marcinkiewicz-type strong law of large numbers for NSD sequences of random variables, which extend and improve the classical Marcinkiewicz-type strong law of large numbers for independent and identically distributed random variables to NSD random variables with non-identical distribution. It also holds for NA random variables with non-identical distribution.

4 Proofs of the main results

Proof of Theorem 2.1. In view of (2.2), we have

$$\begin{aligned} \frac{\sqrt{T_n}}{\sigma_{1n}}(\tilde{\beta}_{1n} - \beta_1) &= \frac{\sqrt{T_n}}{\sigma_{1n}} \cdot \frac{\frac{1}{T_n} \left(\sum_{i=1}^n (\omega_i - \bar{\omega}_n) e_i + \sum_{i=1}^n (x_i - \bar{x}_n)(e_i - \beta_1 \omega_i) - \beta_1 \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2 \right)}{\frac{1}{T_n} \sum_{i=1}^n (U_i - \bar{U}_n)^2} \\ &= \frac{\frac{1}{\sigma_{1n} \sqrt{T_n}} \left(\sum_{i=1}^n (\omega_i - \bar{\omega}_n) e_i + \sum_{i=1}^n (x_i - \bar{x}_n)(e_i - \beta_1 \omega_i) - \beta_1 \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2 \right)}{\frac{1}{T_n} \sum_{i=1}^n (U_i - \bar{U}_n)^2}. \end{aligned}$$

Hence, by Slutsky's theorem, to prove (2.4), it suffices to show that

$$\frac{\sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2}{\sqrt{T_n}} \xrightarrow{P} 0, \quad (4.1)$$

$$\frac{\sum_{i=1}^n (\omega_i - \bar{\omega}_n) e_i}{\sqrt{T_n}} \xrightarrow{P} 0, \quad (4.2)$$

$$\frac{\sqrt{T_n} \sum_{i=1}^n (x_i - \bar{x}_n)(e_i - \beta_1 \omega_i)}{\sigma_{1n} T_n} \xrightarrow{d} N(0, 1), \quad (4.3)$$

and

$$\frac{\sum_{i=1}^n (U_i - \bar{U}_n)^2}{T_n} \xrightarrow{P} 1. \quad (4.4)$$

First, we prove (4.1). Note that

$$\frac{\sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2}{\sqrt{T_n}} \leq \frac{\sum_{i=1}^n \omega_i^2}{\sqrt{T_n}},$$

hence, (4.1) follows from (C₂).

Second, we prove (4.2). Similar to the proof of (4.1), we can get that

$$\frac{\sum_{i=1}^n (e_i - \bar{e}_n)^2}{\sqrt{T_n}} \xrightarrow{P} 0. \quad (4.5)$$

Since

$$\left| \sum_{i=1}^n (\omega_i - \bar{\omega}_n) e_i \right| \leq \frac{1}{2} \left[\sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2 + \sum_{i=1}^n (e_i - \bar{e}_n)^2 \right],$$

and together with (4.1) and (4.5), (4.2) follows.

Third, we prove (4.4). By the Cauchy-Schwarz inequality, we have, for any $m > 0$,

$$\begin{aligned} \sum_{i=1}^n |x_i - \bar{x}_n| \cdot |\omega_i - \bar{\omega}_n| &\leq \sqrt{m \sum_{i=1}^n (x_i - \bar{x}_n)^2} \cdot \sqrt{\frac{1}{m} \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2} \\ &\leq \frac{m \sum_{i=1}^n (x_i - \bar{x}_n)^2 + \frac{1}{m} \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2}{2} \\ &= \frac{m}{2} T_n + \frac{1}{2m} \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2. \end{aligned}$$

Note that

$$\sum_{i=1}^n (U_i - \bar{U}_n)^2 = \sum_{i=1}^n (x_i - \bar{x}_n)^2 + 2 \sum_{i=1}^n (x_i - \bar{x}_n)(\omega_i - \bar{\omega}_n) + \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2.$$

Thus,

$$\begin{aligned} \left| \sum_{i=1}^n (U_i - \bar{U}_n)^2 - T_n \right| &= \left| 2 \sum_{i=1}^n (x_i - \bar{x}_n)(\omega_i - \bar{\omega}_n) + \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2 \right| \\ &\leq 2 \left| \sum_{i=1}^n (x_i - \bar{x}_n)(\omega_i - \bar{\omega}_n) \right| + \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2 \\ &\leq m T_n + \frac{1+2m}{2m} \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2. \end{aligned} \quad (4.6)$$

By (4.1) and (4.6), we have

$$\left| \frac{1}{T_n} \sum_{i=1}^n (U_i - \bar{U}_n)^2 - 1 \right| \leq m + \frac{1+2m}{2m} \cdot \frac{1}{T_n} \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2 \xrightarrow{P} m. \quad (4.7)$$

Since $m > 0$ is arbitrary and from (4.7), it follows that

$$\frac{1}{T_n} \sum_{i=1}^n (U_i - \bar{U}_n)^2 - 1 \xrightarrow{P} 0,$$

which implies (4.4).

Finally, we will prove (4.3). Denote $X_i = e_i - \beta_1 \omega_i$ and $a_{ni} = \frac{x_i - \bar{x}_n}{\sigma_{1n} \sqrt{T_n}}$, then

$$\frac{\sqrt{T_n} \sum_{i=1}^n (x_i - \bar{x}_n)(e_i - \beta_1 \omega_i)}{\sigma_{1n} T_n} = \sum_{i=1}^n a_{ni} X_i.$$

By Lemmas 3.1 and 3.3, we know that $\{X_i, i \geq 1\}$ is still an NSD sequence of random variables. It is easy to check that $\sum_{i=1}^n a_{ni}^2 = O(1)$ and $\max_{1 \leq i \leq n} |a_{ni}| \rightarrow 0$ by (C₂) and (C₃). By the stationarity of $\{X_i, i \geq 1\}$ and $EX_i^2 = E(\varepsilon_i - \beta_1 \delta_i)^2 \leq CE\varepsilon_i^2 + CE\delta_i^2 < \infty$, we know that $\{X_i, i \geq 1\}$ is uniformly integral in L_2 (see Remark 3.1). From (C₄), it follows that

$$\text{Var} \left(\sum_{i=1}^n a_{ni} X_i \right) = \text{Var} \left(\sum_{i=1}^n \frac{x_i - \bar{x}_n}{\sigma_{1n} \sqrt{T_n}} (e_i - \beta_1 \omega_i) \right) = \frac{\text{Var}(\sum_{i=1}^n (x_i - \bar{x}_n)(e_i - \beta_1 \omega_i))}{\sigma_{1n}^2 T_n} = 1.$$

By (C₃) and (C₄), we have

$$\begin{aligned} \sigma_{1n}^2 &= \text{Var} \left(\sum_{i=1}^n \frac{x_i - \bar{x}_n}{\sqrt{T_n}} (e_i - \beta_1 \omega_i) \right) \\ &\leq \left(\max_{1 \leq i \leq n} \frac{|x_i - \bar{x}_n|}{\sqrt{T_n}} \right)^2 \text{Var} \left(\sum_{i=1}^n X_i \right) < \frac{1}{n} E \left(\sum_{i=1}^n X_i \right)^2 \\ &= EX_1^2 + 2 \left(\frac{n-1}{n} EX_1 X_2 + \frac{n-2}{n} EX_1 X_3 + \cdots + \frac{1}{n} EX_1 X_n \right) \\ &\leq EX_1^2 + 2 \left(\frac{n-1}{n} EX_1 X_2 + \cdots + \frac{n-m+1}{n} EX_1 X_m \right). \end{aligned}$$

Hence, for any fixed m ,

$$\sigma_{1n}^2 < EX_1^2 + 2(EX_1 X_2 + \cdots + EX_1 X_m)$$

as $n \rightarrow \infty$. Hence,

$$0 < \sigma_{1n}^2 < EX_1^2 + 2 \sum_{j=2}^{\infty} EX_1 X_j$$

as $m \rightarrow \infty$. Thus, by Remark 3.1, we have $\sup_{j \geq 1} \sum_{i: |i-j| \geq n} |\text{Cov}(X_i, X_j)| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, (4.3) follows from Lemma 3.4.

This completes the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. In view of (2.3), we have

$$\begin{aligned} \frac{\sqrt{n}}{\sigma_{0n}}(\tilde{\beta}_{0n} - \beta_0) &= \frac{\sqrt{n}}{\sigma_{0n}}[(\beta_1 - \tilde{\beta}_{1n})\bar{x}_n + (\beta_1 - \tilde{\beta}_{1n})\bar{\omega}_n + \bar{e}_n - \beta_1\bar{\omega}_n] \\ &= \frac{\sqrt{n}}{\sigma_{0n}}(\bar{e}_n - \beta_1\bar{\omega}_n) + \frac{\sqrt{n}}{\sigma_{0n}}(\bar{x}_n + \bar{\omega}_n)(\beta_1 - \tilde{\beta}_{1n}). \end{aligned}$$

Note that

$$\frac{\sqrt{n}}{\sigma_{0n}}(\bar{e}_n - \beta_1\bar{\omega}_n) = \frac{\sqrt{n}}{\sigma_{0n}} \cdot \frac{1}{n} \sum_{i=1}^n (e_i - \beta_1\omega_i) = \frac{1}{\sigma_{0n}\sqrt{n}} \sum_{i=1}^n (e_i - \beta_1\omega_i).$$

Denote $Y_i = e_i - \beta_1\omega_i$ and $b_{ni} = \frac{1}{\sigma_{0n}\sqrt{n}}$, then $\frac{1}{\sigma_{0n}\sqrt{n}} \sum_{i=1}^n (e_i - \beta_1\omega_i) = \sum_{i=1}^n b_{ni}Y_i$. By the stationarity of $\{Y_i, i \geq 1\}$ and $EY_i^2 = E(\varepsilon_i - \beta_1\delta_i)^2 \leq CE\varepsilon_i^2 + CE\delta_i^2 < \infty$, we know that $\{Y_i, i \geq 1\}$ is uniformly integral in L_2 (see Remark 3.1). From (C₅), it follows that

$$\text{Var}\left(\sum_{i=1}^n b_{ni}Y_i\right) = \text{Var}\left(\sum_{i=1}^n \frac{1}{\sigma_{0n}\sqrt{n}}(e_i - \beta_1\omega_i)\right) = \frac{\text{Var}(\sum_{i=1}^n (e_i - \beta_1\omega_i))}{n\sigma_{0n}^2} = 1.$$

By (C₅), we have

$$\begin{aligned} \sigma_{0n}^2 &= \text{Var}\left(\sum_{i=1}^n \frac{e_i - \beta_1\omega_i}{\sqrt{n}}\right) \\ &= \frac{1}{n} \text{Var}\left(\sum_{i=1}^n Y_i\right) = \frac{1}{n} E\left(\sum_{i=1}^n Y_i\right)^2 \\ &= EY_1^2 + 2\left(\frac{n-1}{n}EY_1Y_2 + \frac{n-2}{n}EY_1Y_3 + \cdots + \frac{1}{n}EY_1Y_n\right) \\ &\leq EY_1^2 + 2\left(\frac{n-1}{n}EY_1Y_2 + \cdots + \frac{n-m+1}{n}EY_1Y_m\right). \end{aligned}$$

Hence, for any fixed m ,

$$\sigma_{0n}^2 \leq EY_1^2 + 2(EY_1Y_2 + \cdots + EY_1Y_m)$$

as $n \rightarrow \infty$. Hence,

$$0 < \sigma_{0n}^2 \leq EY_1^2 + 2 \sum_{j=2}^{\infty} EY_1Y_j$$

as $m \rightarrow \infty$. Thus, by Remark 3.1, we have

$$\sup_{j \geq 1} \sum_{i: |i-j| \geq n} |\text{Cov}(Y_i, Y_j)| \rightarrow 0$$

as $n \rightarrow \infty$. Hence, by Lemma 3.4, one can get that

$$\frac{1}{\sigma_{0n}\sqrt{n}} \sum_{i=1}^n (e_i - \beta_1\omega_i) \xrightarrow{d} N(0, 1).$$

Thus, by Theorem 2.1, to prove (2.5), it suffices to show that

$$\frac{\sqrt{n}}{\sqrt{T_n}}(\bar{x}_n + \bar{\omega}_n) = \frac{\bar{x}_n \sqrt{n}}{\sqrt{T_n}} + \frac{\sqrt{n}}{\sqrt{T_n}} \cdot \frac{1}{n} \sum_{i=1}^n \omega_i \xrightarrow{P} 0. \quad (4.8)$$

By (C₆) and Lemma 3.6, we derive that

$$\frac{\bar{x}_n \sqrt{n}}{\sqrt{T_n}} \rightarrow 0 \quad (4.9)$$

as $n \rightarrow \infty$ and

$$\frac{1}{n} \sum_{i=1}^n \omega_i \xrightarrow{a.s.} 0. \quad (4.10)$$

Therefore, (4.8) follows from (4.9) and (4.10).

This completes the proof of Theorem 2.2. $\square$

Proof of Theorem 2.3. By simple calculation, we have

$$\tilde{\beta}_{1n} - \beta_1 = \frac{\sum_{i=1}^n (U_i - \bar{U}_n) e_i - \beta_1 \sum_{i=1}^n (x_i - \bar{x}_n) \omega_i - \beta_1 \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2}{\sum_{i=1}^n (U_i - \bar{U}_n)^2} = \left[\sum_{i=1}^n (U_i - \bar{U}_n)^2 \right]^{-1} (\Delta_1 + \Delta_2 + \Delta_3).$$

Since

$$|\tilde{\beta}_{1n} - \beta_1|^2 \leq C \left[\sum_{i=1}^n (U_i - \bar{U}_n)^2 \right]^{-2} (\Delta_1^2 + \Delta_2^2 + \Delta_3^2),$$

we have

$$E|\tilde{\beta}_{1n} - \beta_1|^2 \leq CE \left\{ \left[\sum_{i=1}^n (U_i - \bar{U}_n)^2 \right]^{-2} (\Delta_1^2 + \Delta_2^2 + \Delta_3^2) \right\}.$$

Hence, to prove (2.6), it suffices to show that

$$\lim_{n \rightarrow \infty} E \left\{ \left[\sum_{i=1}^n (U_i - \bar{U}_n)^2 \right]^{-2} \Delta_1^2 \right\} = 0, \quad (4.11)$$

$$\lim_{n \rightarrow \infty} E \left\{ \left[\sum_{i=1}^n (U_i - \bar{U}_n)^2 \right]^{-2} \Delta_2^2 \right\} = 0, \quad (4.12)$$

and

$$\lim_{n \rightarrow \infty} E \left\{ \left[\sum_{i=1}^n (U_i - \bar{U}_n)^2 \right]^{-2} \Delta_3^2 \right\} = 0. \quad (4.13)$$

By (C₈), we derive that $\sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2 \leq \sum_{i=1}^n \omega_i^2 \leq nC$. Hence, by the Cauchy-Schwarz inequality, we have

$$\left[\sum_{i=1}^n (x_i - \bar{x}_i) \omega_i \right]^2 \leq \sum_{i=1}^n (x_i - \bar{x}_i)^2 \cdot \sum_{i=1}^n \omega_i^2 \leq nC^2 T_n.$$

Thus,

$$\sum_{i=1}^n (U_i - \bar{U}_n)^2 = T_n + 2 \sum_{i=1}^n (x_i - \bar{x}_n) \omega_i + \sum_{i=1}^n (\omega_i - \bar{\omega}_n)^2 \geq T_n - C\sqrt{nT_n}.$$

From (C₇), it follows that

$$\frac{C\sqrt{nT_n}}{T_n} = C\sqrt{\frac{n}{T_n}} \rightarrow 0$$

as $n \rightarrow \infty$. Thus,

$$\sum_{i=1}^n (U_i - \bar{U}_n)^2 \geq T_n - C\sqrt{nT_n} \geq CT_n$$

as $n \rightarrow \infty$. Hence, by the Cauchy-Schwarz inequality and (C₁), we obtain that

$$E\left\{\left[\sum_{i=1}^n (U_i - \bar{U}_n)^2\right]^{-2} \Delta_1^2\right\} \leq E\left\{\left[\sum_{i=1}^n (U_i - \bar{U}_n)^2\right]^{-2} \sum_{j=1}^n (U_j - \bar{U}_n)^2 \sum_{k=1}^n e_k^2\right\} = E\left\{\left[\sum_{i=1}^n (U_i - \bar{U}_n)^2\right]^{-1} \sum_{k=1}^n e_k^2\right\} \leq CnT_n^{-1}.$$

Therefore, (4.10) follows from (C₇). Similar to the proof of (4.9), one can get (4.12) and (4.13).

This completes the proof of Theorem 2.3. $\square$

Proof of Theorem 2.4. In view of (2.3), we have

$$\tilde{\beta}_{0n} - \beta_0 = (\beta_1 - \tilde{\beta}_{1n})\bar{x}_n + (\beta_1 - \tilde{\beta}_{1n})\bar{\omega}_n + \frac{1}{n} \sum_{i=1}^n (e_i - \beta_1 \omega_i).$$

Hence,

$$|\tilde{\beta}_{0n} - \beta_0|^2 \leq C \left[(\beta_1 - \tilde{\beta}_{1n})^2 \bar{x}_n^2 + (\beta_1 - \tilde{\beta}_{1n})^2 \bar{\omega}_n^2 + \frac{1}{n^2} \left(\sum_{i=1}^n (e_i - \beta_1 \omega_i) \right)^2 \right].$$

Then,

$$E|\tilde{\beta}_{0n} - \beta_0|^2 \leq CE \left[(\beta_1 - \tilde{\beta}_{1n})^2 \bar{x}_n^2 + (\beta_1 - \tilde{\beta}_{1n})^2 \bar{\omega}_n^2 + \frac{1}{n^2} \left(\sum_{i=1}^n (e_i - \beta_1 \omega_i) \right)^2 \right].$$

By Theorem 2.3 and (C₈), we can derive that

$$E[(\beta_1 - \tilde{\beta}_{1n})^2 \bar{x}_n^2] = \bar{x}_n^2 E(\beta_1 - \tilde{\beta}_{1n})^2 \rightarrow 0$$

as $n \rightarrow \infty$ and

$$E[(\beta_1 - \tilde{\beta}_{1n})^2 \bar{\omega}_n^2] \leq CE(\beta_1 - \tilde{\beta}_{1n})^2 \rightarrow 0$$

as $n \rightarrow \infty$. By Lemmas 3.1 and 3.3, we know that $\{e_i - \beta_1 \omega_i, i \geq 1\}$ is still an NSD sequence of random variables. Thus, from (C₁) and Lemma 3.2, we can get that

$$\begin{aligned} E \left[\frac{1}{n^2} \left(\sum_{i=1}^n (e_i - \beta_1 \omega_i) \right)^2 \right] &= \frac{1}{n^2} E \left(\sum_{i=1}^n (e_i - \beta_1 \omega_i) \right)^2 \leq \frac{1}{n^2} E \left[\max_{1 \leq k \leq n} \left(\sum_{i=1}^k (e_i - \beta_1 \omega_i) \right)^2 \right] \\ &\leq \frac{C}{n^2} \sum_{i=1}^n E(e_i - \beta_1 \omega_i)^2 \leq \frac{C}{n^2} \sum_{i=1}^n (Ee_i^2 + \beta_1^2 E\omega_i^2) \leq \frac{C}{n} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. $\square$

5 Numerical simulations

In this section, we verify the asymptotic normality and mean consistency of LS estimators in the EV regression model with NSD errors.

The data are generated from model (1.2). For $0 < u < 0.8$ and $n \geq 4$, let normal random vectors $(e_1, e_2, \dots, e_n) \sim N(0, \Sigma)$ and $(\omega_1, \omega_2, \dots, \omega_n) \sim N(0, \Sigma)$, where 0 represents zero vector and

$$\Sigma = \begin{pmatrix} \frac{4}{5} + u & -u & -u^2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -u & \frac{4}{5} + u & -u & -u^2 & \cdots & 0 & 0 & 0 & 0 \\ -u^2 & -u & \frac{4}{5} + u & -u & \cdots & 0 & 0 & 0 & 0 \\ 0 & -u^2 & -u & \frac{4}{5} + u & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{4}{5} + u & -u & -u^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -u & \frac{4}{5} + u & -u & -u^2 \\ 0 & 0 & 0 & 0 & \cdots & -u^2 & -u & \frac{4}{5} + u & -u \\ 0 & 0 & 0 & 0 & \cdots & 0 & -u^2 & -u & \frac{4}{5} + u \end{pmatrix}_{n \times n}.$$

By the definition of NA random variables (see [17]), we know that $(e_1, e_2, \dots, e_n)$ and $(\omega_1, \omega_2, \dots, \omega_n)$ are NA vectors, and thus NSD vectors.

5.1 Simulation example 1

In order to show the asymptotic normality of LS estimators in the model, we choose $u = 0.6$ and $x_i = i$ for $1 \leq i \leq n$. For the fixed $\beta_1 = 3$ and $\beta_0 = 5$, taking the sample sizes n as $n = 100, 500$ and $1,000$, respectively. We use R software to compute $\sigma_{1n}^{-1} \sqrt{T_n}(\tilde{\beta}_{1n} - \beta_1)$ and $\sigma_{0n}^{-1} \sqrt{n}(\tilde{\beta}_{0n} - \beta_0)$ for 1,000 times, respectively, and present the histograms and Quantile–Quantile (Q–Q) plots of them in Figures 1–6.

Figure 1: Histogram and normal Q–Q plot of $\sigma_{1n}^{-1} \sqrt{T_n}(\tilde{\beta}_{1n} - \beta_1) \triangleq \mu$ with $n = 100$.

Figure 2: Histogram and normal Q-Q plot of $\sigma_{0n}^{-1} \sqrt{n}(\tilde{\beta}_{0n} - \beta_0) \triangleq \eta$ with $n = 100$.

Figure 3: Histogram and normal Q-Q plot of $\sigma_{1n}^{-1} \sqrt{T_n}(\tilde{\beta}_{1n} - \beta_1) \triangleq \mu$ with $n = 500$.

Figure 4: Histogram and normal Q-Q plot of $\sigma_{0n}^{-1} \sqrt{n}(\tilde{\beta}_{0n} - \beta_0) \triangleq \eta$ with $n = 500$.

Figure 5: Histogram and normal Q-Q plot of $\sigma_{1n}^{-1} \sqrt{T_n}(\tilde{\beta}_{1n} - \beta_1) \triangleq \mu$ with $n = 1,000$.

Figure 6: Histogram and normal Q-Q plot of $\sigma_{0n}^{-1} \sqrt{n}(\tilde{\beta}_{0n} - \beta_0) \triangleq \eta$ with $n = 1,000$.

Figure 7: Boxplots of $\tilde{\beta}_{1n} - \beta_1 \triangleq \lambda$ with $\beta_1 = 2$ and $\beta_0 = 4$.

Figure 8: Boxplots of $\tilde{\beta}_{0n} - \beta_0 \triangleq \varphi$ with $\beta_1 = 2$ and $\beta_0 = 4$.

Figure 9: Boxplots of $\tilde{\beta}_{1n} - \beta_1 \triangleq \lambda$ with $\beta_1 = 5$ and $\beta_0 = 3$.

It can be seen from Figures 1–6 that the histograms and Q–Q plots show good fit of the distribution for $\sigma_{1n}^{-1} \sqrt{T_n}(\tilde{\beta}_{1n} - \beta_1)$ and $\sigma_{0n}^{-1} \sqrt{n}(\tilde{\beta}_{0n} - \beta_0)$ to standard normal distribution as the sample size n increases. The simulation results verify the validity of our theoretical conclusions in Theorems 2.1 and 2.2.

5.2 Simulation example 2

In order to verify the mean consistency of LS estimators in the model, we choose $u = 0.4$ and $x_i = i$ for all $1 \leq i \leq n$. For the fixed $\beta_1 = 2$ and $\beta_0 = 4$ and $\beta_1 = 5$ and $\beta_0 = 3$, taking the sample sizes n as $n = 100, 200, 300, 500, 800$ and $1,000$, respectively. We use R software to compute $\tilde{\beta}_{1n} - \beta_1$ and $\tilde{\beta}_{0n} - \beta_0$ for 600 times, respectively, and provide the boxplots of them in Figures 7–10.

Figure 10: Boxplots of $\tilde{\beta}_{0n} - \beta_0 \triangleq \varphi$ with $\beta_1 = 5$ and $\beta_0 = 3$.

It can be seen from Figures 7–10 that $\tilde{\beta}_{1n} - \beta_1$ and $\tilde{\beta}_{0n} - \beta_0$ get closer and closer to zero and the ranges of $\tilde{\beta}_{1n} - \beta_1$ and $\tilde{\beta}_{0n} - \beta_0$ decrease as the sample size n increases. The simulation results directly reflect the conclusions in Theorems 2.3 and 2.4.

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