Research Article

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Some non-commuting solutions of the Yang-Baxter-like matrix equation

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Abstract: Let A be a square matrix satisfying $A^4 = A$. We solve the Yang-Baxter-like matrix equation $AXA = XAX$ to find some solutions, based on analysis of the characteristic polynomial of A and its eigenvalues. We divide the problem into small cases so that we can find the solution easily. Finally, in order to illustrate the results, two numerical examples are presented.

Keywords: diagonalizable matrix, matrix equation, Jordan form, eigenvalue

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1 Introduction

Let A be an $n \times n$ complex matrix. The quadratic matrix equation:

$$AXA = XAX, \quad (1)$$

is often called the Yang-Baxter-like matrix equation since it is similar to the classical parameter-free Yang-Baxter equation in format [1–3]. The original Yang-Baxter equation has many applications in statistical mechanics, integrable systems, quantum theory, knot theory, braid group theory, and so on [3–6]. Although some solutions have been found for the Yang-Baxter equation in the quantum group theory, no systematical study of (1) has appeared in the literature as a purely linear algebra problem [7]. One possible reason is that solving a polynomial system of n^2 quadratic equations with n^2 unknowns is a challenging topic [7]. In the past several years, some special cases of (1) have been obtained for various classes of matrices A with different approaches in [7–26]. Because finding general solutions of the Yang-Baxter-like matrix Eq. (1) is difficult, almost all the works so far have been toward constructing commuting solutions of the equation; see, e.g., [7–17] and references therein. Commuting solutions mean that the unknown matrix X satisfies the commutability condition $AX = XA$. All commuting solutions of (1) have been obtained when A is a matrix with general Jordan structure forms in [7], but finding all non-commuting solutions of (1) is still a challenging task when A is arbitrary.

Up to now, there are only isolated results toward this goal for special classes of the given matrix A , e.g., [18–26]. All solutions have been constructed for rank-1 matrices A in [23], rank-2 matrices A in [24,25], non-diagonalizable elementary matrices A in [26], idempotent matrices A ($A^2 = A$) in [19], $A^2 = I$ in [18,20], $A^3 = A$ in [21], and diagonalizable matrices A with two different eigenvalues in [22]. In this paper, we try to solve the matrix Eq. (1) to obtain some non-commuting solutions when the given matrix A satisfies $A^4 = A$. This is an important step to solve more general matrices.

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We first provide some preliminary results. Then we study some solutions for the Yang-Baxter-like matrix equation of (1) when the given matrix A satisfies $A^4 = A$ through analysis of the characteristic polynomial of A and its eigenvalues. Finally, we give some numerical experiments to illustrate our results.

2 Preliminary results

In this section, we give some results for our further discussion. First, we need the following lemmas.

Lemma 2.1. [21, Lemma 2.1] *Let $A = \text{diag}\{M, P\}$ be an $n \times n$ matrix such that M is $m \times m$. Then the solutions of (1) are*

$$X = \begin{pmatrix} K & C \\ D & Z \end{pmatrix},$$

where the sub-matrices $K \in C^{m \times m}$, $Z \in C^{(n-m) \times (n-m)}$, C , and D satisfy

$$\begin{cases} MKM = KMK + CPD, \\ MCP = KMC + CPZ, \\ PDM = DMK + ZPD, \\ PZP = DMC + ZPZ. \end{cases} \quad (2)$$

In particular, if $P = 0$, then (2) is reduced to

$$\begin{cases} MKM = KMK, \\ KMC = 0, \\ DMK = 0, \\ DMC = 0. \end{cases}$$

Lemma 2.2. [27, Theorem 2.1] *Let $k \leq n$ be two positive integers, and let W and Λ be $n \times n$ and $k \times k$ matrices, respectively. If there exists an $n \times k$ matrix U that satisfies $WU = U\Lambda$, then the identity*

$$p_{W+UV^T}(\lambda) \cdot p_{\Lambda}(\lambda) \equiv p_W(\lambda) \cdot p_{\Lambda+V^TU}(\lambda)$$

is true for any $n \times k$ matrix V .

We assume that A is an $n \times n$ complex matrix with $n \geq 4$. Since $A^4 = A$, the polynomial

$$p(\lambda) = \lambda(\lambda - 1) \left(\lambda + \frac{1 + i\sqrt{3}}{2} \right) \left(\lambda + \frac{1 - i\sqrt{3}}{2} \right)$$

is an annihilator of A , that is, $p(A) = 0$. Thus, the eigenvalues of A constitute a subset of $\left\{0, 1, -\frac{1+i\sqrt{3}}{2}, -\frac{1-i\sqrt{3}}{2}\right\}$ and the minimal polynomial $g(\lambda)$ of A , which is the unique annihilator of A with minimal degree and leading coefficient 1, is a factor of $p(\lambda)$. Therefore, each eigenvalue of A is semi-simple, and then A is the diagonalizable matrix. So there exists a nonsingular matrix S such that $A = SJS^{-1}$, where $J = \text{diag}(\lambda_1 I_{m_1}, \dots, \lambda_r I_{m_r})$, $\lambda_i \in \left\{0, 1, -\frac{1+i\sqrt{3}}{2}, -\frac{1-i\sqrt{3}}{2}\right\}$, $i = 1, 2, \dots, r$. Here, each I_{m_i} ($i = 1, 2, \dots, r$) denotes the $m_i \times m_i$ identity matrix.

Let $Y = S^{-1}XS$, clearly that X is the solution of (1) if and only if Y is the solution of the equation:

$$JYJ = YJY. \quad (3)$$

Moreover, X is a commuting solution if and only if Y is a commuting solution. Thus, in the following we just solve (3) to get solutions of (1).

Now, we have to consider the Yang-Baxter-like matrix Eq. (1) through several cases of the given matrix A . As we can see, in some trivial case of the minimal polynomial of A , we immediately obtained the solutions of the Yang-Baxter-like matrix Eq. (1). If $g(\lambda) = 0$, then $A = 0$. Thus, all $n \times n$ matrices are the solutions. If $g(\lambda) = \lambda - 1$, thus, $A = I$. Then all idempotent matrices are the solutions. Else if $g(\lambda) = \lambda + \frac{1+i\sqrt{3}}{2}$, thus, $A = \left(-\frac{1+i\sqrt{3}}{2}\right)I$. Then equation (1) has the form $-\frac{2}{1+i\sqrt{3}}X = \left(-\frac{2}{1+i\sqrt{3}}X\right)^2$. These cases are similar to the case $A = I$, but just an coefficient differs. So we just need to consider the following remaining nontrivial cases.

$$\text{Case 1. } g(\lambda) = \lambda \left(\lambda + \frac{1+i\sqrt{3}}{2} \right).$$

$$\text{Case 2. } g(\lambda) = \lambda \left(\lambda + \frac{1-i\sqrt{3}}{2} \right).$$

$$\text{Case 3. } g(\lambda) = \lambda(\lambda - 1).$$

$$\text{Case 4. } g(\lambda) = (\lambda - 1) \left(\lambda + \frac{1+i\sqrt{3}}{2} \right).$$

$$\text{Case 5. } g(\lambda) = (\lambda - 1) \left(\lambda + \frac{1-i\sqrt{3}}{2} \right).$$

$$\text{Case 6. } g(\lambda) = \left(\lambda + \frac{1+i\sqrt{3}}{2} \right) \left(\lambda + \frac{1-i\sqrt{3}}{2} \right).$$

$$\text{Case 7. } g(\lambda) = \lambda(\lambda - 1) \left(\lambda + \frac{1+i\sqrt{3}}{2} \right).$$

$$\text{Case 8. } g(\lambda) = \lambda(\lambda - 1) \left(\lambda + \frac{1-i\sqrt{3}}{2} \right).$$

$$\text{Case 9. } g(\lambda) = \lambda \left(\lambda + \frac{1+i\sqrt{3}}{2} \right) \left(\lambda + \frac{1-i\sqrt{3}}{2} \right).$$

$$\text{Case 10. } g(\lambda) = (\lambda - 1) \left(\lambda + \frac{1+i\sqrt{3}}{2} \right) \left(\lambda + \frac{1-i\sqrt{3}}{2} \right).$$

$$\text{Case 11. } g(\lambda) = \lambda(\lambda - 1) \left(\lambda + \frac{1+i\sqrt{3}}{2} \right) \left(\lambda + \frac{1-i\sqrt{3}}{2} \right).$$

In this paper, an analysis of all cases except Case 10 and Case 11 is given in the next section. It is too difficult to solve all the solutions of the Yang-Baxter-like matrix Eq. (1) when A is a diagonalizable complex matrix with three distinct nonzero eigenvalues. So Case 10 and Case 11 are still challenging tasks that are not easy to solve in short term and will be further studied in the future.

3 Some solutions of the matrix equation

3.1 Case 1: $A^2 = -\frac{1+i\sqrt{3}}{2}A$

Under the assumption that $A^2 = -\frac{1+i\sqrt{3}}{2}A$, A is diagonalizable with two eigenvalues 0 and $-\frac{1+i\sqrt{3}}{2}$. There exists a nonsingular matrix S , such that $A = SJS^{-1}$ with

$$J = \text{diag} \left\{ -\frac{1+i\sqrt{3}}{2}I_m, 0 \right\}.$$

Partition Y as

$$Y = \begin{pmatrix} K & C \\ D & Z \end{pmatrix},$$

where K is $m \times m$ and Z is $(n - m) \times (n - m)$. By applying Lemma 2.1, we have the following conclusion.

Theorem 3.1. Let A be an $n \times n$ complex matrix such that $A^2 = -\frac{1+i\sqrt{3}}{2}A$ with rank m . Then all solutions of (1) are

$$X = S \begin{pmatrix} K & C \\ D & Z \end{pmatrix} S^{-1} \quad (4)$$

for some $n \times n$ nonsingular matrix S . Here, Z is an arbitrary $(n-m) \times (n-m)$ matrix. For any $m \times m$ nonsingular matrix U partitioned as

$$U = (U_1 \ U_2) \quad (5)$$

and its inverse partitioned as

$$U^{-1} = \begin{pmatrix} \widetilde{U}_1 \\ \widetilde{U}_2 \end{pmatrix}, \quad (6)$$

where U_1 is $m \times s$ and $\widetilde{U}_1$ is $s \times m$ ($s \leq m$). $K = -\frac{1+i\sqrt{3}}{2}U_1\widetilde{U}_1$, $C = U_2W_2$, and $D = H_2\widetilde{U}_2$ with arbitrary $(m-s) \times (n-m)$ matrix W_2 and $(n-m) \times (m-s)$ matrix H_2 satisfying $H_2W_2 = 0$. In addition, X is a commuting solution if and only if $W_2 = 0$ and $H_2 = 0$.

Proof. Applying Lemma 2.1, we have

$$\begin{cases} -\frac{1-i\sqrt{3}}{2}K = \left(-\frac{1-i\sqrt{3}}{2}K\right)^2, \\ KC = 0, \\ DK = 0, \\ DC = 0. \end{cases} \quad (7)$$

So Z is the $(n-m) \times (n-m)$ arbitrary matrix. According to the first condition of (7), we have $K = -\frac{1+i\sqrt{3}}{2}U\Sigma U^{-1}$ for any $m \times m$ nonsingular matrix U , where $\Sigma = \text{diag}(I_s, 0)$ with $s \leq m$ the rank of K . Applying this result to the last three conditions of (7), we get

$$\begin{cases} \Sigma U^{-1}C = 0, \\ DU\Sigma = 0, \\ DUU^{-1}C = 0. \end{cases}$$

Denoting $W = U^{-1}C$ and $H = DU$, we have

$$\Sigma W = 0, \quad H\Sigma = 0, \quad HW = 0.$$

According to the block structure of Σ , partition W and H as

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \quad (H_1 \ H_2),$$

respectively. Then

$$\Sigma W = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} W_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

hence $W_1 = 0$.

$$H\Sigma = (H_1 \ H_2) \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} = (H_1 \ 0) = (0 \ 0),$$

hence $H_1 = 0$. Thus,

$$HW = (0 \ H_2) \begin{pmatrix} 0 \\ W_2 \end{pmatrix} = H_2W_2 = 0.$$

Partition U as (5) and its inverse partition as (6), all solutions of (1) are (4), where $K = -\frac{1+i\sqrt{3}}{2}U_1\widetilde{U}_1$, $C = U_2W_2$, and $D = H_2\widetilde{U}_2$ with arbitrary $(m-s) \times (n-m)$ matrix W_2 and $(n-m) \times (m-s)$ matrix H_2 satisfying $H_2W_2 = 0$.

In addition, X is a commuting solution if and only if $W_2 = 0$ and $H_2 = 0$. $\square$

3.2 Case 2: $A^2 = -\frac{1-i\sqrt{3}}{2}A$

Suppose $A^2 = -\frac{1-i\sqrt{3}}{2}A$. Then A is diagonalizable with two eigenvalues 0 and $-\frac{1-i\sqrt{3}}{2}$. There exists a nonsingular matrix S such that $A = SJS^{-1}$ with $J = \text{diag}\left\{-\frac{1-i\sqrt{3}}{2}I_m, 0\right\}$. Partition Y as

$$Y = \begin{pmatrix} K & C \\ D & Z \end{pmatrix},$$

where K is $m \times m$ and Z is $(n-m) \times (n-m)$. By applying Lemma 2.1, we have the following conclusion.

Theorem 3.2. *Let A be an $n \times n$ complex matrix such that $A^2 = -\frac{1-i\sqrt{3}}{2}A$ with rank m . Then all solutions of (1) are*

$$X = S \begin{pmatrix} K & C \\ D & Z \end{pmatrix} S^{-1}$$

for some $n \times n$ nonsingular matrix S . Here, Z is an arbitrary $(n-m) \times (n-m)$ matrix, the other sub-matrices K , C , and D are constructed as follows. For any $m \times m$ nonsingular matrix U partitioned as $(U_1 \ U_2)$, and its inverse partitioned as $U^{-1} = \begin{pmatrix} \widetilde{U}_1 \\ \widetilde{U}_2 \end{pmatrix}$, where U_1 is $m \times s$ and $\widetilde{U}_1$ is $s \times m$ ($s \leq m$). $K = -\frac{1-i\sqrt{3}}{2}U_1\widetilde{U}_1$, $C = U_2W$, and

$D = H\widetilde{U}_2$ is $(n-m) \times m$ with arbitrary $(m-s) \times (n-m)$ matrix W and $(n-m) \times (m-s)$ matrix H satisfying $HW = 0$. In addition, X is a commuting solution if and only if $W = 0$ and $H = 0$.

Proof. The proof is similar to the proof of Theorem 3.1 and is omitted here. $\square$

3.3 Case 3: $A^2 = A$

In this case, A is idempotent. So A has two eigenvalues 0 and 1. Let $A = SJS^{-1}$, where S is a $n \times n$ nonsingular matrix, $J = \text{diag}\{I_m, 0\}$. Eq. (1) in this case has been studied previously by [21]. But for the completeness of the presentation, we summarize the main results of [21] in the following theorem.

Theorem 3.3. *Let A be an $n \times n$ idempotent with rank m . Then all solutions of (1) are*

$$X = S \begin{pmatrix} K & C \\ D & Z \end{pmatrix} S^{-1},$$

for some $n \times n$ nonsingular matrix S . Here, Z is an arbitrary $(n-m) \times (n-m)$ matrix, the other sub-matrices K , C , and D are constructed as follows. For any $m \times m$ nonsingular matrix U partitioned as

$$(U_1 \ U_2),$$

and its inverse partitioned as

$$U^{-1} = \begin{pmatrix} \widetilde{U}_1 \\ \widetilde{U}_2 \end{pmatrix},$$

where U_1 is $m \times s$ and $\widetilde{U}_1$ is $s \times m$ ($s \leq m$), the $m \times m$ matrix $K = U_1 \widetilde{U}_1$, the $m \times (n - m)$ matrix $C = U_2 W$, and the $(n - m) \times m$ matrix $D = H \widetilde{U}_2$ with arbitrary $(m - s) \times (n - m)$ matrix W and $(n - m) \times (m - s)$ matrix H satisfying $HW = 0$. In addition, X is a commuting solution if and only if $W = 0$ and $H = 0$.

3.4 Case 4: $A^2 = \frac{1-i\sqrt{3}}{2}A + \frac{1+i\sqrt{3}}{2}I_n$

In this case, A is nonsingular with two eigenvalues 1 and $-\frac{1+i\sqrt{3}}{2}$. Let m be the multiplicity of eigenvalue 1, and let

$$J = \text{diag} \left\{ I_m, -\frac{1+i\sqrt{3}}{2} I_{n-m} \right\}$$

be the Jordan form of A . Then there exists a nonsingular matrix S such that $A = SJS^{-1}$. Let

$$Y = \begin{pmatrix} K & C \\ D & Z \end{pmatrix},$$

where K is $m \times m$, Z is $(n - m) \times (n - m)$, C is $m \times (n - m)$, and D is $(n - m) \times m$. Applying Lemma 2.1, we obtain all commuting solutions and non-commuting solutions of (1) in the following theorem.

Theorem 3.4. Let A be an $n \times n$ complex matrix such that $A^2 = \frac{1-i\sqrt{3}}{2}A + \frac{1+i\sqrt{3}}{2}I_n$ with eigenvalue 1 of multiplicity m .

(1) All the commuting solutions of Eq. (1) are

$$X = S \begin{pmatrix} K & 0 \\ 0 & Z \end{pmatrix} S^{-1},$$

where K and Z satisfy $K = K^2$ and $-\frac{1-i\sqrt{3}}{2}Z = \left(-\frac{1-i\sqrt{3}}{2}Z\right)^2$.

(2) All the non-commuting solutions of Eq. (1) are

$$X = S \begin{pmatrix} K & C \\ D & Z \end{pmatrix} S^{-1},$$

where K is any $m \times m$ diagonalizable matrix and Z is any $(n - m) \times (n - m)$ diagonalizable matrix such that

(i) the nonzero matrices C and D have the same rank r such that

$$CDC = \frac{1+i\sqrt{3}}{3}C, \quad DCD = \frac{1+i\sqrt{3}}{3}D;$$

(ii) K and Z have eigenvalues $-\frac{i\sqrt{3}}{3}$ and $\frac{3-i\sqrt{3}}{6}$ of multiplicity r , respectively;

(iii) the nonzero columns of C and nonzero rows of D are eigenvectors and left eigenvectors of K , respectively, associated with eigenvalue $-\frac{i\sqrt{3}}{3}$, and the nonzero columns of D and nonzero rows of C are eigenvectors and left eigenvectors of Z , respectively, associated with eigenvalue $\frac{3-i\sqrt{3}}{6}$;

(iv) the other eigenvalues of K and Z belong to $\{0, 1\}$ and $\left\{0, -\frac{1+i\sqrt{3}}{2}\right\}$, respectively.

Proof. By applying Lemma 2.1, Yang-Baxter-like matrix equation (3) is equivalent to

$$\begin{cases} K^2 - K = \frac{1+i\sqrt{3}}{2}CD, \\ \left(-\frac{1-i\sqrt{3}}{2}Z\right)^2 + \frac{1-i\sqrt{3}}{2}Z = -DC, \\ KC = -\frac{1+i\sqrt{3}}{2}C + \frac{1+i\sqrt{3}}{2}CZ, \\ DK = -\frac{1+i\sqrt{3}}{2}D + \frac{1+i\sqrt{3}}{2}ZD. \end{cases} \quad (8)$$

We first look for all commuting solutions of Eq. (1), which correspond to all commuting solutions of (3). If $JY = YJ$, then

$$\begin{cases} -\frac{1+i\sqrt{3}}{2}D = D, \\ -\frac{1+i\sqrt{3}}{2}C = C. \end{cases}$$

We get $C = 0$ and $D = 0$. If $C = 0$ and $D = 0$, it is easy to prove that $JY = YJ$. Thus, (8) implies that all commuting solutions of (1) are $X = SYS^{-1}$ with $Y = \text{diag}(K, Z)$, where K and Z satisfy $K = K^2$ and $-\frac{1-i\sqrt{3}}{2}Z = \left(-\frac{1-i\sqrt{3}}{2}Z\right)^2$, respectively.

We show that there are no solutions of (8) such that $C = 0$ and $D \neq 0$ or $C \neq 0$ and $D = 0$. If $C = 0$ and $D \neq 0$ satisfy (8) for some matrices K and Z . From the first two equations of (8), we get $K = K^2$ and $-\frac{1-i\sqrt{3}}{2}Z = \left(-\frac{1-i\sqrt{3}}{2}Z\right)^2$. So all the possible eigenvalues of K and Z are 0, 1 and 0, $-\frac{1+i\sqrt{3}}{2}$, respectively. According to the last equation of (8), we know that $D \neq 0$ is a solution of the Sylvester equation $D\left(\frac{1-i\sqrt{3}}{2}K + I\right) = ZD$. This means $\frac{1-i\sqrt{3}}{2}K + I$ and Z have common eigenvalues. Note that $\frac{1-i\sqrt{3}}{2} \times 0 + 1 = 1$ and $\frac{1-i\sqrt{3}}{2} \times 1 + 1 = \frac{3-i\sqrt{3}}{2}$ are not eigenvalues of Z . This is a contradiction. The same results can also be obtained with assumption $C \neq 0$ and $D = 0$. Thus, any solutions of (8) with $C = 0$ or $D = 0$ is a commuting one. So all non-commuting solutions of (8) must have $C \neq 0$ and $D \neq 0$.

From the first two equations of (8), we have

$$(K^2 - K)C = \frac{1+i\sqrt{3}}{2}CDC = -\frac{1+i\sqrt{3}}{2}C\left(-\frac{1+i\sqrt{3}}{2}Z^2 + \frac{1-i\sqrt{3}}{2}Z\right) = -CZ - \frac{1-i\sqrt{3}}{2}CZ^2.$$

From the first equation and the third equation of (8), we obtain

$$\begin{aligned} (K^2 - K)C &= (K - I_m)KC \\ &= (K - I_m)\left(-\frac{1+i\sqrt{3}}{2}C + \frac{1+i\sqrt{3}}{2}CZ\right) \\ &= -\frac{1+i\sqrt{3}}{2}KC + \frac{1+i\sqrt{3}}{2}KCZ + \frac{1+i\sqrt{3}}{2}C - \frac{1+i\sqrt{3}}{2}CZ \\ &= -\frac{1-i\sqrt{3}}{2}C + \frac{1-i\sqrt{3}}{2}CZ + \frac{1-i\sqrt{3}}{2}CZ - \frac{1-i\sqrt{3}}{2}CZ^2 + \frac{1+i\sqrt{3}}{2}C - \frac{1+i\sqrt{3}}{2}CZ \\ &= -\frac{1-i\sqrt{3}}{2}CZ^2 + \frac{1-i\sqrt{3}}{2}CZ + i\sqrt{3}C. \end{aligned}$$

Combining these results we have

$$CZ = \frac{3-i\sqrt{3}}{6}C. \quad (9)$$

Similarly,

$$ZD = \frac{3-i\sqrt{3}}{6}D. \quad (10)$$

Substituting (9) into the right of the third equation of (8), we get

$$KC = -\frac{i\sqrt{3}}{3}C. \quad (11)$$

Substituting (10) into the right of the fourth equation of (8), we have

$$DK = -\frac{i\sqrt{3}}{3}D. \quad (12)$$

Multiplying C to the first equation of (8) from the right and using (11), we obtain

$$CDC = \frac{1 + i\sqrt{3}}{3}C.$$

From which, we have

$$r(C) = r(CDC) \leq r(CD) \leq r(C).$$

Hence, $r(C) = r(CD)$. Multiplying D to the second equation of (8) from the right and using (10), we get

$$DCD = \frac{1 + i\sqrt{3}}{3}D.$$

So

$$r(D) = r(DCD) \leq r(CD) \leq r(D).$$

Hence, $r(D) = r(CD)$. Therefore,

$$r(C) = r(D).$$

This means that (i) is true. From (10) and (11), we know that all nonzero columns of D and C are eigenvectors of Z and K associated with eigenvalues $\frac{3-i\sqrt{3}}{6}$ and $-\frac{i\sqrt{3}}{3}$, respectively. From (9) and (12), we know that all nonzero rows of C and D are left eigenvectors of Z and K associated with eigenvalues $\frac{3-i\sqrt{3}}{6}$ and $-\frac{i\sqrt{3}}{3}$, respectively. So for any non-commuting solution of (8), $-\frac{i\sqrt{3}}{3}$ and $\frac{3-i\sqrt{3}}{6}$ must be an eigenvalue of K and Z , respectively. Furthermore, they are semi-simple eigenvalues of K and Z , respectively. If eigenvalue $-\frac{i\sqrt{3}}{3}$ of K is not semi-simple, then there exists a nonzero vector v satisfying $u \equiv \left(K + \frac{i\sqrt{3}}{3}I\right)v \neq 0$ and $\left(K + \frac{i\sqrt{3}}{3}I\right)u = \left(K + \frac{i\sqrt{3}}{3}I\right)^2v = 0$. The eigenvector u and the generalized eigenvector v must be linearly independent. In fact, if $au + bv = 0$ for some $a, b \in \mathbb{C}$, then via multiplying this equality by $\left(K + \frac{i\sqrt{3}}{3}I\right)$ from the left we get $b\left(K + \frac{i\sqrt{3}}{3}I\right)v = bu = 0$, from which $b = 0$ and so $a = 0$. Since

$$\frac{1 + i\sqrt{3}}{2}CDv = (K^2 - K)v = \left(K + \frac{i\sqrt{3}}{3}I\right)^2v - \frac{3 + i2\sqrt{3}}{3}\left(K + \frac{i\sqrt{3}}{3}I\right)v - \frac{1 - i\sqrt{3}}{3}v = -\frac{3 + i2\sqrt{3}}{3}u - \frac{1 - i\sqrt{3}}{3}v,$$

we get

$$u = \frac{3}{3 + i2\sqrt{3}}\left(-\frac{1 + i\sqrt{3}}{2}CDv - \frac{1 - i\sqrt{3}}{3}v\right).$$

Since $KC = -\frac{i\sqrt{3}}{3}C$, we get

$$\begin{aligned} 0 &= \left(K + \frac{i\sqrt{3}}{3}I\right)^2v = \left(K + \frac{i\sqrt{3}}{3}I\right)u = \left(K + \frac{i\sqrt{3}}{3}I\right)\frac{3}{3 + i2\sqrt{3}}\left(-\frac{1 + i\sqrt{3}}{2}CDv - \frac{1 - i\sqrt{3}}{3}v\right) \\ &= -\frac{3(1 + i\sqrt{3})}{2(3 + i2\sqrt{3})}\left(KCDv + \frac{i\sqrt{3}}{3}CDv\right) - \frac{1 - i\sqrt{3}}{3 + i2\sqrt{3}}\left(K + \frac{i\sqrt{3}}{3}I\right)v \\ &= -\frac{3(1 + i\sqrt{3})}{2(3 + i2\sqrt{3})}\left(-\frac{i\sqrt{3}}{3}CDv + \frac{i\sqrt{3}}{3}CDv\right) - \frac{1 - i\sqrt{3}}{3 + i2\sqrt{3}}u = -\frac{1 - i\sqrt{3}}{3 + i2\sqrt{3}}u \neq 0. \end{aligned}$$

This is a contradiction. Thus, $-\frac{i\sqrt{3}}{3}$ is a semi-simple eigenvalue of K . Similarly, $\frac{3-i\sqrt{3}}{6}$ is a semi-simple eigenvalue of Z . So (ii) and (iii) are true.

Since $KC = -\frac{i\sqrt{3}}{3}C$, we obtain

$$\left(\frac{1-i\sqrt{3}}{2}K\right)C = C\left(-\frac{3+i\sqrt{3}}{6}I_{n-m}\right).$$

Applying Lemma 2.2 to the first equation of (8), it follows that

$$p_{\frac{1-i\sqrt{3}}{2}K+CD}(\alpha) \cdot p_{-\frac{3+i\sqrt{3}}{6}I_{n-m}}(\alpha) \equiv p_{\frac{1-i\sqrt{3}}{2}K}(\alpha) \cdot p_{DC-\frac{3+i\sqrt{3}}{6}I_{n-m}}(\alpha).$$

According to the second equation of (8) and the fact that the eigenvalues of the square of a matrix are the squares of the eigenvalues of the matrix, the aforementioned identity can be written as

$$\prod_{l=1}^m \left(\alpha - \frac{1-i\sqrt{3}}{2}a_l^2\right) \left(\alpha + \frac{3+i\sqrt{3}}{6}\right)^{n-m} \equiv \prod_{j=1}^m \left(\alpha - \frac{1-i\sqrt{3}}{2}a_j\right) \prod_{k=1}^{n-m} \left[\alpha - \left(-\frac{1-i\sqrt{3}}{2}b_k + \frac{1+i\sqrt{3}}{2}b_k^2 - \frac{3+i\sqrt{3}}{6}\right)\right],$$

where $a_1, a_2, \dots, a_m$ are the eigenvalues of K and $b_1, b_2, \dots, b_{n-m}$ are the eigenvalues of Z , all counting algebraic multiplicity. Since K and Z have eigenvalues $-\frac{i\sqrt{3}}{3}$ and $\frac{3-i\sqrt{3}}{6}$ of multiplicity r , respectively, let $a_j = -\frac{i\sqrt{3}}{3}$ and $b_j = \frac{3-i\sqrt{3}}{6}$, for $j = 1, \dots, r$. Thus,

$$\alpha - \frac{1-i\sqrt{3}}{2}a_j = \alpha + \frac{3+i\sqrt{3}}{6}, \quad \text{for } j = 1, \dots, r.$$

Dividing both sides by $\left(\alpha + \frac{3+i\sqrt{3}}{6}\right)^r$, we obtain

$$\prod_{l=1}^m \left(\alpha - \frac{1-i\sqrt{3}}{2}a_l^2\right) \left(\alpha + \frac{3+i\sqrt{3}}{6}\right)^{n-m-r} \equiv \prod_{j=r+1}^m \left(\alpha - \frac{1-i\sqrt{3}}{2}a_j\right) \prod_{k=1}^{n-m} \left[\alpha - \left(-\frac{1-i\sqrt{3}}{2}b_k + \frac{1+i\sqrt{3}}{2}b_k^2 - \frac{3+i\sqrt{3}}{6}\right)\right].$$

Since $\frac{1-i\sqrt{3}}{2}a_j^2 = -\frac{1-i\sqrt{3}}{6}$, $-\frac{1-i\sqrt{3}}{2}b_j + \frac{1+i\sqrt{3}}{2}b_j^2 - \frac{3+i\sqrt{3}}{6} = -\frac{1-i\sqrt{3}}{6}$, for $j = 1, \dots, r$, dividing both sides by $\left(\alpha + \frac{1-i\sqrt{3}}{6}\right)^r$, the aforementioned identity can be further simplified to

$$\prod_{l=r+1}^m \left(\alpha - \frac{1-i\sqrt{3}}{2}a_l^2\right) \left(\alpha + \frac{3+i\sqrt{3}}{6}\right)^{n-m-r} \equiv \prod_{j=r+1}^m \left(\alpha - \frac{1-i\sqrt{3}}{2}a_j\right) \prod_{k=r+1}^{n-m} \left[\alpha - \left(-\frac{1-i\sqrt{3}}{2}b_k + \frac{1+i\sqrt{3}}{2}b_k^2 - \frac{3+i\sqrt{3}}{6}\right)\right].$$

This implies that

$$a_j^2 = a_j, \quad \text{for } j = r+1, \dots, m$$

and

$$b_k^2 = -\frac{1+i\sqrt{3}}{2}b_k, \quad \text{for } k = r+1, \dots, n-m.$$

Consequently, $a_l = 0$ or 1 , for $l = r+1, \dots, m$, and $b_k = 0$ or $-\frac{1+i\sqrt{3}}{2}$, for $k = r+1, \dots, n-m$. Next, we show that such eigenvalues are semi-simple. If 0 is an eigenvalue of K that is not semi-simple, then there exists a vector $v \neq 0$ such that $u \equiv Kv \neq 0$ and $K^2v = 0$. Multiplying v to the first equation of (8) from the right, we get

$$\frac{1+i\sqrt{3}}{2}CDv = (K^2 - K)v = K^2v - Kv = -Kv = -u.$$

Combining this with (11), we obtain

$$0 = Ku = -\frac{1+i\sqrt{3}}{2}KCDv = -\frac{1+i\sqrt{3}}{2} \cdot -\frac{i\sqrt{3}}{3}CDv = -\frac{i\sqrt{3}}{3}u \neq 0.$$

This is a contradiction. If 1 is an eigenvalue of K that is not semi-simple, then there exists a vector $v \neq 0$ such that $u \equiv (K - I)v \neq 0$ and $(K - I)u = (K - I)^2v = 0$. Multiplying v to the first equation of (8) from the right, we have

$$\frac{1 + i\sqrt{3}}{2}CDv = (K^2 - K)v = K^2v - Kv = (K - I)^2v + (K - I)v = u.$$

Combining this with (11), we obtain

$$\begin{aligned} 0 &= (K - I)^2v = (K - I)(K - I)v = (K - I)u \\ &= \frac{1 + i\sqrt{3}}{2}(K - I)CDv = \frac{1 + i\sqrt{3}}{2}(KCDv - CDv) \\ &= \frac{1 + i\sqrt{3}}{2}\left(\frac{-i\sqrt{3}}{3}CDv - CDv\right) = -\frac{3 + i\sqrt{3}}{3} \cdot \frac{1 + i\sqrt{3}}{2}CDv \\ &= -\frac{3 + i\sqrt{3}}{3}u \neq 0. \end{aligned}$$

This is another contradiction. Therefore, the eigenvalues 0 and 1 are semi-simple. Similarly, the eigenvalues 0 and $-\frac{1+i\sqrt{3}}{2}$ of Z are semi-simple. Hence, K and Z are diagonalizable.

Conversely, suppose that K is an $m \times m$ diagonalizable matrix, Z is an $(n - m) \times (n - m)$ diagonalizable matrix, C is an $m \times (n - m)$ matrix, and D is an $(n - m) \times m$ matrix such that (K, C, D, Z) satisfies (i)–(iv). We show that it is a solution of (8). According to (iii), we have (9), (10), (11), and (12). Combining (9) and (11), we have the third equality of (8). Combining (10) and (12), we have the fourth equality of (8). Then from (11) and (ii), we obtain

$$(K^2 - K)C = -\frac{1 - i\sqrt{3}}{3}C = \frac{1 + i\sqrt{3}}{2}CDC.$$

Thus,

$$(K^2 - K)\hat{C} = \frac{1 + i\sqrt{3}}{2}CD\hat{C},$$

where $\hat{C}$ is the $m \times r$ matrix consisting of r linearly independent columns of C . Let $\tilde{C} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_{m-r})$ be an $m \times (m - r)$ matrix whose columns $\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_{m-r}$ are linearly independent eigenvectors of K associated with eigenvalues 0 or 1. If $K\tilde{c}_j = \tilde{c}_j$ for some columns of $\tilde{C}$, from the equality (12), we have

$$D\tilde{c}_j = -\frac{3}{i\sqrt{3}}DK\tilde{c}_j = -\frac{3}{i\sqrt{3}}D\tilde{c}_j.$$

Therefore, $D\tilde{c}_j = 0$. Similarly, if $K\tilde{c}_j = 0$ for some columns of $\tilde{C}$, we also have $D\tilde{c}_j = 0$. Thus, $D\tilde{C} = 0$. Then

$$(K^2 - K)\tilde{C} = 0 = D\tilde{C} = \frac{1 + i\sqrt{3}}{2}CD\tilde{C}.$$

Since the columns of $\hat{C}$ and $\tilde{C}$ form a basis of $\mathbb{C}^m$, then

$$K^2 - K = \frac{1 + i\sqrt{3}}{2}CD.$$

By the same token and under the assumption $DCD = \frac{1+i\sqrt{3}}{3}D$, we can obtain

$$-\frac{1 + i\sqrt{3}}{2}Z^2 + \frac{1 - i\sqrt{3}}{2}Z = -DC.$$

Therefore, (K, C, D, Z) is a solution of (8). $\square$

3.5 Case 5: $A^2 = \frac{1+i\sqrt{3}}{2}A + \frac{1-i\sqrt{3}}{2}I_n$

In this case, A is nonsingular with two eigenvalues 1 and $-\frac{1-i\sqrt{3}}{2}$. Let m be the multiplicity of eigenvalue 1 , and let

$$J = \text{diag}\left\{I_m, -\frac{1-i\sqrt{3}}{2}I_{n-m}\right\}$$

be the Jordan form of A . Then there exists a nonsingular matrix S such that $A = SJS^{-1}$. In order to solve (1), we partition Y as

$$Y = \begin{pmatrix} K & C \\ D & Z \end{pmatrix},$$

where K is $m \times m$, Z is $(n-m) \times (n-m)$, C is $m \times (n-m)$, and D is $(n-m) \times m$. Applying Lemma 2.1 and the technique in Theorem 3.4, we summarize the main results as the following theorem.

Theorem 3.5. Let A be an $n \times n$ complex matrix such that $A^2 = \frac{1+i\sqrt{3}}{2}A + \frac{1-i\sqrt{3}}{2}I_n$ with eigenvalue 1 of multiplicity m .

(1) All the commuting solutions of (1) are

$$X = S \begin{pmatrix} K & 0 \\ 0 & Z \end{pmatrix} S^{-1},$$

where K and Z satisfy $K = K^2$ and $-\frac{1+i\sqrt{3}}{2}Z = \left(-\frac{1+i\sqrt{3}}{2}Z\right)^2$.

(2) All the non-commuting solutions of (1) are

$$X = S \begin{pmatrix} K & C \\ D & Z \end{pmatrix} S^{-1},$$

where K is any $m \times m$ diagonalizable matrix and Z is any $(n-m) \times (n-m)$ diagonalizable matrix such that

(i) the nonzero matrices C and D have the same rank r and satisfy

$$CDC = \frac{1-i\sqrt{3}}{3}C, \quad DCD = \frac{1-i\sqrt{3}}{3}D;$$

- (ii) K and Z have eigenvalues $\frac{i\sqrt{3}}{3}$ and $\frac{3+i\sqrt{3}}{6}$ of multiplicity r , respectively;
- (iii) the nonzero columns of C and nonzero rows of D are eigenvectors and left eigenvectors of K , respectively, associated with eigenvalue $\frac{i\sqrt{3}}{3}$, and the nonzero columns of D and nonzero rows of C are eigenvectors and left eigenvectors of Z , respectively, associated with eigenvalue $\frac{3+i\sqrt{3}}{6}$;
- (iv) the other eigenvalues of K and Z belong to $\{0, 1\}$ and $\left\{0, -\frac{1-i\sqrt{3}}{2}\right\}$, respectively.

Proof. The proof is similar to the proof of Theorem 3.4 and is omitted here. $\square$

3.6 Case 6: $A^2 = -A - I_n$

If $A^2 = -A - I_n$, then A is a diagonalizable matrix with two eigenvalues $-\frac{1+i\sqrt{3}}{2}$ and $-\frac{1-i\sqrt{3}}{2}$. Let m and $n-m$ be the multiplicity of eigenvalues $-\frac{1+i\sqrt{3}}{2}$ and $-\frac{1-i\sqrt{3}}{2}$, respectively. Then there exists a nonsingular matrix S such that $A = SJS^{-1}$, where

$$J = \text{diag} \left\{ -\frac{1+i\sqrt{3}}{2} I_m, -\frac{1-i\sqrt{3}}{2} I_{n-m} \right\}.$$

Applying Lemma 2.1 and the technique in Theorem 3.4, we obtain all commuting solutions and non-commuting solutions of (1) in the following theorem.

Theorem 3.6. *Let A be an $n \times n$ complex matrix such that $A^2 = -A - I_n$. If the multiplicity of eigenvalues $-\frac{1+i\sqrt{3}}{2}$ and $-\frac{1-i\sqrt{3}}{2}$ are m and $n - m$, respectively.*

(1) *All the commuting solutions of (1) are*

$$X = S \begin{pmatrix} K & 0 \\ 0 & Z \end{pmatrix} S^{-1},$$

$$\text{where } K \text{ and } Z \text{ satisfy } -\frac{1-i\sqrt{3}}{2}K = \left(-\frac{1-i\sqrt{3}}{2}K\right)^2 \text{ and } -\frac{1+i\sqrt{3}}{2}Z = \left(-\frac{1+i\sqrt{3}}{2}Z\right)^2.$$

(2) *All the non-commuting solutions of (1) are*

$$X = S \begin{pmatrix} K & C \\ D & Z \end{pmatrix} S^{-1},$$

where K is any $m \times m$ diagonalizable matrix and Z is any $(n - m) \times (n - m)$ diagonalizable matrix such that

(i) *the nonzero matrices C and D have the same rank r and satisfy*

$$CDC = -\frac{2}{3}C, \quad DCD = -\frac{2}{3}D;$$

(ii) *K and Z have eigenvalues $\frac{-3+i\sqrt{3}}{3}$ and $\frac{-3+i\sqrt{3}}{6}$ of multiplicity r , respectively;*

(iii) *the nonzero columns of C and nonzero rows of D are eigenvectors and left eigenvectors of K , respectively, associated with eigenvalue $\frac{-3+i\sqrt{3}}{3}$, and the nonzero columns of D and nonzero rows of C are eigenvectors and left eigenvectors of Z , respectively, associated with eigenvalue $\frac{-3+i\sqrt{3}}{6}$;*

(iv) *the other eigenvalues of K and Z belong to $\left\{0, -\frac{1+i\sqrt{3}}{2}\right\}$ and $\left\{0, -\frac{1-i\sqrt{3}}{2}\right\}$, respectively.*

Proof. The proof is similar to the proof of Theorem 3.4 and is omitted here. $\square$

3.7 Case 7: $A^3 = \frac{1-i\sqrt{3}}{2}A^2 + \frac{1+i\sqrt{3}}{2}A$

Now we consider the case that $A^3 = \frac{1-i\sqrt{3}}{2}A^2 + \frac{1+i\sqrt{3}}{2}A$. So the minimal polynomial of A is $g(\lambda) = \lambda(\lambda - 1)\left(\lambda + \frac{1+i\sqrt{3}}{2}\right)$. In this case, A has three distinct eigenvalues: 0, 1, and $-\frac{1+i\sqrt{3}}{2}$. Assume that the rank of A is m and the multiplicity of eigenvalue 1 is k . Then there exists a nonsingular matrix S such that $A = SJS^{-1}$, where

$$J = \text{diag} \left\{ I_k, -\frac{1+i\sqrt{3}}{2} I_{m-k}, 0 \right\}.$$

We partition matrix Y as

$$Y = \begin{bmatrix} K & F & C_1 \\ E & T & C_2 \\ D_1 & D_2 & Z \end{bmatrix}$$

accordingly. Then we have

$$JYJ = \begin{bmatrix} K & -\frac{1+i\sqrt{3}}{2}F & 0 \\ -\frac{1+i\sqrt{3}}{2}E & -\frac{1-i\sqrt{3}}{2}T & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$YJY = \begin{bmatrix} K^2 - \frac{1+i\sqrt{3}}{2}FE & KF - \frac{1+i\sqrt{3}}{2}FT & KC_1 - \frac{1+i\sqrt{3}}{2}FC_2 \\ EK - \frac{1+i\sqrt{3}}{2}TE & EF - \frac{1+i\sqrt{3}}{2}T^2 & EC_1 - \frac{1+i\sqrt{3}}{2}TC_2 \\ D_1K - \frac{1+i\sqrt{3}}{2}D_2E & D_1F - \frac{1+i\sqrt{3}}{2}D_2T & D_1C_1 - \frac{1+i\sqrt{3}}{2}D_2C_2 \end{bmatrix}.$$

According to Eq. (3), we know that Z is any $(n-m) \times (n-m)$ matrix. We first look for all commuting solutions of (3). If $YJ = JY$, then

$$\begin{bmatrix} K & F & C_1 \\ -\frac{1+i\sqrt{3}}{2}E & -\frac{1+i\sqrt{3}}{2}T & -\frac{1+i\sqrt{3}}{2}C_2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} K & -\frac{1+i\sqrt{3}}{2}F & 0 \\ E & -\frac{1+i\sqrt{3}}{2}T & 0 \\ D_1 & -\frac{1+i\sqrt{3}}{2}D_2 & 0 \end{bmatrix}.$$

Thus,

$$E = 0, \quad F = 0, \quad D_1 = 0, \quad D_2 = 0, \quad C_1 = 0, \quad C_2 = 0.$$

So, all commuting solutions of Eq. (3) must be satisfied

$$\begin{cases} K^2 = K, \\ \left(-\frac{1-i\sqrt{3}}{2}T\right)^2 = -\frac{1-i\sqrt{3}}{2}T. \end{cases}$$

Hence, we have the following result.

Theorem 3.7. Let A be an $n \times n$ complex matrix such that $A^3 = \frac{1-i\sqrt{3}}{2}A^2 + \frac{1+i\sqrt{3}}{2}A$. Suppose that the rank of matrix A is m and the multiplicity of eigenvalue 1 is k . Then all commuting solutions of Eq. (1) are given by

$$X = S \operatorname{diag}\{K, T, Z\}S^{-1},$$

where Z is any $(n-m) \times (n-m)$ matrix, K is the $k \times k$ idempotent matrix, and T is the $(m-k) \times (m-k)$ matrix satisfying $-\frac{1-i\sqrt{3}}{2}T = \left(-\frac{1-i\sqrt{3}}{2}T\right)^2$.

Next, we will find all the non-commuting solutions of (1). Write matrix $J = \operatorname{diag}\{M, 0\}$ with $M = \operatorname{diag}\left\{I_k, -\frac{1+i\sqrt{3}}{2}I_{m-k}\right\}$. Partition Y as

$$Y = \begin{pmatrix} \widehat{K} & C \\ D & Z \end{pmatrix},$$

where $\widehat{K}$ has the same size as M . Then by applying Lemma 2.1, we have

$$\begin{cases} M\widehat{K}M = \widehat{K}M\widehat{K}, \\ \widehat{K}MC = 0, \\ DM\widehat{K} = 0, \\ DMC = 0. \end{cases} \quad (13)$$

So Z is an arbitrary $(n-m) \times (n-m)$ matrix for all of its solutions. Next, we can get some results for several special cases as follows.

Theorem 3.8. Let A be an $n \times n$ complex matrix such that $A^3 = \frac{1-i\sqrt{3}}{2}A^2 + \frac{1+i\sqrt{3}}{2}A$. Suppose that the rank of matrix A is m and the multiplicity of eigenvalue 1 is k .

- (1) If $\widehat{K} = 0$, then all solutions of (13) are $(0, C, D, Z)$ such that $DMC = 0$. If in addition $C = 0$ or $D = 0$, then all solutions are $(0, 0, D, Z)$ or $(0, C, 0, Z)$, respectively.
- (2) If $C = 0$, then all solutions of (13) are $(\widehat{K}, 0, D, Z)$ such that $\widehat{K}$ is a solution of the Yang-Baxter-like matrix equation $M\widehat{K}M = \widehat{K}M\widehat{K}$ and all rows of D belong to the left null space of $M\widehat{K}$. If in addition $D = 0$, then all solutions are commuting.
- (3) If $D = 0$, then all solutions of (13) are $(\widehat{K}, C, 0, Z)$ such that $\widehat{K}$ is a solution of the Yang-Baxter-like matrix equation $M\widehat{K}M = \widehat{K}M\widehat{K}$ and all rows of C belong to the null space of $\widehat{K}M$. If in addition $C = 0$, then all solutions are commuting.

Proof.

- (1) Clearly that if $\widehat{K} = 0$, then the equivalent systems of (13) are $DMC = 0$. Thus, all solutions of (13) are $(0, C, D, Z)$. In addition, if $C = 0$, then all solutions are $(0, 0, D, Z)$, else if $D = 0$, then all solutions are $(0, C, 0, Z)$.
- (2) If $C = 0$, then the equivalent systems of (13) are

$$\begin{cases} M\widehat{K}M = \widehat{K}M\widehat{K}, \\ DM\widehat{K} = 0. \end{cases}$$

Thus, $\widehat{K}$ is a solution of the Yang-Baxter-like matrix equation $M\widehat{K}M = \widehat{K}M\widehat{K}$. All rows of D belong to the left null space of $M\widehat{K}$. If in addition $D = 0$, then we just have only one equivalent $M\widehat{K}M = \widehat{K}M\widehat{K}$. Hence, all solutions are commuting.

- (3) The proof is similar to the proof of (2) and is omitted here. $\square$

Now we will solve (13) for all non-commuting solutions of (3). We give the main result when the case $A^3 = \frac{1-i\sqrt{3}}{2}A^2 + \frac{1+i\sqrt{3}}{2}A$.

Theorem 3.9. Let A be an $n \times n$ complex matrix such that $A^3 = \frac{1-i\sqrt{3}}{2}A^2 + \frac{1+i\sqrt{3}}{2}A$. Suppose that the rank of matrix A is m and the multiplicity of eigenvalue 1 is k . Then all solutions of Eq. (1) are given by

$$X = S \begin{bmatrix} K & F & C_1 \\ E & T & C_2 \\ D_1 & D_2 & Z \end{bmatrix} S^{-1},$$

where K is any $k \times k$ diagonalizable matrix and T is any $(m-k) \times (m-k)$ diagonalizable matrix such that

- (i) the nonzero matrices F and E have the same rank r such that

$$FEF = \frac{1+i\sqrt{3}}{3}F, \quad EFE = \frac{1+i\sqrt{3}}{3}E;$$

- (ii) K and T have eigenvalues $-\frac{i\sqrt{3}}{3}$ and $\frac{3-i\sqrt{3}}{6}$ of multiplicity r , respectively;
- (iii) the nonzero columns of F and nonzero rows of E are eigenvectors and left eigenvectors of K , respectively, associated with eigenvalue $-\frac{i\sqrt{3}}{3}$, and the nonzero columns of E and nonzero rows of F are eigenvectors and left eigenvectors of T , respectively, associated with eigenvalue $\frac{3-i\sqrt{3}}{6}$;
- (iv) the other eigenvalues of K and T belong to $\{0, 1\}$ and $\left\{0, -\frac{1+i\sqrt{3}}{2}\right\}$, respectively;
- (v) $\begin{bmatrix} K & F \\ E & T \end{bmatrix}$ is a commuting solution with M if and only if $E = 0$ and $F = 0$, and in this case $K = K^2$ and $-\frac{1-i\sqrt{3}}{2}T = \left(-\frac{1-i\sqrt{3}}{2}T\right)^2$.

Any nonzero column vector $c = (c_1^T \ c_2^T)^T$ of the $m \times (n - m)$ matrix $[C_1^T \ C_2^T]^T$ and any nonzero row vector $d = (d_1 \ d_2)$ of the $(n - m) \times m$ matrix $[D_1 \ D_2]$ are an eigenvector and a left eigenvector of the matrices

$$\begin{bmatrix} K & -\frac{1+i\sqrt{3}}{2}F \\ E & -\frac{1+i\sqrt{3}}{2}T \end{bmatrix} \text{ and } \begin{bmatrix} K & F \\ -\frac{1+i\sqrt{3}}{2}E & -\frac{1+i\sqrt{3}}{2}T \end{bmatrix},$$

respectively, such that $d_1 c_1 - \frac{1+i\sqrt{3}}{2} d_2 c_2 = 0$. Z is an arbitrary $(n - m) \times (n - m)$ matrix.

Proof. The first equation of (13) is just the Yang-Baxter-like matrix equation for the nonsingular matrix $M = \text{diag} \left\{ I_k, -\frac{1+i\sqrt{3}}{2} I_{m-k} \right\}$ that satisfies the condition $M^2 = \frac{1-i\sqrt{3}}{2} M + \frac{1+i\sqrt{3}}{2} I_m$. Its general solution has been constructed in Theorem 3.4. Thus, K is any $k \times k$ diagonalizable matrix and T is any $(m - k) \times (m - k)$ diagonalizable matrix such that

(i) the nonzero matrices F and E have the same rank r such that

$$FEF = \frac{1+i\sqrt{3}}{3} F, \quad EFE = \frac{1+i\sqrt{3}}{3} E;$$

- (ii) K and T have eigenvalues $-\frac{i\sqrt{3}}{3}$ and $\frac{3-i\sqrt{3}}{6}$ of multiplicity r , respectively;
- (iii) the nonzero columns of F and nonzero rows of E are eigenvectors and left eigenvectors of K , respectively, associated with eigenvalue $-\frac{i\sqrt{3}}{3}$, and the nonzero columns of E and nonzero rows of F are eigenvectors and left eigenvectors of T , respectively, associated with eigenvalue $\frac{3-i\sqrt{3}}{6}$;
- (iv) the other eigenvalues of K and T belong to $\{0, 1\}$ and $\left\{0, -\frac{1+i\sqrt{3}}{2}\right\}$, respectively;
- (v) $\begin{bmatrix} K & F \\ E & T \end{bmatrix}$ is a commuting solution with M if and only if $E = 0$ and $F = 0$, and in this case $K = K^2$ and $-\frac{1-i\sqrt{3}}{2} T = \left(-\frac{1-i\sqrt{3}}{2} T\right)^2$.

We solve the remaining three equations of (13) to get C and D for each such obtained solution (K, F, E, T) . Then the last three equations of (13) are as follows:

$$\begin{cases} \begin{bmatrix} K & -\frac{1+i\sqrt{3}}{2}F \\ E & -\frac{1+i\sqrt{3}}{2}T \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0, \\ [D_1 \ D_2] \begin{bmatrix} K & F \\ -\frac{1+i\sqrt{3}}{2}E & -\frac{1+i\sqrt{3}}{2}T \end{bmatrix} = 0, \\ [D_1 \ D_2] \begin{bmatrix} C_1 \\ -\frac{1+i\sqrt{3}}{2}C_2 \end{bmatrix} = 0. \end{cases} \quad (14)$$

The first equation of (14) implies that any nonzero column vector $c = [c_1^T \ c_2^T]^T$ of the $m \times (n - m)$ matrix

$[C_1^T \ C_2^T]^T$ is the eigenvector of the matrix $\begin{bmatrix} K & -\frac{1+i\sqrt{3}}{2}F \\ E & -\frac{1+i\sqrt{3}}{2}T \end{bmatrix}$. The second equation of (14) implies that any

nonzero row vector $d = [d_1 \ d_2]$ of the $(n - m) \times m$ matrix $[D_1 \ D_2]$ is the left eigenvector of the matrix

$\begin{bmatrix} K & F \\ -\frac{1+i\sqrt{3}}{2}E & -\frac{1+i\sqrt{3}}{2}T \end{bmatrix}$. From the last equation of (14), we get $d_1 c_1 - \frac{1+i\sqrt{3}}{2} d_2 c_2 = 0$. $\square$

3.8 Case 8: $A^3 = \frac{1+i\sqrt{3}}{2}A^2 + \frac{1-i\sqrt{3}}{2}A$

Now we consider the case that $A^3 = \frac{1+i\sqrt{3}}{2}A^2 + \frac{1-i\sqrt{3}}{2}A$. So the minimal polynomial of A is $g(\lambda) = \lambda(\lambda - 1)\left(\lambda + \frac{1-i\sqrt{3}}{2}\right)$. In this case, A have three distinct eigenvalues: 0, 1, and $-\frac{1-i\sqrt{3}}{2}$. Assume that the rank of matrix A is m and the multiplicity of eigenvalue 1 is k . Then there exists a nonsingular matrix S such that $A = SJS^{-1}$, where

$$J = \text{diag}\left\{I_k, -\frac{1-i\sqrt{3}}{2}I_{m-k}, 0\right\}.$$

We partition matrix Y as

$$Y = \begin{bmatrix} K & F & C_1 \\ E & T & C_2 \\ D_1 & D_2 & Z \end{bmatrix}$$

accordingly. We have the following main results in this case.

Theorem 3.10. Let A be an $n \times n$ complex matrix such that $A^3 = \frac{1+i\sqrt{3}}{2}A^2 + \frac{1-i\sqrt{3}}{2}A$. Suppose that the rank of matrix A is m and the multiplicity of eigenvalue 1 is k .

(1) All commuting solutions of (1) are given by

$$X = S \text{diag}\{K, T, Z\} S^{-1},$$

where Z is any $(n-m) \times (n-m)$ matrix, K is the $k \times k$ idempotent matrix, T is the $(m-k) \times (m-k)$ matrix satisfying $-\frac{1+i\sqrt{3}}{2}T = \left(-\frac{1+i\sqrt{3}}{2}T\right)^2$.

(2) All non-commuting solutions of (1) are given by

$$X = S \begin{bmatrix} K & F & C_1 \\ E & T & C_2 \\ D_1 & D_2 & Z \end{bmatrix} S^{-1},$$

where K is any $k \times k$ diagonalizable matrix and T is any $(m-k) \times (m-k)$ diagonalizable matrix such that

(i) the nonzero matrices F and E have the same rank r and satisfy

$$FEF = \frac{1-i\sqrt{3}}{3}F, \quad EFE = \frac{1-i\sqrt{3}}{3}E;$$

- (ii) K and T have eigenvalues $\frac{i\sqrt{3}}{3}$ and $\frac{3+i\sqrt{3}}{6}$ of multiplicity r , respectively;
- (iii) the nonzero columns of F and nonzero rows of E are eigenvectors and left eigenvectors of K , respectively, associated with eigenvalue $\frac{i\sqrt{3}}{3}$, and the nonzero columns of E and nonzero rows of F are eigenvectors and left eigenvectors of T , respectively, associated with eigenvalue $\frac{3+i\sqrt{3}}{6}$;
- (iv) the other eigenvalues of K and T belong to $\{0, 1\}$ and $\left\{0, -\frac{1-i\sqrt{3}}{2}\right\}$, respectively;
- (v) $\begin{bmatrix} K & F \\ E & T \end{bmatrix}$ is a commuting solution with $\text{diag}\left\{I_k, -\frac{1-i\sqrt{3}}{2}I_{m-k}\right\}$ if and only if $E = 0$ and $F = 0$, and in this case $K = K^2$ and $-\frac{1+i\sqrt{3}}{2}T = \left(-\frac{1+i\sqrt{3}}{2}T\right)^2$.

Any nonzero column vector $c = (c_1^T \ c_2^T)^T$ of the $m \times (n-m)$ matrix $[C_1^T \ C_2^T]^T$ and any nonzero row vector $d = (d_1 \ d_2)$ of the $(n-m) \times m$ matrix $[D_1 \ D_2]$ are an eigenvector and a left eigenvector of the matrices

$$\begin{bmatrix} K & -\frac{1-i\sqrt{3}}{2}F \\ E & -\frac{1-i\sqrt{3}}{2}T \end{bmatrix} \text{ and } \begin{bmatrix} K & F \\ -\frac{1-i\sqrt{3}}{2}E & -\frac{1-i\sqrt{3}}{2}T \end{bmatrix},$$

respectively, such that $d_1c_1 - \frac{1-i\sqrt{3}}{2}d_2c_2 = 0$. Z is an arbitrary $(n-m) \times (n-m)$ matrix.

Proof. The proof is similar to the proof of Theorems 3.7 and 3.9 and is omitted here. $\square$

3.9 Case 9: $A^3 = -A^2 - A$

If $A^3 = -A^2 - A$, then the minimal polynomial of A is $g(\lambda) = \lambda\left(\lambda + \frac{1+i\sqrt{3}}{2}\right)\left(\lambda + \frac{1-i\sqrt{3}}{2}\right)$. In this case, A have three distinct eigenvalues: 0 , $-\frac{1+i\sqrt{3}}{2}$, and $-\frac{1-i\sqrt{3}}{2}$. Assume that the rank of matrix A is m and the multiplicity of eigenvalue $-\frac{1+i\sqrt{3}}{2}$ is k . Then there exists a nonsingular matrix S such that $A = SJS^{-1}$, where

$$J = \text{diag}\left\{-\frac{1+i\sqrt{3}}{2}I_k, -\frac{1-i\sqrt{3}}{2}I_{m-k}, 0\right\}.$$

We partition matrix Y as

$$Y = \begin{bmatrix} K & F & C_1 \\ E & T & C_2 \\ D_1 & D_2 & Z \end{bmatrix}$$

accordingly. We have the following main results in this case.

Theorem 3.11. Let A be an $n \times n$ complex matrix such that $A^3 = -A^2 - A$. Suppose the rank of matrix A is m and the multiplicity of eigenvalue $-\frac{1+i\sqrt{3}}{2}$ is k .

(1) All commuting solutions of (1) are given by

$$X = S \text{ diag}\{K, T, Z\}S^{-1},$$

where Z is any $(n-m) \times (n-m)$ matrix, K is the $k \times k$ matrix satisfying $-\frac{1-i\sqrt{3}}{2}K = \left(-\frac{1-i\sqrt{3}}{2}K\right)^2$, and T is the $(m-k) \times (m-k)$ matrix satisfying $-\frac{1+i\sqrt{3}}{2}T = \left(-\frac{1+i\sqrt{3}}{2}T\right)^2$.

(2) All non-commuting solutions of (1) are given by

$$X = S \begin{bmatrix} K & F & C_1 \\ E & T & C_2 \\ D_1 & D_2 & Z \end{bmatrix} S^{-1},$$

where K is any $k \times k$ diagonalizable matrix and T is any $(m-k) \times (m-k)$ diagonalizable matrix such that

(i) the nonzero matrices F and E have the same rank r and satisfy

$$FEF = -\frac{2}{3}F, \quad EFE = -\frac{2}{3}E;$$

(ii) K and T have eigenvalues $\frac{-3+i\sqrt{3}}{3}$ and $\frac{-3+i\sqrt{3}}{6}$ of multiplicity r , respectively;

(iii) the nonzero columns of F and nonzero rows of E are eigenvectors and left eigenvectors of K , respectively, associated with eigenvalue $\frac{-3+i\sqrt{3}}{3}$, and the nonzero columns of E and nonzero rows of F are eigenvectors and left eigenvectors of T , respectively, associated with eigenvalue $-\frac{3+i\sqrt{3}}{6}$;

- (iv) the other eigenvalues of K and T belong to $\left\{0, -\frac{1+i\sqrt{3}}{2}\right\}$ and $\left\{0, -\frac{1-i\sqrt{3}}{2}\right\}$, respectively;
- (v) $\begin{bmatrix} K & F \\ E & T \end{bmatrix}$ is a commuting solution with $\text{diag}\left\{-\frac{1+i\sqrt{3}}{2}I_k, -\frac{1-i\sqrt{3}}{2}I_{m-k}\right\}$ if and only if $E = 0$ and $F = 0$, and in this case $-\frac{1-i\sqrt{3}}{2}K = \left(-\frac{1-i\sqrt{3}}{2}K\right)^2$, T is $(m-k) \times (m-k)$, and $-\frac{1+i\sqrt{3}}{2}T = \left(-\frac{1+i\sqrt{3}}{2}T\right)^2$.

Any nonzero column vector $c = (c_1^T \ c_2^T)^T$ of the $m \times (n-m)$ matrix $[C_1^T \ C_2^T]^T$ and any nonzero row vector $d = (d_1 \ d_2)$ of the $(n-m) \times m$ matrix $[D_1 \ D_2]$ are an eigenvector and a left eigenvector of the matrices

$$\begin{bmatrix} -\frac{1+i\sqrt{3}}{2}K & -\frac{1-i\sqrt{3}}{2}F \\ -\frac{1+i\sqrt{3}}{2}E & -\frac{1-i\sqrt{3}}{2}T \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1+i\sqrt{3}}{2}K & -\frac{1+i\sqrt{3}}{2}F \\ -\frac{1-i\sqrt{3}}{2}E & -\frac{1-i\sqrt{3}}{2}T \end{bmatrix},$$

respectively, such that $-\frac{1+i\sqrt{3}}{2}d_1c_1 - \frac{1-i\sqrt{3}}{2}d_2c_2 = 0$. Z is an arbitrary $(n-m) \times (n-m)$ matrix.

Proof. The proof is similar to the proof of Theorems 3.7 and 3.9 and is omitted here. $\square$

4 Numerical examples

We present two numerical examples to illustrate our results.

Example 4.1. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $A^2 = A$. This is Case 3. There exists a nonsingular matrix

$$S = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

such that $A = SJS^{-1}$, $J = \text{diag}\{1, 1, 1, 0\}$. By Theorem 3.3, all solutions of (1) are

$$X = SYS^{-1} = S \begin{pmatrix} K & C \\ D & z \end{pmatrix} S^{-1},$$

where z is an arbitrary number, K is any 3×3 matrix satisfying

$$K = S\Sigma S^{-1}, \quad \Sigma = \text{diag}(I_s, 0), \quad s \leq 3;$$

$C = U_2 W_2$; $D = H_2 \widetilde{U}_2$ with any 3×3 nonsingular matrix U . Let

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}, \quad W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad H = (h_1 \ h_2 \ h_3).$$

Denote

$$U^{-1} = \frac{1}{\det U} \begin{pmatrix} U_{11} & U_{21} & U_{31} \\ U_{12} & U_{22} & U_{32} \\ U_{13} & U_{23} & U_{33} \end{pmatrix},$$

where $\det U \neq 0$, $U_{11} = u_{22}u_{33} - u_{23}u_{32}$, $U_{12} = -(u_{21}u_{33} - u_{23}u_{31})$, $U_{13} = u_{12}u_{32} - u_{22}u_{31}$, $U_{21} = -(u_{12}u_{33} - u_{13}u_{32})$, $U_{22} = u_{11}u_{33} - u_{13}u_{31}$, $U_{23} = -(u_{11}u_{32} - u_{12}u_{31})$, $U_{31} = u_{12}u_{23} - u_{31}u_{22}$, $U_{32} = -(u_{11}u_{23} - u_{13}u_{21})$, and $U_{33} = u_{11}u_{22} - u_{12}u_{21}$. Depending on the rank of K , we have the following expressions of K , C , and D .

(1) $s = 0$, then $K = UOU^{-1} = 0$. So

$$Y = \begin{pmatrix} 0 & 0 & 0 & u_{11}w_1 + u_{12}w_2 + u_{13}w_3 \\ 0 & 0 & 0 & u_{21}w_1 + u_{22}w_2 + u_{23}w_3 \\ 0 & 0 & 0 & u_{31}w_1 + u_{32}w_2 + u_{33}w_3 \\ h_1U_{11} + h_2U_{12} + h_3U_{13} & h_1U_{21} + h_2U_{22} + h_3U_{23} & h_1U_{31} + h_2U_{32} + h_3U_{33} & z \end{pmatrix},$$

for all numbers u_{ij} , w_i , h_j ($i = 1, 2, 3$; $j = 1, 2, 3$) such that $\det U \neq 0$ and $h_1w_1 + h_2w_2 + h_3w_3 = 0$.

(2) $s = 1$, then $K = U\Sigma U^{-1}$, where $\Sigma = \text{diag}(I_1, 0)$. So

$$K = \frac{1}{\det U} \begin{pmatrix} u_{11}U_{11} & u_{11}U_{21} & u_{11}U_{31} \\ u_{21}U_{11} & u_{21}U_{21} & u_{21}U_{31} \\ u_{31}U_{11} & u_{31}U_{21} & u_{31}U_{31} \end{pmatrix}, \quad C = \begin{pmatrix} u_{12}w_2 + u_{13}w_3 \\ u_{22}w_2 + u_{23}w_3 \\ u_{32}w_2 + u_{33}w_3 \end{pmatrix},$$

$$D = \frac{1}{\det U} (h_2U_{12} + h_3U_{13} \quad h_2U_{22} + h_3U_{23} \quad h_2U_{32} + h_3U_{33}),$$

for all numbers u_{ij} ($i = 1, 2, 3$; $j = 1, 2, 3$), h_2 , h_3 , w_2 , w_3 such that $\det U \neq 0$ and $h_2w_2 + h_3w_3 = 0$.

(3) $s = 2$, then $K = U\Sigma U^{-1}$, where $\Sigma = \text{diag}(I_2, 0)$. So

$$K = \frac{1}{\det U} \begin{pmatrix} u_{11}U_{11} + u_{12}U_{12} & u_{11}U_{21} + u_{12}U_{22} & u_{11}U_{31} + u_{12}U_{32} \\ u_{21}U_{11} + u_{22}U_{12} & u_{21}U_{21} + u_{22}U_{22} & u_{21}U_{31} + u_{22}U_{32} \\ u_{31}U_{11} + u_{32}U_{12} & u_{31}U_{21} + u_{32}U_{22} & u_{31}U_{31} + u_{32}U_{32} \end{pmatrix}.$$

Since $h_3w_3 = 0$, we have

- $h_3 = 0$ from which $D = 0$ and

$$C = \begin{pmatrix} u_{13}w_3 \\ u_{23}w_3 \\ u_{33}w_3 \end{pmatrix},$$

for all numbers u_{ij} ($i = 1, 2, 3$; $j = 1, 2, 3$), w_3 such that $\det U \neq 0$.

- $w_3 = 0$ from which $D = 0$ and

$$D = \frac{1}{\det U} (h_3U_{13} \quad h_3U_{23} \quad h_3U_{33}),$$

for all numbers u_{ij} ($i = 1, 2, 3$; $j = 1, 2, 3$), h_3 such that $\det U \neq 0$.

(4) $s = 3$, then $K = UIU^{-1} = I$, $C = 0$, and $D = 0$, so

$$Y = \text{diag}(1, 1, 1, z), \quad \forall z.$$

Example 4.2. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{3-i\sqrt{3}}{2} & -\frac{1-i\sqrt{3}}{2} & 0 \\ -1 & -\frac{1-i\sqrt{3}}{2} & -\frac{3-i\sqrt{3}}{2} & 1 \end{pmatrix},$$

then $A^3 = \frac{1+i\sqrt{3}}{2}A^2 + \frac{1-i\sqrt{3}}{2}A$. This is Case 8. There exists a nonsingular matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix},$$

such that $A = SJ S^{-1}$ with $J = \text{diag}\left\{1, 1, -\frac{1-i\sqrt{3}}{2}, 0\right\}$.

By Theorem 3.10, all commuting solutions of (1) are $X = S \text{diag}\{K, t, z\} S^{-1}$, where K is any 2×2 idempotent matrix, t equals 0 or $-\frac{1-i\sqrt{3}}{2}$, and z is any number.

All non-commuting solutions of (1) are

$$X = S \begin{bmatrix} K & f & c_1 \\ e & \frac{3+i\sqrt{3}}{6} & c_2 \\ d_1 & d_2 & z \end{bmatrix} S^{-1},$$

where K is any 2×2 diagonalizable matrix with a simple eigenvalue $\frac{i\sqrt{3}}{3}$. The other simple eigenvalue is either 0 or 1 with a nonzero column vector f . The nonzero row vector e are eigenvector and left eigenvector of K associated with eigenvalue $\frac{i\sqrt{3}}{3}$ such that $ef = -\frac{2i\sqrt{3}}{6}$. z is an arbitrary number. Any nonzero column vector $c = (c_1^T \ c_2^T)^T$ and any nonzero row vector $d = (d_1 \ d_2)$ are an eigenvector and a left eigenvector of the matrices

$$\begin{bmatrix} K & -\frac{1-i\sqrt{3}}{2}f \\ e & -\frac{3-i\sqrt{3}}{6} \end{bmatrix} \text{ and } \begin{bmatrix} K & f \\ -\frac{1-i\sqrt{3}}{2}e & -\frac{3-i\sqrt{3}}{6} \end{bmatrix},$$

respectively, such that $d_1 c_1 - \frac{1-i\sqrt{3}}{2} d_2 c_2 = 0$.

We write matrix K as

$$K = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix},$$

then we will find that the explicit expressions of K in two cases corresponding to the eigenvalues of K are $\left\{\frac{i\sqrt{3}}{3}, 0\right\}$ and $\left\{\frac{i\sqrt{3}}{3}, 1\right\}$, respectively.

- If the eigenvalues of K are $\frac{i\sqrt{3}}{3}$ and 0, we have

$$\begin{cases} s_1 s_4 - \frac{i\sqrt{3}}{3}(s_1 + s_4) - \frac{1}{3} - s_2 s_3 = 0, \\ s_1 s_4 = s_2 s_3. \end{cases}$$

Solving the aforementioned equations, we have

$$K = \begin{bmatrix} -\frac{i\sqrt{3}}{6} \pm \frac{1}{2}\sqrt{-\frac{1}{3} - 4s_2 s_3} & s_2 \\ s_3 & -\frac{i\sqrt{3}}{6} \mp \frac{1}{2}\sqrt{-\frac{1}{3} - 4s_2 s_3} \end{bmatrix}, \quad \forall s_2, s_3.$$

- If the eigenvalues of K are $\frac{i\sqrt{3}}{3}$ and 1, we have

$$\begin{cases} s_1 s_4 - \frac{i\sqrt{3}}{3}(s_1 + s_4) - \frac{1}{3} - s_2 s_3 = 0, \\ s_1 s_4 - s_2 s_3 - (s_1 + s_4) + 1 = 0. \end{cases}$$

Solving the aforementioned system equation, we have

$$K = \begin{bmatrix} -\frac{3+i\sqrt{3}}{6} \pm \frac{1}{2}\sqrt{\frac{1-3i\sqrt{3}}{3} - 4s_2 s_3} & s_2 \\ s_3 & -\frac{3+i\sqrt{3}}{6} \pm \frac{1}{2}\sqrt{\frac{1-3i\sqrt{3}}{3} - 4s_2 s_3} \end{bmatrix}, \quad \forall s_2, s_3.$$

5 Conclusions

In this paper, we have found some solutions of the Yang-Baxter-like matrix Eq. (1) when the given matrix A satisfies $A^4 = A$, which has extended the previous results of [18–21]. Our approach here is to use the Jordan decomposition of A to obtain a simplified Yang-Baxter-like matrix equation with A replaced by a simple block diagonal matrix, and then we solve a system of several matrix equations for the smaller sized solution blocks. The same idea and technique in this paper can be applied to find all solutions of (1) when A satisfies the condition $A^4 = -A$ or when $A^k = A$ for some $k \in \mathbb{N}$. Once we obtain all the solutions of (1) when A is a diagonalizable complex matrix with three distinct nonzero eigenvalues, we can solve Cases 10 and 11. However, their commuting solution can be obtained by the same way used for these cases before. Finding all the non-commuting solutions of the Yang-Baxter-like matrix Eq. (1) for a general matrix A is a hard task, which will be further studied in the future.

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