Research Article

Yibin Feng and Yanping Zhou*

General (p, q) -mixed projection bodies

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Abstract: In this article, the general (p, q) -mixed projection bodies are introduced. Then, some basic properties of the general (p, q) -mixed projection bodies are discussed, and the extreme values of volumes of the general (p, q) -mixed projection bodies and their polar bodies are established.

Keywords: projection bodies, general (p, q) -mixed projection bodies, extreme value

MSC 2020: 52A20, 52A40

1 Introduction

Projection bodies, introduced by Minkowski at the end of nineteenth century, are a central notion in the classical Brunn-Minkowski theory. In recent years, projection bodies and their polars have received considerable attention, see, e.g., [1–9].

The classical Brunn-Minkowski theory has a natural extension called the L_p Brunn-Minkowski theory, which was initiated in the early 1960s when Firey introduced his concept of L_p composition of convex bodies (see [10]) and was born in the works of Lutwak [11, 12]. Since Lutwak's two seminal works, this theory has attracted increasing interest in recent years, see, e.g., [13–20]. One of the central concepts in the L_p Brunn-Minkowski theory is the L_p projection bodies introduced by Lutwak et al. [17]. Afterward, Ludwig [21] (see also Haberl and Schuster [22]) extended Lutwak, Yang and Zhang's L_p projection bodies to an entire class which may be called general L_p projection bodies. See, e.g., [23–30] for the L_p and general L_p projection bodies and their extensions.

The dual Brunn-Minkowski theory, developed by Lutwak [31] in 1975, is a dual concept of the classical Brunn-Minkowski theory. It is a theory of dual mixed volumes of star bodies and has already attracted considerable interest, see, e.g., [32–40]. In particular, a recent groundbreaking work of Huang et al. [41] showed that there exists a new family of geometry measures, in the dual Brunn-Minkowski theory, called dual curvature measures which are the long-sought duals of Federer's curvature measures. Now, the dual curvature measures become the core of the dual Brunn-Minkowski theory.

Very recently, a unification of the L_p Brunn-Minkowski theory and the dual Brunn-Minkowski theory was discovered by Lutwak et al. [42], where they introduced the L_p dual curvature measures which include L_p surface area measures and L_p integral curvatures in the L_p Brunn-Minkowski theory as well as the dual curvature measures in the dual Brunn-Minkowski theory. In the same way that the L_p surface area and dual curvature measures, respectively, play a critical role in the L_p and dual Brunn-Minkowski theories, the L_p dual curvature measures can be seen to be a central concept within the unifying theory. See, e.g., [43–46] for some recent developments related to the L_p dual curvature measures.

In this article, motivated by the works of Lutwak et al. [42], Ludwig [21] and Haberl and Schuster [22], we introduce a more general definition of projection bodies according to the L_p dual curvature measures,

Yibin Feng: School of Mathematics and Statistics, Hexi University, Zhangye, 734000, China, e-mail: fengyibin001@163.com

* **Corresponding author: Yanping Zhou,** Department of Mathematics, China Three Gorges University, Yichang, 443002, China; Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, 443002, China, e-mail: zhouyp@1.shu.edu.cn

which is called the general (p, q) -mixed projection bodies. Special cases include the classical projection bodies, L_p projection bodies and general L_p projection bodies. Then, we establish the extreme values of volumes for the general (p, q) -mixed projection bodies and their polar bodies. The detailed descriptions for the definition and main results are provided below.

Throughout $\mathbb{R}^n$ denotes the n -dimensional Euclidean space. A convex body is a compact convex subset of $\mathbb{R}^n$ with nonempty interior. We denote by $\mathcal{K}^n$ the set of convex bodies and by $\mathcal{K}_o^n$ the set of convex bodies containing the origin in their interiors. For a convex body $K \in \mathcal{K}^n$, let ∂K and $V(K)$ be its boundary and n -dimensional volume, respectively. The unit sphere in $\mathbb{R}^n$ will be denoted by S^{n-1} . For $x \in \mathbb{R}^n$, $|x| = \sqrt{x \cdot x}$ denotes the Euclidean norm of x , and for $x \in \mathbb{R}^n \setminus \{0\}$, the unit vector $x/|x| \in S^{n-1}$ will be abbreviated by $\langle x \rangle$.

For all $x \in \mathbb{R}^n$, the support function of $K \in \mathcal{K}^n$ is defined by

$$h(K, x) = h_K(x) = \max\{x \cdot y : y \in K\},$$

where $x \cdot y$ denotes the standard inner product of x and y .

The radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$, of a compact and star-shaped set, with respect to the origin, $K \subset \mathbb{R}^n$, is defined by

$$\rho(K, x) = \max\{\lambda : \lambda x \in K\}.$$

If ρ_K is positive and continuous, then K is called a star body with respect to the origin. The set of all star bodies about the origin in $\mathbb{R}^n$ is denoted by $\mathcal{S}_o^n$. Two star bodies K and L are dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$. For a star body $Q \in \mathcal{S}_o^n$, $\|\cdot\|_Q : \mathbb{R}^n \rightarrow [0, \infty)$ is a continuous and positively homogeneous function of degree 1, which was defined, in [42], by

$$\|x\|_Q = \begin{cases} 1/\rho_Q(x) & x \neq 0; \\ 0 & x = 0. \end{cases}$$

For a convex body $K \in \mathcal{K}_o^n$, its polar body K^* is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

Suppose $p, q \in \mathbb{R}$. If $K \in \mathcal{K}_o^n$ while $Q \in \mathcal{S}_o^n$, then the L_p dual curvature measure, $\tilde{C}_{p,q}(K, Q, \cdot)$, on S^{n-1} was defined, in [42], by

$$\int_{S^{n-1}} g(v) d\tilde{C}_{p,q}(K, Q, v) = \frac{1}{n} \int_{S^{n-1}} g(\alpha_K(u)) h_K(\alpha_K(u))^{-p} \rho_K(u)^q \rho_Q(u)^{n-q} du \quad (1.1)$$

for each continuous $g : S^{n-1} \rightarrow \mathbb{R}$, where α_K is the radial Gauss map that associates with almost each $u \in S^{n-1}$ the unique outer unit normal at the point $\rho_K(u)u \in \partial K$.

In [42], the L_p surface area measures were shown to be special cases of the L_p dual curvature measures:

$$\tilde{C}_{p,q}(K, K, \cdot) = \frac{1}{n} S_p(K, \cdot); \quad (1.2)$$

$$\tilde{C}_{p,n}(K, Q, \cdot) = \frac{1}{n} S_p(K, \cdot). \quad (1.3)$$

For $\tau \in [-1, 1]$, the function $\varphi_\tau : \mathbb{R} \rightarrow [0, \infty)$ was defined, in [22], by

$$\varphi_\tau(t) = |t| + \tau t. \quad (1.4)$$

Now, we begin to define the general (p, q) -mixed projection bodies.

Definition 1.1. Suppose $p \geq 1$, $q \in \mathbb{R}$ and $\tau \in [-1, 1]$. If $K \in \mathcal{K}_o^n$ while $Q \in \mathcal{S}_o^n$, then for $u \in S^{n-1}$, define the general (p, q) -mixed projection body, $\Pi_{p,q}^\tau(K, Q)$, to be the convex body whose support function is given by

$$h\left(\Pi_{p,q}^\tau(K, Q), u\right)^p = n c_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p d\tilde{C}_{p,q}(K, Q, v). \quad (1.5)$$

Here,

$$c_{n,p}(\tau) = \frac{c_{n,p}}{(1+\tau)^p + (1-\tau)^p}, \quad \text{where } c_{n,p} = \frac{\Gamma\left(\frac{n+p}{2}\right)}{\pi^{(n-1)/2}\Gamma\left(\frac{1+p}{2}\right)}.$$

If we take $Q = K$ or $q = n$ in (1.5), then from (1.2) or (1.3) we have

$$\Pi_{p,q}^{\tau}(K, Q) = \Pi_p^{\tau}K,$$

where $\Pi_p^{\tau}K$ is the general L_p projection body [22] of K , whose support function is given by

$$h\left(\Pi_p^{\tau}K, u\right)^p = c_{n,p}(\tau) \int_{S^{n-1}} \varphi_{\tau}(u \cdot v)^p dS_p(K, v). \quad (1.6)$$

For $\tau = 0$ in (1.6), $\Pi_p^{\tau}K = \Pi_p K$ which is the L_p projection body [17]. For $\tau = 0$ and $p = 1$ in (1.6), the body $\Pi_p^{\tau}K$ is the classical projection body, ΠK , of K .

Our main results are as follows for the general (p, q) -mixed projection bodies. The first result is to establish the extreme values of $V(\Pi_{p,q}^{\tau,*})$. Note that $\Pi_{p,q}^{\tau,*}$ is used to denote the polar body of $\Pi_{p,q}^{\tau}$.

Theorem 1.1. Suppose $p \geq 1$ and $q \in \mathbb{R}$. If $K \in \mathcal{K}_o^n$ while $L \in S_o^n$, then for every $\tau \in [-1, 1]$,

$$V\left(\Pi_{p,q}^{*}\right) \leq V\left(\Pi_{p,q}^{\tau,*}\right) \leq V\left(\Pi_{p,q}^{\pm,*}\right). \quad (1.7)$$

Suppose that both K and Q are not origin-symmetric. If $\tau \neq 0$, then equality holds in the left inequality if and only if $\Pi_{p,q}^{\tau}(K, Q)$ is origin-symmetric; if $\tau \neq \pm 1$, then equality holds in the right inequality if and only if both $\Pi_{p,q}^{\pm}(K, Q)$ are origin-symmetric.

Here, $\Pi_{p,q}^{+}(K, Q)$ is the nonsymmetric (p, q) -mixed projection body, which is the convex body defined by

$$h\left(\Pi_{p,q}^{+}(K, Q), u\right)^p = nc_{n,p} \int_{S^{n-1}} (u \cdot v)_+^p d\tilde{C}_{p,q}(K, Q, v) \quad (1.8)$$

for $u \in S^{n-1}$, where $(u \cdot v)_+ = \max\{u \cdot v, 0\}$. Moreover, $\Pi_{p,q}(K, Q)$ and $\Pi_{p,q}^{-}(K, Q)$ are, respectively, defined by

$$\Pi_{p,q}(K, Q) := \Pi_{p,q}^0(K, Q) \quad (1.9)$$

and

$$\Pi_{p,q}^{-}(K, Q) := \Pi_{p,q}^{+}(-K, -Q). \quad (1.10)$$

The special case $Q = K$ or $q = n$ of Theorem 1.1 can be found in [22], which gives rise to the strongest L_p Petty projection inequality. L_p Petty projection inequality is one of the crucial tools used for establishing the sharp affine L_p Sobolev inequalities and the affine Pólya-Szegő principle which are stronger than the usual sharp Sobolev inequalities and the usual Pólya-Szegő principle in the Euclidean space, see, e.g., [24–27].

The following is to provide the extreme values of $V(\Pi_{p,q}^{\tau})$.

Theorem 1.2. Suppose $p \geq 1$ and $q \in \mathbb{R}$. If $K \in \mathcal{K}_o^n$ while $L \in S_o^n$, then for every $\tau \in [-1, 1]$,

$$V\left(\Pi_{p,q}(K, Q)\right) \geq V\left(\Pi_{p,q}^{\tau}(K, Q)\right) \geq V\left(\Pi_{p,q}^{\pm}(K, Q)\right). \quad (1.11)$$

Suppose that both K and Q are not origin-symmetric. If $\tau \neq 0$, then equality holds in the left inequality if and only if $\Pi_{p,q}^\tau(K, Q)$ is origin-symmetric; if $\tau \neq \pm 1$, then equality holds in the right inequality if and only if both $\Pi_{p,q}^\pm(K, Q)$ are origin-symmetric.

When $Q = K$ or $q = n$, Theorem 1.2 was given by Haberl and Schuster [22].

For quick later reference, we list in Section 2 some basic and well-known facts of convex and star bodies, radial and reverse radial Gauss images and L_p dual curvature measures. The basic properties of the general and nonsymmetric (p, q) -mixed projection bodies are developed in Section 3. Section 4 is devoted to prove Theorems 1.1 and 1.2.

2 Preliminaries

2.1 Basics regarding convex and star bodies

Good general references about the theory of convex bodies are the books of Gardner [47] and Schneider [10].

Let $SL(n)$ denote the group of special linear transformation. If $\phi \in SL(n)$, then we write ϕ^t for the transpose of ϕ , ϕ^{-1} for the inverse of ϕ and ϕ^{-t} for the inverse of the transpose of ϕ . From the definitions of the support function, radial function and polar body, it is easy to see that for $\phi \in SL(n)$, $K \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$,

$$h(\phi K, x) = h(K, \phi^t x), \quad x \in \mathbb{R}^n; \quad (2.1)$$

$$\rho(\phi K, x) = \rho(K, \phi^{-1} x), \quad x \in \mathbb{R}^n \setminus \{0\}; \quad (2.2)$$

$$(\phi K)^* = \phi^{-t} K^*. \quad (2.3)$$

Moreover, it is easy to verify that for $K \in \mathcal{K}_o^n$ and $c > 0$,

$$(cK)^* = \frac{1}{c} K^*. \quad (2.4)$$

Suppose $p \in \mathbb{R}$. If μ is a Borel measure on S^{n-1} and $\phi \in SL(n)$, then the L_p image of μ under ϕ , $\phi_{p^*} \mu$, is a Borel measure defined, in [42], by

$$\int_{S^{n-1}} f(u) d\phi_{p^*} \mu(u) = \int_{S^{n-1}} |\phi^{-1} u|^p f(\langle \phi^{-1} u \rangle) d\mu(u), \quad (2.5)$$

for each Borel $f : S^{n-1} \rightarrow \mathbb{R}$.

The support and radial functions of a convex body $K \in \mathcal{K}_o^n$ and its polar body are related by

$$\rho_K = 1/h_{K^*} \quad \text{and} \quad h_K = 1/\rho_{K^*}. \quad (2.6)$$

If $K_i \in \mathcal{K}^n$, we say that $K_i \rightarrow K_0 \in \mathcal{K}^n$ provided

$$|h_{K_i} - h_{K_0}|_\infty := \max_{u \in S^{n-1}} |h_{K_i}(u) - h_{K_0}(u)| \rightarrow 0.$$

For $K, L \in \mathcal{K}^n$ and $\alpha, \beta \geq 0$ (both not zero), the Minkowski combination $\alpha K + \beta L$ is defined by

$$\alpha K + \beta L = \{\alpha x + \beta y : x \in K, y \in L\}$$

whose support function is

$$h_{\alpha K + \beta L} = \alpha h_K + \beta h_L. \quad (2.7)$$

In the early 1960s, Firey [48] introduced the L_p Minkowski combination, which is also known as the Minkowski-Firey combination. For $p \geq 1$, $K, L \in \mathcal{K}_o^n$ and $\alpha, \beta \geq 0$ (both not zero), the L_p Minkowski combination, $\alpha \cdot K +_p \beta \cdot L$, was defined by

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p, \quad (2.8)$$

where $\alpha \cdot K = \alpha^{\frac{1}{p}} K$.

In [48], Firey also established the L_p Brunn-Minkowski inequality: if $p \geq 1$ and $K, L \in \mathcal{K}_o^n$, then

$$V(K +_p L)^{\frac{p}{n}} \geq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}, \quad (2.9)$$

with equality if and only if K and L are dilates.

The L_p harmonic radial combination of two star bodies was introduced by Lutwak [12]. For $p \geq 1$, $K, L \in \mathcal{S}_o^n$ and $\alpha, \beta \geq 0$ (both not zero), the L_p harmonic radial combination, $\alpha \circ K \tilde{+}_p \beta \circ L$, is a star body whose radial function is given by

$$\rho(\alpha \circ K \tilde{+}_p \beta \circ L, \cdot)^{-p} = \alpha \rho(K, \cdot)^{-p} + \beta \rho(L, \cdot)^{-p}. \quad (2.10)$$

Note that $\alpha \circ K = \alpha^{-\frac{1}{p}} K$.

Lutwak's L_p dual Brunn-Minkowski inequality, see [12], is as follows: if $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, then

$$V(K \tilde{+}_p L)^{-\frac{p}{n}} \geq V(K)^{-\frac{p}{n}} + V(L)^{-\frac{p}{n}}, \quad (2.11)$$

with equality if and only if K and L are dilates.

2.2 Radial and reverse radial Gauss images

In the following, we state some necessary facts with regard to the radial and reverse radial Gauss images, cf. [41, 42].

For $K \in \mathcal{K}_o^n$ and $v \in S^{n-1}$, the set

$$H_K(v) = \{x \in \mathbb{R}^n : x \cdot v = h_K(v)\} \quad (2.12)$$

is called the supporting hyperplane of K with the outer unit normal v .

The spherical image of $\sigma \subset \partial K$, $K \in \mathcal{K}_o^n$, is defined by

$$\mathbf{v}_K(\sigma) = \{v \in S^{n-1} : x \in H_K(v) \text{ for some } x \in \sigma\} \subset S^{n-1}.$$

Let $\sigma_K \subset \partial K$ be the set consisting of all $x \in \partial K$ for which the set $\mathbf{v}_K(\{x\})$, often abbreviated as $\mathbf{v}_K(x)$, contains more than a single element. Then, $\mathcal{H}^{n-1}(\sigma_K) = 0$ based on Schneider [10, p. 84], where $\mathcal{H}^{n-1}$ is the $(n-1)$ -dimensional Hausdorff measure on ∂K . For $x \in \partial K \setminus \sigma_K$, $\mathbf{v}_K(x)$ has the unique element denoted by $v_K(x)$ called the spherical image map of K . For convenience, we abbreviate $\partial K \setminus \sigma_K$ as $\partial'K$. If the integration is with respect to $\mathcal{H}^{n-1}$, then it follows from $\mathcal{H}^{n-1}(\sigma_K) = 0$ that it will be immaterial over subsets of $\partial'K$ or ∂K .

For $K \in \mathcal{K}_o^n$ and $\omega \subset S^{n-1}$, the radial Gauss image of ω , $\alpha_K(\omega)$, is defined by

$$\alpha_K(\omega) = \{v \in S^{n-1} : \rho_K(u)u \in H_K(v) \text{ for some } u \in \omega\}.$$

Thus for $u \in S^{n-1}$,

$$\alpha_K(u) = \{v \in S^{n-1} : \rho_K(u)u \in H_K(v)\}. \quad (2.13)$$

Let $\omega_K = \{u \in S^{n-1} : \rho_K(u)u \in \sigma_K\} \subset S^{n-1}$. Clearly, if $u \in S^{n-1} \setminus \omega_K$, then $\alpha_K(u)$ contains only a single element denoted by $\alpha_K(u)$, i.e.,

$$\alpha_K : S^{n-1} \setminus \omega_K \rightarrow S^{n-1},$$

which is named as the radial Gauss map of K . Combining (2.12) and (2.13), we obtain two essential facts needed: for $\lambda > 0$,

$$\alpha_{\lambda K} = \alpha_K; \quad (2.14)$$

$$\alpha_{-K}(u) = -\alpha_K(-u), \quad \text{for } u \in S^{n-1}. \quad (2.15)$$

Let $K \in \mathcal{K}_o^n$ and $\eta \subset S^{n-1}$. The reverse radial Gauss image of η , $\alpha_K^*(\eta)$, is defined by

$$\alpha_K^*(\eta) = \{u \in S^{n-1} : \rho_K(u)u \in H_K(v) \text{ for some } v \in \eta\}. \quad (2.16)$$

Together (2.12) with (2.16), it is easy to see that for $\lambda > 0$,

$$\alpha_{\lambda K}^* = \alpha_K^*. \quad (2.17)$$

2.3 L_p dual curvature measures

The following are some key properties, shown in [42] and needed in next section, of the L_p dual curvature measures.

In [42], Lutwak et al. showed that for $p, q \in \mathbb{R}$, $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, definition (1.1) can also be written as:

$$\int_{S^{n-1}} g(v) d\tilde{C}_{p,q}(K, Q, v) = \frac{1}{n} \int_{\partial' K} g(v_K(x)) (x \cdot v_K(x))^{1-p} \|x\|_Q^{q-n} d\mathcal{H}^{n-1}(x). \quad (2.18)$$

The L_p dual curvature measures have the following integral representation (see [42]): if $p, q \in \mathbb{R}$, $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, then

$$\tilde{C}_{p,q}(K, Q, \eta) = \frac{1}{n} \int_{\alpha_K^*(\eta)} h_K(\alpha_K(u))^{-p} \rho_K^q(u) \rho_Q^{n-q}(u) du \quad (2.19)$$

for each Borel set $\eta \subseteq S^{n-1}$. Glancing (2.14) and (2.17), it immediately follows from (2.19) that for $\lambda > 0$,

$$\tilde{C}_{p,q}(\lambda K, Q, \eta) = \lambda^{q-p} \tilde{C}_{p,q}(K, Q, \eta). \quad (2.20)$$

It was shown, in [42], that if $p \neq 0$ and $q \neq 0$, then for $K \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$ and all $\phi \in SL(n)$,

$$\tilde{C}_{p,q}(\phi K, \phi Q, \cdot) = \phi_p^t \tilde{C}_{p,q}(K, Q, \cdot). \quad (2.21)$$

The L_p dual curvature measures are weakly convergent (see [42]): if $p, q \in \mathbb{R}$, $Q \in \mathcal{S}_o^n$ and $K_i \in \mathcal{K}_o^n$ with $K_i \rightarrow K_0 \in \mathcal{K}_o^n$, then for each continuous function $g : S^{n-1} \rightarrow \mathbb{R}$,

$$\lim_{i \rightarrow \infty} \int_{S^{n-1}} g(v) d\tilde{C}_{p,q}(K_i, Q, v) = \int_{S^{n-1}} g(v) d\tilde{C}_{p,q}(K_0, Q, v). \quad (2.22)$$

Moreover, they are a valuation (see [42]), i.e., if $K, L \in \mathcal{K}_o^n$ are such that $K \cup L \in \mathcal{K}_o^n$, then

$$\tilde{C}_{p,q}(K, Q, \cdot) + \tilde{C}_{p,q}(L, Q, \cdot) = \tilde{C}_{p,q}(K \cap L, Q, \cdot) + \tilde{C}_{p,q}(K \cup L, Q, \cdot). \quad (2.23)$$

3 General and nonsymmetric (p, q) -mixed projection bodies

In this section, we first show the relation between general and nonsymmetric (p, q) -mixed projection bodies. Then, some of their basic properties are established.

Let $g(v) = \varphi_\tau(u \cdot v)^p$, with $p \geq 1$, in (2.18). Then, it will be easy to see that definition (1.5) can be rewritten as:

$$h\left(\prod_{p,q}^\tau(K, Q), u\right)^p = c_{n,p}(\tau) \int_{\partial K} \varphi_\tau(u \cdot v_K(x))^p (x \cdot v_K(x))^{1-p} \|x\|_Q^{q-n} d\mathcal{H}^{n-1}(x). \quad (3.1)$$

From the definition of $\Pi_{p,q}^-(K, Q)$, (1.8), (1.1) and (2.15), it follows that for $p \geq 1$, $K \in \mathcal{K}_o^n$ and $Q \in S_o^n$,

$$h\left(\bar{\Pi}_{p,q}^-(K, Q), u\right)^p = nc_{n,p} \int_{S^{n-1}} (-u \cdot v)_+^p d\tilde{C}_{p,q}(K, Q, v), \quad u \in S^{n-1}. \quad (3.2)$$

Since for any $u, v \in S^{n-1}$,

$$\varphi_\tau(u \cdot v)^p = (1 + \tau)^p (u \cdot v)_+^p + (1 - \tau)^p (-u \cdot v)_+^p,$$

by (1.5), (1.8), (3.2) and (2.8) we have

$$\bar{\Pi}_{p,q}^\tau(K, Q) = f_1(\tau) \cdot \Pi_{p,q}^+(K, Q) +_p f_2(\tau) \cdot \bar{\Pi}_{p,q}^-(K, Q), \quad (3.3)$$

where

$$f_1(\tau) = \frac{(1 + \tau)^p}{(1 + \tau)^p + (1 - \tau)^p} \quad \text{and} \quad f_2(\tau) = \frac{(1 - \tau)^p}{(1 + \tau)^p + (1 - \tau)^p}.$$

Note that

$$f_1(\tau) + f_2(\tau) = 1,$$

$$f_1(-\tau) = f_2(\tau) \quad \text{and} \quad f_2(-\tau) = f_1(\tau).$$

From (3.3), we see that if $\tau = \pm 1$, then

$$\bar{\Pi}_{p,q}^{+1}(K, Q) = \Pi_{p,q}^+(K, Q) \quad \text{and} \quad \bar{\Pi}_{p,q}^{-1}(K, Q) = \bar{\Pi}_{p,q}^-(K, Q); \quad (3.4)$$

if $\tau = 0$, then

$$\Pi_{p,q}(K, Q) = \frac{1}{2} \cdot \Pi_{p,q}^+(K, Q) +_p \frac{1}{2} \cdot \bar{\Pi}_{p,q}^-(K, Q). \quad (3.5)$$

We first show the property of $\Pi_{p,q}^\pm$.

Proposition 3.1. Suppose $p \geq 1$, $q \in \mathbb{R}$. If $K \in \mathcal{K}_o^n$ while $Q \in S_o^n$, then

$$-\bar{\Pi}_{p,q}^+(K, Q) = \bar{\Pi}_{p,q}^-(K, Q) = \bar{\Pi}_{p,q}^+(-K, -Q). \quad (3.6)$$

Proof. We just need to prove the left equality. From (3.2) and (1.8), it follows that for any $u \in S^{n-1}$,

$$h\left(\bar{\Pi}_{p,q}^-(K, Q), u\right)^p = nc_{n,p} \int_{S^{n-1}} (-u \cdot v)_+^p d\tilde{C}_{p,q}(K, Q, v) = h\left(\Pi_{p,q}^+(K, Q), -u\right)^p = h\left(-\Pi_{p,q}^+(K, Q), u\right)^p,$$

i.e.,

$$\bar{\Pi}_{p,q}^-(K, Q) = -\Pi_{p,q}^+(K, Q).$$

This finishes the proof. □

The following is to show the properties of $\Pi_{p,q}^\tau$.

Proposition 3.2. Suppose $p \geq 1$, $q \in \mathbb{R}$, $K \in \mathcal{K}_o^n$ while $Q \in S_o^n$.

(i) If $\tau \in [-1, 1]$, then

$$\bar{\Pi}_{p,q}^\tau(-K, -Q) = \bar{\Pi}_{p,q}^{-\tau}(K, Q) = -\Pi_{p,q}^\tau(K, Q); \quad (3.7)$$

(ii) If $\tau \neq 0$, then

$$\overset{\tau}{\Pi}_{p,q}(K, Q) = \overset{-\tau}{\Pi}_{p,q}(K, Q) \Leftrightarrow \overset{+}{\Pi}_{p,q}(K, Q) = \overset{-}{\Pi}_{p,q}(K, Q). \quad (3.8)$$

Proof.

(i) By (3.3) and (3.6), we have

$$\begin{aligned} \overset{-\tau}{\Pi}_{p,q}(K, Q) &= f_2(\tau) \cdot \overset{+}{\Pi}_{p,q}(K, Q) +_p f_1(\tau) \cdot \overset{-}{\Pi}_{p,q}(K, Q) \\ &= f_2(\tau) \cdot \overset{-}{\Pi}_{p,q}(-K, -Q) +_p f_1(\tau) \cdot \overset{+}{\Pi}_{p,q}(-K, -Q) \\ &= \overset{\tau}{\Pi}_{p,q}(-K, -Q). \end{aligned} \quad (3.9)$$

Moreover, from (3.3), (2.8), (3.6) and (2.8) again, we get that for any $u \in S^{n-1}$,

$$\begin{aligned} h\left(\overset{-\tau}{\Pi}_{p,q}(K, Q), u\right)^p &= f_2(\tau) h\left(\overset{+}{\Pi}_{p,q}(K, Q), u\right)^p + f_1(\tau) h\left(\overset{-}{\Pi}_{p,q}(K, Q), u\right)^p \\ &= f_2(\tau) h\left(\overset{-}{\Pi}_{p,q}(K, Q), -u\right)^p + f_1(\tau) h\left(\overset{+}{\Pi}_{p,q}(K, Q), -u\right)^p \\ &= h\left(f_1(\tau) \cdot \overset{+}{\Pi}_{p,q}(K, Q) +_p f_2(\tau) \cdot \overset{-}{\Pi}_{p,q}(K, Q), -u\right)^p \\ &= h\left(\overset{\tau}{\Pi}_{p,q}(K, Q), u\right)^p, \end{aligned}$$

i.e.,

$$\overset{-\tau}{\Pi}_{p,q}(K, Q) = -\overset{\tau}{\Pi}_{p,q}(K, Q). \quad (3.10)$$

Together (3.9) with (3.10) gives

$$\overset{\tau}{\Pi}_{p,q}(-K, -Q) = \overset{-\tau}{\Pi}_{p,q}(K, Q) = -\overset{\tau}{\Pi}_{p,q}(K, Q).$$

(ii) Let $\tau \neq 0$ and $\overset{+}{\Pi}_{p,q}(K, Q) = \overset{-}{\Pi}_{p,q}(K, Q)$. We know that for any $u \in S^{n-1}$,

$$h\left(\overset{\tau}{\Pi}_{p,q}(K, Q), u\right)^p = f_1(\tau) h\left(\overset{+}{\Pi}_{p,q}(K, Q), u\right)^p + f_2(\tau) h\left(\overset{-}{\Pi}_{p,q}(K, Q), u\right)^p \quad (3.11)$$

and

$$h\left(\overset{-\tau}{\Pi}_{p,q}(K, Q), u\right)^p = f_2(\tau) h\left(\overset{+}{\Pi}_{p,q}(K, Q), u\right)^p + f_1(\tau) h\left(\overset{-}{\Pi}_{p,q}(K, Q), u\right)^p. \quad (3.12)$$

Together (3.11) with (3.12), it follows that

$$\overset{\tau}{\Pi}_{p,q}(K, Q) = \overset{-\tau}{\Pi}_{p,q}(K, Q).$$

Moreover, let $\overset{\tau}{\Pi}_{p,q}(K, Q) = \overset{-\tau}{\Pi}_{p,q}(K, Q)$. Combining (3.11) with (3.12), we have

$$[f_1(\tau) - f_2(\tau)] h\left(\overset{+}{\Pi}_{p,q}(K, Q), u\right)^p = [f_1(\tau) - f_2(\tau)] h\left(\overset{-}{\Pi}_{p,q}(K, Q), u\right)^p. \quad (3.13)$$

Since $f_1(\tau) - f_2(\tau) \neq 0$ when $\tau \neq 0$, it follows from (3.13) that

$$\overset{+}{\Pi}_{p,q}(K, Q) = \overset{-}{\Pi}_{p,q}(K, Q).$$

□

Next, we will show that the operator $\Pi_{p,q}^\tau(\cdot, \cdot)$ is $SL(n)$ contravariant and homogeneous of $q/p - 1$ with respect to the first part.

Proposition 3.3. Suppose $p \geq 1$, $q \in \mathbb{R}$, $\tau \in [-1, 1]$, $K \in \mathcal{K}_o^n$ while $Q \in S_o^n$.

(i) For all $\phi \in SL(n)$,

$$\Pi_{p,q}^\tau(\phi K, \phi Q) = \phi^{-t} \Pi_{p,q}^\tau(K, Q); \quad (3.14)$$

(ii) For $\lambda > 0$,

$$\Pi_{p,q}^\tau(\lambda K, Q) = \lambda^{q/p-1} \Pi_{p,q}^\tau(K, Q). \quad (3.15)$$

Proof.

(i) From (1.5), (2.21), (2.5), (1.5) again and (2.1), we have that for any $u \in S^{n-1}$ and all $\phi \in SL(n)$,

$$\begin{aligned} h\left(\Pi_{p,q}^\tau(\phi K, \phi Q), u\right)^p &= nc_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p d\tilde{C}_{p,q}(\phi K, \phi Q, v) \\ &= nc_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p d\phi_{p^{-1}}^t \tilde{C}_{p,q}(K, Q, v) \\ &= nc_{n,p}(\tau) \int_{S^{n-1}} |\phi^{-t}v|^p \varphi_\tau(u \cdot \langle \phi^{-t}v \rangle)^p d\tilde{C}_{p,q}(K, Q, v) \\ &= nc_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot \phi^{-t}v)^p d\tilde{C}_{p,q}(K, Q, v) \\ &= nc_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(\phi^{-1}u \cdot v)^p d\tilde{C}_{p,q}(K, Q, v) \\ &= h\left(\Pi_{p,q}^\tau(K, Q), \phi^{-1}u\right)^p \\ &= h\left(\phi^{-t} \Pi_{p,q}^\tau(K, Q), u\right)^p, \end{aligned}$$

i.e.,

$$\Pi_{p,q}^\tau(\phi K, \phi Q) = \phi^{-t} \Pi_{p,q}^\tau(K, Q).$$

(ii) By (1.5) and (2.20), it follows that for any $u \in S^{n-1}$ and $\lambda > 0$,

$$h\left(\Pi_{p,q}^\tau(\lambda K, Q), u\right)^p = nc_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p d\tilde{C}_{p,q}(\lambda K, Q, v) = \lambda^{q-p} h\left(\Pi_{p,q}^\tau(K, Q), u\right)^p = h\left(\lambda^{q/p-1} \Pi_{p,q}^\tau(K, Q), u\right)^p.$$

That is,

$$\Pi_{p,q}^\tau(\lambda K, Q) = \lambda^{q/p-1} \Pi_{p,q}^\tau(K, Q). \quad \square$$

For a fixed $Q \in S_o^n$, we will prove that $\Pi_{p,q}^\tau(\cdot, Q) : \mathcal{K}_o^n \rightarrow \mathcal{K}_o^n$ is continuous.

Proposition 3.4. Suppose $p \geq 1$, $q \in \mathbb{R}$ and $Q \in S_o^n$. If $K_i \in \mathcal{K}_o^n$ with $K_i \rightarrow K_0 \in \mathcal{K}_o^n$, then for every $\tau \in [-1, 1]$,

$$\Pi_{p,q}^\tau(K_i, Q) \rightarrow \Pi_{p,q}^\tau(K_0, Q). \quad (3.16)$$

Proof. Suppose $u_0 \in S^{n-1}$. From (1.5) and (2.22), we get for a fixed $Q \in S^n_o$,

$$h\left(\frac{\tau}{\Pi_{p,q}}(K_i, Q), u_0\right) \rightarrow h\left(\frac{\tau}{\Pi_{p,q}}(K_0, Q), u_0\right), \quad \text{as } i \rightarrow \infty.$$

The support functions $h_{\Pi_{p,q}^\tau(K_i, Q)} \rightarrow h_{\Pi_{p,q}^\tau(K_0, Q)}$ pointwise on S^{n-1} imply that they converge uniformly (see [10, Theorem 1.8.15]). This finishes the proof. $\square$

An operator $Z : \mathcal{K}_o^n \rightarrow \mathcal{K}_o^n$ is called an L_p Minkowski valuation if

$$ZK_1 +_p ZK_2 = Z(K_1 \cup K_2) +_p Z(K_1 \cap K_2),$$

whenever $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}_o^n$.

The theory of real valued valuations lies at the very core of geometry, see, e.g., [49, 50]. In the 1970s, Schneider [51] first obtained the results on a special class of Minkowski valuations. Recently, since the seminal work of Ludwig [3, 21], the investigations of these Minkowski valuations have become the focus of increased attention, see, e.g., [52–59].

An immediate result of (2.23) is that for a fixed $Q \in S^n_o$, $\Pi_{p,q}^\tau(\cdot, Q) : \mathcal{K}_o^n \rightarrow \mathcal{K}_o^n$ is an L_p Minkowski valuation.

Proposition 3.5. Suppose, $q \in \mathbb{R}$ and $Q \in S^n_o$. Then, for every $\tau \in [-1, 1]$,

$$\frac{\tau}{\Pi_{p,q}}(\cdot, Q) : \mathcal{K}_o^n \rightarrow \mathcal{K}_o^n$$

is an L_p Minkowski valuation, i.e., if $K, L \in \mathcal{K}_o^n$ are such that $K \cup L \in \mathcal{K}_o^n$, then

$$\frac{\tau}{\Pi_{p,q}}(K, Q) +_p \frac{\tau}{\Pi_{p,q}}(L, Q) = \frac{\tau}{\Pi_{p,q}}(K \cup L, Q) +_p \frac{\tau}{\Pi_{p,q}}(K \cap L, Q).$$

4 Proofs of Theorems 1.1–1.2

This section is dedicated to give the proofs of Theorems 1.1–1.2.

Proof of Theorem 1.1. Suppose that both K and Q are not origin-symmetric (otherwise the result is trivial). We first show the right inequality. Let $-1 < \tau < 1$. From (3.3), (2.8), (2.6) and (2.10), it follows that

$$\frac{\tau, *}{\Pi_{p,q}}(K, Q) = f_1(\tau) \circ \frac{+, *}{\Pi_{p,q}}(K, Q) \tilde{+}_p f_2(\tau) \circ \frac{-, *}{\Pi_{p,q}}(K, Q). \quad (4.1)$$

Using the L_p dual Brunn-Minkowski inequality (2.11), (3.6) and (2.4), and note that $\alpha \circ K = \alpha^{-\frac{1}{p}}K$, we have

$$V\left(\frac{\tau, *}{\Pi_{p,q}}(K, Q)\right) \leq V\left(\frac{+, *}{\Pi_{p,q}}(K, Q)\right),$$

with equality if and only if $\Pi_{p,q}^{+,*}(K, Q)$ and $\Pi_{p,q}^{-,*}(K, Q)$ are dilates, which is equivalent to $\Pi_{p,q}^{+,*}(K, Q) = \Pi_{p,q}^-(K, Q)$. Since $\Pi_{p,q}^-(K, Q) = -\Pi_{p,q}^+(K, Q)$ and $\Pi_{p,q}^{+,*}(K, Q) = -\Pi_{p,q}^-(K, Q)$, $\Pi_{p,q}^{+,*}(K, Q) = \Pi_{p,q}^-(K, Q)$ implies that both $\Pi_{p,q}^+(K, Q)$ and $\Pi_{p,q}^-(K, Q)$ are origin-symmetric and *vice versa*.

Next, we prove the left inequality. Let $\tau \neq 0$. From (4.1) and (2.10), it follows that for any $u \in S^{n-1}$,

$$\rho\left(\frac{\tau, *}{\Pi_{p,q}}(K, Q), u\right)^{-p} = f_1(\tau)\rho\left(\frac{+, *}{\Pi_{p,q}}(K, Q), u\right)^{-p} + f_2(\tau)\rho\left(\frac{-, *}{\Pi_{p,q}}(K, Q), u\right)^{-p} \quad (4.2)$$

and

$$\rho\left(\overset{-\tau,*}{\Pi}_{p,q}(K, Q), u\right)^{-p} = f_2(\tau)\rho\left(\overset{+,*}{\Pi}_{p,q}(K, Q), u\right)^{-p} + f_1(\tau)\rho\left(\overset{-,*}{\Pi}_{p,q}(K, Q), u\right)^{-p}. \quad (4.3)$$

Combining (4.2) and (4.3), we get

$$\frac{1}{2}\rho\left(\overset{\tau,*}{\Pi}_{p,q}(K, Q), u\right)^{-p} + \frac{1}{2}\rho\left(\overset{-\tau,*}{\Pi}_{p,q}(K, Q), u\right)^{-p} = \frac{1}{2}\rho\left(\overset{+,*}{\Pi}_{p,q}(K, Q), u\right)^{-p} + \frac{1}{2}\rho\left(\overset{-,*}{\Pi}_{p,q}(K, Q), u\right)^{-p}.$$

This together (2.10), (2.8) with (3.5) yields

$$\overset{*}{\Pi}_{p,q}(K, Q) = \frac{1}{2} \circ \overset{\tau,*}{\Pi}_{p,q}(K, Q) \dot{+}_p \frac{1}{2} \circ \overset{-\tau,*}{\Pi}_{p,q}(K, Q). \quad (4.4)$$

By inequalities (2.11), (3.7) and (2.4), this has

$$V\left(\overset{*}{\Pi}_{p,q}(K, Q)\right) \leq V\left(\overset{\tau,*}{\Pi}_{p,q}(K, Q)\right),$$

with equality if and only if $\overset{\tau,*}{\Pi}_{p,q}(K, Q)$ and $\overset{-\tau,*}{\Pi}_{p,q}(K, Q)$ are dilates, which is equivalent to $\overset{\tau}{\Pi}_{p,q}(K, Q) = \overset{-\tau}{\Pi}_{p,q}(K, Q)$. From (3.7), $\overset{\tau}{\Pi}_{p,q}(K, Q) = \overset{-\tau}{\Pi}_{p,q}(K, Q)$ yields that $\overset{\tau}{\Pi}_{p,q}(K, Q)$ is origin-symmetric and *vice versa*. $\square$

We will use the following consequence (see [22]).

Lemma 5.1. *If $K \in \mathcal{K}_o^n$ and p is not an odd integer, then*

$$\overset{+}{\Pi}_p K = \overset{-}{\Pi}_p K \Leftrightarrow K \text{ is origin-symmetric}.$$

Here, $\overset{\pm}{\Pi}_p K$ denote the nonsymmetric L_p projection bodies (see [22]).

Let $Q = K$ or $q = n$ in Theorem 1.1. Then, we immediately have the following result given by Haberl and Schuster [22], where the equality conditions follow from Lemma 5.1 and the case $Q = K$ or $q = n$ of (3.8).

Corollary 5.1. [22] *For $p \geq 1$, $\tau \in [-1, 1]$ and $K \in \mathcal{K}_o^n$,*

$$V\left(\overset{*}{\Pi}_p K\right) \leq V\left(\overset{\tau,*}{\Pi}_p K\right) \leq V\left(\overset{\pm,*}{\Pi}_p K\right).$$

If K is not origin-symmetric and p is not an odd integer, equality holds in the left inequality if and only if $\tau = 0$ and equality holds in the right inequality if and only if $\tau = \pm 1$.

Proof of Theorem 1.2. Assume that both K and Q are not origin-symmetric. First, we deduce the right inequality. Let $-1 < \tau < 1$. Then it follows from (3.3), the L_p Brunn-Minkowski inequality (2.9) and the fact that $\alpha \cdot K = \alpha^{\frac{1}{p}}K$ that

$$V\left(\overset{\tau}{\Pi}_{p,q}(K, Q)\right) \geq V\left(\overset{\pm}{\Pi}_{p,q}(K, Q)\right),$$

with equality if and only if $\overset{+}{\Pi}_{p,q}(K, Q)$ and $\overset{-}{\Pi}_{p,q}(K, Q)$ are dilates, which is equivalent to $\overset{+}{\Pi}_{p,q}(K, Q) = \overset{-}{\Pi}_{p,q}(K, Q)$, i.e., $\overset{\pm}{\Pi}_{p,q}(K, Q)$ are origin-symmetric.

In order to see the left inequality, we suppose $\tau \neq 0$. From (4.4), (2.6) and (2.8), we have

$$\overset{\tau}{\Pi}_{p,q}(K, Q) = \frac{1}{2} \cdot \overset{\tau}{\Pi}_{p,q}(K, Q) \dot{+}_p \frac{1}{2} \cdot \overset{-\tau}{\Pi}_{p,q}(K, Q).$$

Using inequality (2.9) and equality (3.7), this gets

$$V\left(\overset{\tau}{\Pi}_{p,q}(K, Q)\right) \geq V\left(\overset{\tau}{\Pi}_{p,q}(K, Q)\right),$$

with equality if and only if $\Pi_{p,q}^\tau(K, Q)$ and $\Pi_{p,q}^{-\tau}(K, Q)$ are dilates, which is equivalent to $\Pi_{p,q}^\tau(K, Q) = \Pi_{p,q}^{-\tau}(K, Q)$. That is, $\Pi_{p,q}^\tau(K, Q)$ is origin-symmetric.

As a special case of Theorem 1.2, the following is a direct result. Note that the proof of the equality conditions is similar to that of Corollary 5.1.

Corollary 5.2. [22] For $p \geq 1$, $\tau \in [-1, 1]$ and $K \in \mathcal{K}_o^n$,

$$V\left(\Pi_p K\right) \geq V\left(\Pi_p^\tau K\right) \geq V\left(\Pi_p^\pm K\right).$$

If K is not origin-symmetric and p is not an odd integer, equality holds in the left inequality if and only if $\tau = 0$ and equality holds in the right inequality if and only if $\tau = \pm 1$.

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