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On regular polynomial endomorphisms of $\mathbb{C}^2$ without bounded critical orbits

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Abstract: We study conditions involving the critical set of a regular polynomial endomorphism $f : \mathbb{C}^2 \mapsto \mathbb{C}^2$ under which all complete external rays from infinity for f have well defined endpoints.

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Among polynomial maps on $\mathbb{C}^k$, $k \geq 2$, regular polynomial endomorphisms are those whose dynamical behavior most resembles that of polynomials in one complex variable. Specifically, a regular polynomial endomorphism $f : \mathbb{C}^k \mapsto \mathbb{C}^k$ is a mapping $f = (f_1, ..., f_k)$, such that $f_1, ..., f_k$ are polynomial maps of degree $d \geq 2$ and $\hat{f}_1^{-1}(0) \cap ... \cap \hat{f}_k^{-1}(0) = \{0\}$ (i.e., $\hat{f}_j$ have no common components) , where $\hat{f}_j$ is the homogeneous part of degree d of f_j, $\quad j = 1, ..., k$. Such a map extends holomorphically to $\mathbb{P}^k$. With f we can associate the following objects:

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^n(z)\|, \quad z \in \mathbb{C}^k,$$

$$T = \frac{1}{2\pi} dd^c G,$$

$$J = \mathrm{supp}(T \wedge ... \wedge T)(k \text{ times }),$$

$$K = \{z \in \mathbb{C}^k : \{z, f(z), f^2(z), ...\}\text{is bounded}\},$$

Recall that G is a continuous plurisubharmonic function in $\mathbb{C}^k$ satisfying the equation $G(f(z)) = d \cdot G(z)$ for all $z \in \mathbb{C}^k$, T is a positive closed $(1, 1)$-current in $\mathbb{C}^k$ extending

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trivially to $\mathbb{P}^k$, $\mu := T \wedge ... \wedge T$ (k times) is an f-invariant Borel finite measure and K is a compact set in $\mathbb{C}^k$ with $J \subset \partial K$. (cf. [1], [2], [6], [9], [10], [12], [20])

Let C denote the critical set of f, i.e., $C = \{z \in \mathbb{C}^k : \det(Df(z)) = 0\}$. Critical sets play an important role in the dynamics of non-invertible maps. For example, it is well known that if $f : \mathbb{C} \mapsto \mathbb{C}$ is a polynomial such that $K \cap C = \emptyset$, then f is uniformly expanding on K (see e.g. [16], theorem 19.1; a recent paper [11] gives an algorithmic construction of an expanding metrics). It turns out that this result can be extended to regular polynomial endomorphisms of $\mathbb{C}^k$([8], [19]). Recall first the following notion, coming from [17]:

Definition 1. Let K be a compact subset of a smooth Riemannian manifold M such that $f(K) = K$ for a $\mathcal{C}^{1+\alpha}$-map $f : M \mapsto M$. K is a mixing repeller for f if it satisfies the following three conditions:

(i) there exists an open neighborhood V of K (called a *basin*) such that $K = \{x \in V : f^n(x) \in V \text{ for all } n \geq 0\}$;

(ii) there exists $c > 0$ and $\lambda > 1$ such that $\|Df_x^n v\| \geq c\lambda^n \|v\|$ for all $x \in K$, $v \in T_xM$ and $n \geq 1$ (with respect to a Riemannian metric on M), i.e., f is *uniformly expanding* on K;

(iii) for any open set U intersecting K there is a natural number $N > 0$ such that $K \subset f^N(U)$.

Theorem 2. *Let $f : \mathbb{C}^k \mapsto \mathbb{C}^k$ be a regular polynomial endomorphism, C its critical set and K its set of bounded forward orbits. If $\mathcal{C} \cap K = \emptyset$, then K is a mixing repeller for f.*

Proof. (cf. [19], theorem 4.0.1): (i)If K is as above, then $K = f^{-1}(K)$, so any bounded open neighborhood V of K can be taken as a basin.

(ii) We will work with a special kind of basin. Note that by the definition of the function G, $K = G^{-1}(0)$. By the continuity of G, $\{\{G < \varepsilon\} : \varepsilon > 0\}$ is a neighborhood base for K. Let $\epsilon > 0$. Each component of the open set $V = \{G < \varepsilon\}$ is a bounded domain, hence it admits a hyperbolic (e.g., Kobayashi) metric κ_ε(see [13] for background on complex hyperbolic geometry). We will show that f strictly expands this metric. Note that by the maximum principle and functional equation for the Green function G we have

$$\partial\{G \leq \varepsilon\} = \partial\{G < \varepsilon\} = \{G = \varepsilon\}$$

and

$$f^{-1}(\{G = \varepsilon\}) = \{G = \varepsilon/d\} = \partial\{G \leq \varepsilon/d\}.$$

Therefore, $\{G < \varepsilon/d\}$ lies strictly inside $\{G < \varepsilon\}$. Since $C \cap K = \emptyset$, we can choose $\varepsilon > 0$ so that $f : \{G < \varepsilon/d\} \mapsto \{G < \varepsilon\}$ is a covering map (recall that f is proper, being a regular polynomial endomorphism). Hence $\kappa_\varepsilon(f(z), Df(z)(v)) = \kappa_{\varepsilon/d}(z, v)$ for $z \in \{G < \varepsilon\}$. Since $\{G < \varepsilon/d\}$ lies within a positive Euclidean distance from the boundary of $\{G < \varepsilon\}$, there is a constant $\lambda > 1$ such that $\kappa_{\varepsilon/d}(z, v) \geq \lambda\kappa_\varepsilon(z, v)$.

To prove (iii), recall that by [3] repelling periodic points of f are dense in J (which is here equal to K; see [8] for the proof), so U can be taken to satisfy $U \subset f(U)$. Take a smooth test function $0 \leq \varphi \leq 1$, which is positive on U, 0 outside U and has $\int \varphi d\mu > 0$. Note that for all $n \geq 0$, $0 \leq A^n\varphi \leq 1$, where $(A\varphi)(x) = \frac{1}{d^k}\sum_{f(y)=x}\varphi(y)$. By a theorem of Ueda [20], $\{A^n\varphi\}$ has a subsequence that converges uniformly on compact subsets of some basin V (which can be chosen so that $C \cap V = \emptyset$) to a continuous function h. Observe that $h(z) = 0$ for $z \notin \bigcup_{n=0}^\infty f^n(U)$. Suppose there is a $z_0 \in J \setminus \bigcup_{n=0}^\infty f^n(U)$. Then $W = \{z : h(z) < 1/2\}$ is an open neighborhood of z_0. Furthermore,

$$\int (\chi_W \circ f^n)\varphi \, \mathrm{d}\mu = \int (\chi_W \circ f^n)\varphi \, \mathrm{d}\frac{1}{d^{nk}}f^{n*}\mu = \int \chi_W A^n\varphi \, \mathrm{d}\mu$$

Since the measure μ is f-mixing (see [6]), the integrals in the above equation converge to $\mu(W)\int\varphi d\mu$ as $n \to \infty$. On the other hand, by Lebesgue dominated convergence, the limit over some subsequence of n's is equal to $\int_W h d\mu$, which does not exceed $(1/2)\mu(W)$. Hence $\mu(W) = 0$. But W is a neighborhood of $z_0 \in$ supp μ, which gives a contradiction.
□

From now on we consider only $k = 2$. The holomorphic extension of f to $\mathbb{P}^2$ acts as a rational map on the complex line Π at infinity. Let J_Π denote the Julia set of the restricted map $f|_\Pi$. We have the following:

Proposition 3. *If $f : \mathbb{P}^2 \mapsto \mathbb{P}^2$ is a holomorphic map arising from a regular polynomial endomorphism of $\mathbb{C}^2$ such that $K \cap C = \emptyset$ and $f|_\Pi$ is uniformly expanding on J_Π, then f satisfies Axiom A.*

Proof. Let Ω_f be the nonwandering set for f. Note that by condition (ii) of theorem 2, K is the intersection of Ω_f with the affine space $\mathbb{C}^2$. Under the assumptions of the present proposition, K equals J, hence $\Omega_f = S_0 \cup J_\Pi \cup J$, where S_0 is the set of attracting periodic points of $f|_\Pi$ in Π (see [16], problem 19-a, for the characterization of the nonwandering set for a rational map on $\mathbb{P}^1$). This describes the decomposition of Ω_f into subsets with unstable index 0, 1 and 2 respectively, so f is hyperbolic on Ω_f. Periodic points of f are dense in Ω_f by [3]. □

For a regular polynomial endomorphism f of $\mathbb{C}^2$ Bedford and Jonsson introduced the set of external rays $\mathcal{E}$ and the endpoint map $e : \mathcal{E} \mapsto \partial K$, in analogy to one-dimensional theory developed in [5](cf. [1], theorem 7.3 and the preceding discussion). They also studied conditions under which e is a continuous surjection.(This property is useful in investigating the topology of J.) They proved that e maps $\mathcal{E}$ Hölder continuously onto J if f is Axiom A, $f^{-1}(S_2) = S_2$ (where S_2 is the subset of Ω_f with unstable index 2) and $W^s(J_\Pi) \cap C = \emptyset$, where

$$W^s(J_\Pi) = \{z \in \mathbb{P}^2 : \mathrm{dist}(f^n(z), J_\Pi) \to 0 \text{ as } n \to \infty\}.$$

Below we prove an analogous result for the maps satisfying the assumptions of Proposition 3. We do this in two steps. First we show that the maps satisfying the assumptions of Proposition 3 do not satisfy $W^s(J_\Pi) \cap C = \emptyset$ (unlike the maps studied in [1]). Then we show that all complete external rays for such maps have uniquely defined endpoints.

Proposition 4. *Let f be as in Proposition 3. Then $W^s(J_\Pi) \cap C \neq \emptyset$.*

Proof. If $f\ |_\Pi$ is uniformly expanding on J_Π, then by [1], corollary 5.2, $W^s(J_\Pi) =$ supp $T \cap \mathbb{P}^2 \setminus K$. The closure $\overline{C}$ of C in $\mathbb{P}^2$ is an algebraic set (cf. [15], theorem VII.6.3), so the set reg $\overline{C}$ of nonsingular points of $\overline{C}$ supports a positive closed $(1,1)$-current (the current of integration). By [7], theorem 4.4, supp $T \cap \overline{C} \neq \emptyset$. If $K \cap C = \emptyset$, then $\overline{C} \subset \mathbb{P}^2 \setminus K$, so $W^s(J_\Pi) \cap \overline{C} =$ supp $T \cap \overline{C} \neq \emptyset$. It remains to show this intersection is not a subset of Π. But if $f\ |_\Pi$ is expanding on J_Π, then $W^s(J_\Pi) \cap \Pi = J_\Pi$. On the other hand, $\overline{C} \cap \Pi$ consists of critical points of $f\ |_\Pi$, which cannot belong to J_Π if $f\ |_\Pi$ is expanding there. Hence $W^s(J_\Pi) \cap C \neq \emptyset$. □

Before formulating the next theorem, let us briefly recall the construction of external rays for a regular polynomial endomorphism f. A detailed exposition can be found in [1], an overview in [2]. (For the theory and applications in dimension one, see [16] and the references given there.) One starts with constructing Riemann surfaces W_a, $a \in J_\Pi$, by pasting together connected components of successive preimages by f of local stable manifolds of points in J_Π. Corollary 6.8 in [1] states that each W_a is a simply connected Riemann surface and the topology of W_a as a manifold coincides with its topology as a subspace of $\mathbb{P}^2 \setminus K$. The restriction of G to W_a is a harmonic function with a logarithmic pole at a. One defines external rays for f in W_a as gradient lines of $G \mid W_a$ (i.e., trajectories of the vector field grad $G \mid W_a$). The set $\mathcal{E}$ is the union of all external rays in all W_a, $a \in J_\Pi$. The endpoint map e is defined as follows: for $\gamma \in \mathcal{E}$ and $r > 0$, $e_r(\gamma) := \gamma \cap \{G = r\}$ and $e(\gamma) = \lim_{r \to 0^+} e_r(\gamma)$. For each a, the set of the external rays in W_a for which e_r cannot be defined is finite. Moreover, by theorem 7.4 in [1], if f is expanding on J_Π, the set of points contained in incomplete gradient lines is closed and nowhere dense in $W^s(J_\Pi)$. Let us consider the set $\mathcal{E}'$ of external rays γ such that $e_r(\gamma)$ exists for all $r > 0$. Then the following holds:

Theorem 5. *If f is as in Proposition 3 and $\gamma \in \mathcal{E}'$, then $e(\gamma)$ exists.*

Proof. Recall the following result due to Ueda [20] (reformulated in the present version in [18], Remarques 3.2.3(ii)): let f be a holomorphic map of degree $d \geq 2$ on $\mathbb{CP}^k$ and let a belong to a hyperbolic set for f. Let $W^s(a) = \{z : \text{ dist } (f^n(z), f^n(a)) \to 0 \text{ as } n \to \infty\}$. Then $\tilde{G} \mid_{\pi^{-1}(W^s(a))}$ is pluriharmonic, where $\tilde{G}(z,t) = \lim_{n\to\infty} d^{-n} \log |\tilde{f}^n(z,t)|$, $\tilde{f}(z,t) = (t^d f(z/t), t^d)$ and $\pi(z,t) = [z : t]$.

In our setting, $a \in J_\Pi$, which by assumption is hyperbolic (of stable index 1), so $W^s(a) \setminus \{a\} = W^s(a) \cap \mathbb{C}^2$, and $W^s{}_{loc}(a)$ is a 1-dimensional complex manifold. Also, $\tilde{G}(z,1) = G(z), \quad z \in \mathbb{C}^2$.

Note that $\tilde{G}$, being harmonic, is a real analytic function on $W_a \subset W^s(a)$. $W^s_{loc}(a), \quad a \in J_\Pi$ are plaques of a Riemann surface lamination with a metric g introducing its complex structure (see [4] for background on laminations) By a result of [14], there is a neighborhood of a in $W^s(a)$ with respect to the metric g (which is equivalent to the Fubini- Study metric on $\mathbb{CP}^2$ near J_Π), in which each trajectory of the gradient field of $\tilde{G}$ has finite length. In fact, the lengths of external rays for f will be uniformly bounded in a neighborhood U of J_Π, since the local stable manifolds depend continuously on a and J_Π is a compact set. Extending the complete external rays we see that there is a uniform upper bound on Euclidean lenghts of pieces of all rays in each set $\{\rho' \leq G \leq \rho\}, \quad 0 < \rho' < \rho$: Observe that a piece of an external ray not contained in U is a smooth image of a compact interval in $\mathbb{R}$; the Fubini- Study lengths of external rays above the level $\{G = \rho'\}$ are uniformly bounded, and away from Π the Fubini- Study metric is equivalent to the Euclidean one. We need to choose suitable $\rho, \rho' > 0$, since the bound on lenghts depends on these numbers. Take $\rho > 0$ so that in the neighborhood $V = \{G < \rho\}$ of K we have $\|Df(z)(v)\| \geq \lambda\|v\|$ for all $z \in V, \quad v \in T_z\mathbb{C}^2$ (we can adjust the expanding metric from Theorem 2 so that $c = 1$; this metric is equivalent to the Euclidean one in a neighborhood of K). Let $\rho' = \rho/d$ and let L be the upper bound on lengths of all pieces of external rays in $\{\rho' \leq G \leq \rho\}$. The lenghts of f- preimages of these rays are uniformly bounded above by L/λ. Since $G(f(z)) = d \cdot G(z)$, we can represent each complete external ray in V as a union of preimages of other rays by successive iterates of f. Summing the lenghts of these preimages gives a uniform upper bound $L/(\lambda - 1)$ on the length of each piece of an external ray in V. Hence each complete external ray has exactly one limit point on K. (The part of the argument involving infinite summation of geometrically decreasing lengths of rays was used in [5] to prove landing of external rays for a polynomial in $\mathbb{C}$ that is subhyperbolic on K.) □

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