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Bayoumi Quasi-differential is different from Fréchet-differential

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Abstract: We prove that the Quasi Differential of Bayoumi of maps between locally bounded F-spaces may not be Fréchet-Differential and vice versa. So a new concept has been discovered with rich applications (see [1–6]). Our F-spaces here are not necessarily locally convex

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1 Introduction

In this paper we study the relationship between the concepts of Quasi-Differentiability (Q-differentiability) and Fréchet-Differentiability (F-Differentiability) of maps between locally bounded F-spaces E and F. By an F-space we mean a space metrized by an F-norm. Their classes will be denoted by $QD(E,F)$ and $FD(E,F)$ respectively.

Let E and F be p-normed and q-normed spaces respectively ($0 < p, q \leq 1$), and U a nonempty open subset of E. For $f, g\colon U \to F$, and $a \in U$, we say that f and g are *quasi tangent* (*or pq-tangent*) to each other at a if

$$\lim_{x\to a} \|f(x) - g(x)\|^{1/q} / \|x - a\|^{1/p} = 0. \qquad (1)$$

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In 1995 we gave the following definition depending on (1): Let E and F be p-normed and q-normed spaces, respectively ($0 < p, q \leq 1$) and U a nonempty open set in E. A mapping $f\colon U \to F$ is said to be *quasi-differentiable* (*Bayoumi differential* or *pq-differentiable*) at $a \in U$, if there exists a continuous linear map $T_a \in L(E, F)$ such that f and the continuous affine linear mapping $E \ni x \mapsto f(a) + T_a(x - a)$ are pq-tangent at a, that is,

$$\lim_{x \to a} \|f(x) - f(a) - T_a(x - a)\|^p / \|x - a\|^q = 0. \tag{2}$$

T_a is called the *quasi-differential* of f at a (or pq-differential) and is denoted by $Df(a)$.

It is to be noted that if E and F are both p-normed spaces or quasi-normed spaces with the same quasi-norm constants, then condition (2) turns out to be as the following classical one:

$$\lim_{x \to a} \|f(x) - f(a) - T_a(x - a)\| / \|x - a\| = 0.$$

We have seen in [2] that quasi-differentiability for a mapping between locally bounded F-spaces is a continuous linear operator. This is similar to that for the Fréchet ones for normed spaces.

However, our goal here is to prove that:

The two concepts of differentiability are totally different.

But the new one, that is the quasi-differentiability, is more suitable for all F-spaces which are locally convex or not. It may play roles in analysis and applied mathematics which cannot be done by the classical Fréchet differentiability.

It is worthwhile to point out that this new concept of differentiability has been discovered while the author was working on finding theorems for the mean value in real and complex locally bounded F-spaces; see [2, 3].

2 Finite-dimensional case

The following lemma shows the equivalence between the quasi-differentiability (Q-differentiability) and Fréchet-differentiability (F-differentiability) in finite-dimensional spaces.

Lemma 2.1. *Let E and F be any two finite-dimensional p-normed spaces ($1 > p > 0$) and let f be a map between them. Then f is Q-differentiable if and only if f is F-differentiable.*

Proof. Note that

$$\lim_{x \to a} \frac{\|f(x) - f(a) - T_a(x - a)\|^{1/p}}{\|x - a\|^{1/p}} = 0 \tag{3}$$

if and only if

$$\lim_{x \to a} \frac{\|f(x) - f(a) - T_a(x - a)\|}{\|x - a\|} = 0$$

for a continuous linear mapping $T_a \in L(E, F)$, where $a, x \in E$. We know that finite-dimensional topologies are equivalent. □

The following theorem shows that we still have the linear transformation, defined by the Jacobian matrix, to find quasi-differentials when we deal with finite dimensional spaces.

Theorem 2.2. *Let E and F be any two finite-dimensional p-normed and q-normed spaces, whose dimensions are n and m respectively $(1 > p, q > 0)$. Let $U \subset E$ be open and $f\colon U \to \mathbf{R}^m$ a function defined by*

$$f(a) = (f_1(a), \dots, f_m(a)),$$

where $f_i\colon U \to \mathbf{R}$ $(1 \le i \le m)$. Then if f is quasi-differentiable at $a \in U$, each of the partial derivatives $\frac{\partial f_i}{\partial x_j}(a)$ exists, $(1 \le i \le m, 1 \le j \le n)$. Furthermore, the quasi-differential $Df(a)\colon \mathbf{R}^n \to \mathbf{R}^m$ is the linear transformation defined by the Jacobian matrix of f at a.

Proof. Let $L = Df(a)\colon \mathbf{R}^n \to \mathbf{R}^m$. Let $e_1, \dots, e_n$ and $e'_1, \dots, e'_m$ be the standard basis for $\mathbf{R}^n$ and $\mathbf{R}^m$. Assume (a_{ij}) is the Jacobian matrix of f at a, so that

$$Le_j = \sum_{i=1}^{m} a_{ij} e'_i, \qquad 1 \le j \le n$$

Since f is quasi-differentiable at a, it follows that

$$\lim_{x \to a} \frac{\|f(a + he_j) - f(a) - L(he_j)\|^{1/q}}{\|he_j\|^{1/p}} = 0, \qquad 1 \le j \le n$$

Now

$$\lim_{x \to a} \frac{\|f(a + he_j) - f(a) - hL(e_j)\|^{1/q}}{|h|} = \lim_{x \to a} \left[\sum_{i=1}^{n} \left| \frac{f_i(a + he_j) - f_i(a)}{h} - a_{ij} \right|^q \right]^{1/q} = 0;$$

hence each term in the sum must tend to zero. Therefore

$$\frac{\partial f_i(a)}{\partial x_j} = \lim_{x \to a} \frac{f_i(a + he_j) - f_i(a)}{h} = a_{ij}, \qquad 1 \le i \le m, 1 \le j \le n,$$

This proves that $\frac{\partial f_i(a)}{\partial x_j}$ exists and equals a_{ij}, completing the proof of the theorem. □

The following gives an example of a non quasi-differentiable function.

Example 2.3. Let $E = l^p$, $1 > p > 0$. The function $f(x)\colon l^p \to \mathbf{R}$ defined by

$$f(x) = |x_1| \tag{4}$$

is not quasi-differentiable at 0. Notice that

$$\lim_{x \to 0} \frac{[|\|x\| - \|0\| - T_0(x)|^q]^{1/q}}{\|x - 0\|^{1/p}} = \lim_{x \to 0} \frac{||x_1| - \alpha x_1|}{(\sum |x_i|^p)^{1/p}}.$$

This shows that the limit does not exist for any constant α since it depends on the way that x_1 approaches 0

That is, f is not quasi-differentiable at 0; i.e., $f \notin QD(l_p; \mathbf{R})$.

Remark 2.4. For $E = l_p^1 = l_p \cap \mathbf{R}$, we can also take

$$f(x) = |x_1|^p : l_p^1 \to \mathbf{R}$$

to have

$$\lim_{x \to 0} \frac{||x_1|^p - \alpha x_1|}{|x_1|} = \lim_{x_1 \to 0} \left| \frac{|x_1|^p}{x_1} - \alpha \right|$$

which does not also exist if p equals, say, 1/2 .

2.1 Maxima and minima at interior points

We shall be concerned here with maxima and minima at points of an open set in a finite-dimensional p-normed space E $(0 < p < 1)$. We show that we still have the quadratic forms defined by the Hessian matrix of the second order quasi-differentials of the given function

$$f : U \to \mathbf{R}$$

to find out local maximum and local minimum of f.

As in the classical case, we say that a point $v_0 \in U$ is a *local maximum* for f if

$$f(v) \leq f(v_0) \text{ for all } v \text{ in some ball about } v_0$$

Similarly, v_1 is a *local minimum* for f if

$$f(v) \geq f(v_1) \text{ for all } x \text{ in some ball about } v_1.$$

Just as in the classical case, we say that the interior point v_0 is a *critical point* of f if the quasi-differential,

$$f'(v_0) = D_{v_0} f = 0,$$

the zero transformation. One reason for this definition is the following theorem.

Theorem 2.5. *If v_0 is an interior local maximum or local minimum point of*

$$f : U \to \mathbf{R}$$

and f is quasi differentiable near v_0, then v_0 is a critical point of f.

Proof. This is similar to the classical proof for differentiability of functions. □

Remark 2.6. According to Theorem 2.5, it suffices to find the maximum and minimum of some functions f. We know that for some purposes U may be a compact set, and $f(v) = 0$ if v is a boundary point for U and that $f > 0$ somewhere in U.

2.2 Criteria for local maxima and minima

Let U be an open subset of $\mathbf{R}^n$, let $f\colon U \to \mathbf{R}$ be a function with a continuous second order quasi-derivative, and let v_0 be a point of U which is a critical point of f. We would like to have a method of determining whether v_0 is a local maximum, or a local minimum or neither. We shall look at the second quasi-derivatives.

We use the Hessian of f at v, $H(f)(v)$, which is a matrix formed from the second partial derivatives. If the partial derivatives are continuous as we are assuming, then $H(f)(v)$ is a symmetric matrix.

By the arc segment $A_{v_0}^v$ from v_0 to v we mean the set

$$A_{v_0}^v = \left\{ w \in U; w = (1-\lambda)^{1/p} v_0 + \lambda^{1/p} v,\ 0 \le \lambda \le 1 \right\}, \qquad 0 < p < 1.$$

If $p = 1$, $A_{v_0}^v$ becomes the line segment from v_0 to v. It is proved in [3] that the unit ball B_E of a locally bounded space E contains $A_{v_0}^v$ if v_0 and v are in B_E.

Theorem 2.7. *Let U be an open set in $\mathbf{R}^n$ and*

$$f\colon U \to \mathbf{R}$$

a function having continuous second order partial derivatives. Let v_0 be a critical point of f and let $H(f)(v_0)$ be the Hessian of f at v_0.

(i) If $H(f)(v_0)$ is positive definite, then v_0 is a local minimum.

(ii) If $H(f)(v_0)$ is negative definite, then v_0 is a local maximum.

(iii) If $H(f)(v_0)$ is indefinite, then v_0 is neither a local maximum nor a local minimum.

Proof. The proof is like that for functions on $\mathbf{R}$; see [8]. □

3 Infinite-dimensional case

The following example is for a non Fréchet and non quasi-differentiable function.

Example 3.1. Put $E = l_p$, $F = \mathbf{R}$, $a = 0$. We are going to establish a function $f\colon E \to F$ which is neither Fréchet differentiable nor quasi-differentiable at $a = 0$, i.e.,

$$f \notin QD \cup FD.$$

That is,

$$\lim_{x \to a} \frac{\|f(x) - f(a) - T_a(x-a)\|^{1/q}}{\|x-a\|^{1/p}} \neq 0 \tag{5}$$

even when $p = q = 1$. Consider the following function

$$f(x) = \sum_{1}^{\infty} \left(\frac{1}{n}\right)^{1-1/p} \cdot x_n, x = (x_n) \in l_p,$$

and the following set which approaches zero in l_1

$$A_1 = \left\{x^{(n)}; x^{(n)} = (0, \dots, 0, n^{-1/p}, 0, \dots), n \in \mathbf{N}\right\}.$$

Now

$$\begin{aligned}\lim_{x\to 0} \frac{\|f(x) - T_0(x)\|}{\|x\|} &= \lim_{x^n\to 0} \frac{\|\sum_1^\infty (\frac{1}{n})^{(1-1/p)}.x_n - \sum_1^\infty \alpha_n x_n\|}{\sum |x_i|} \\ &= \lim_{n\to\infty} \frac{\|(\frac{1}{n})^{(1-1/p)} \cdot n^{-1/p} - \sum \alpha_n x_n\|}{(\frac{1}{n})^{1/p}} \\ &= \lim_{n\to\infty} \frac{|\frac{1}{n} - \alpha_n (\frac{1}{n})^{1/p}|}{(\frac{1}{n})^{1/p}} = \lim_{n\to\infty} \left|\alpha_n - n^{-1+1/p}\right| = \infty\end{aligned}$$

which never goes to zero. That is, $f \notin FD$. Notice that $x^{(n)} \to 0$ in l_1.

In the same way we can show that $f \notin QD$. In fact, let f be given above as a function on l_p.

$$\begin{aligned}\lim_{x^n\to 0} \frac{\|f(x^n) - T_0(x^n)\|^{1/q}}{\|x^n\|^{1/p}} &= \lim_{x_n\to 0} \frac{\|\sum_1^\infty (\frac{1}{n})^{1-1/p}.x_n - \sum \beta_n x_n\|^{1/q}}{(\sum |x_i|^p)^{1/p}} \\ &= \lim_{n\to\infty} \frac{|\frac{1}{n} - \beta_n (\frac{1}{n})^{1/p}|}{(\frac{1}{n})^{1/p}} = \lim_{n\to\infty} |n^{1/p-1} - \beta_n| = \infty;\end{aligned}$$

that is, $f \notin QD$, *where* $T_0(x) = \sum \beta_n x_n$, $(\beta_n) \in l_\infty$, and $x_n \to 0$ in l_p.▲

The following example shows that the class of F-differentiable maps may not be a subset of the class of quasi-differentiable maps.

Example 3.2. There is a Fréchet-differentiable map $f\colon E \to F$ from a p-normed space E into a q-normed space F $(1 > p, q > 0)$ which is not quasi-differentiable. That is

$$FD(E;F) \nsubseteq QD(E;F). \tag{6}$$

More precisely, the function $G\colon L^{1/2}(I) \to \mathbf{R}$ given by

$$G(f) = \|f\| = \int_0^1 |f|^{1/2} dx$$

has this property.

Proof. Take first $E = L^p(I)$, the space of integrable functions on $I = [0, 1]$, and $F = \mathbf{R}$, the real field with the q-norm, $f = 0$, the zero function in $L^p(I)$. Consider the function $G\colon L^p(I) \to \mathbf{R}$ given by

$$G(f) = \|f\| = \int_0^1 |f|^p dx \tag{7}$$

Then

$$\lim_{f\to 0}\frac{\|G(f)-G(0)-0)\|^{1/q}}{\|f-0\|^{1/p}}=\lim_{f\to 0}\frac{\|f\|^{1/q}}{\|f\|^{1/p}}=\lim_{f\to 0}\frac{1}{\|f\|^{1/p-1/q}},$$

So the limit does not exist when $\frac{1}{p}>\frac{1}{q}$, that is, for all values of $(1>q>p>0)$, say for $p=1/2$. Notice that the dual space of $L^p(I)$ equals $\{0\}$. Therefore, the function G is not quasi-differentiable at $f=0$.

However, we notice that

$$L^1(0,1)\subset L^{1l2}(0,1), \tag{8}$$

For if $f\in L^1(0,1)$, then

$$\int_0^1 |f^{1/2}.1|dx\le\left(\int_0^1|f|dx\right)^{1/2}\cdot\left(\int_0^1 1dx\right)^{1/2}$$

that is, $f\in L^{1/2}(0,1)$. Therefore we have considered our function f in $L^1(0,1)\cap L^{1/2}(0,1)$.

We claim that $G(f)=\|f\|$ is a Fréchet differentiable function at 0. Notice that $T_0(x)\in(L^1)'$, the dual space which is isomorphic to L_∞. Now

$$\begin{aligned}\lim_{f\to o}\frac{\|G(f)-G(0)-T_0(f-0)\|}{\|f-0\|_{L_1}}&=\lim_{f\to 0}\frac{|G(f)-T_0(f)|}{\|f\|_{L_1}}\\&=\lim_{f\to 0}\frac{|\|f\|-T_0(f)|}{\|f\|_{L_1}}=0\end{aligned}$$

as $f\to 0$ in L^1, where $T_0(f)\in(L^1)'\simeq L^\infty$ may be taken equal to $\|f\|=\int_0^1|f|^{1/2}dx$, since f is considered as an element in L^1. This completes the proof of the example. □

In what follows we prove that the class of Q-differentiable maps may not be contained in the class of F-differentiable maps.

Theorem 3.3. *Not every quasi-differentiable map f from a p-normed space E into a q-normed space F is Fréchet-differentiable, that is*

$$QD(E;F)\nsubseteqq FD(E;F). \tag{9}$$

Proof. As in the above example, we take $E=L^p(I)$, the space of integrable functions on $I=[0,1]$, $F=\mathbf{R}$, with the q-norm, and $f=0$, the zero function in $L^p(I)$. Consider the function

$$G\colon L^p(I)\to\mathbf{R}$$

given by

$$G(f)=\|f\|=\int_0^1|f|^p dx. \tag{10}$$

We have seen that

$$\begin{aligned}\lim_{x\to a}\frac{\|G(f)-G(0)-0)\|^{1lq}}{\|f-0\|^{1/p}}&=\lim_{f\to 0}\frac{\|f\|^{1/q}}{\|f\|^{1/p}}\\&=\lim_{f\to 0}\frac{1}{\|f\|^{1/p-1/q}}.\end{aligned}$$

Now if $\frac{1}{q} - \frac{1}{p} > 0$, that is, when $q < p$, the limit will approach to zero and $G(f)$ will be a quasi-differentiable function. But if $q = p$ (say, $p = q = 1/2$) we have non-zero limit since

$$\lim_{f \to 0} \frac{\|f\|^{1/p}}{\|f\|^{1/p}} = 1. \tag{11}$$

So $G(f)$ will never be a Fréchet-differentiable function. In addition, we note that for $q > p$, f is not quasi-differentiable. □

Remark 3.4. The relation between the classes of Q-differentials and F-differentials may be summarized so far by the following diagram. (We point out here that the term *quasi-differentiable* has been used in a quite a different way by some mathematicians, see Dem'yanov/Vasil'ev, Nondifferentiable Optimization, Optimization Software, Inc., 1985. Therefore the term "pq-differentiable" or "super-differentiable" or "Bayoumi differentiable" may also be used here instead of; see the Mathematical Reviews 2004).

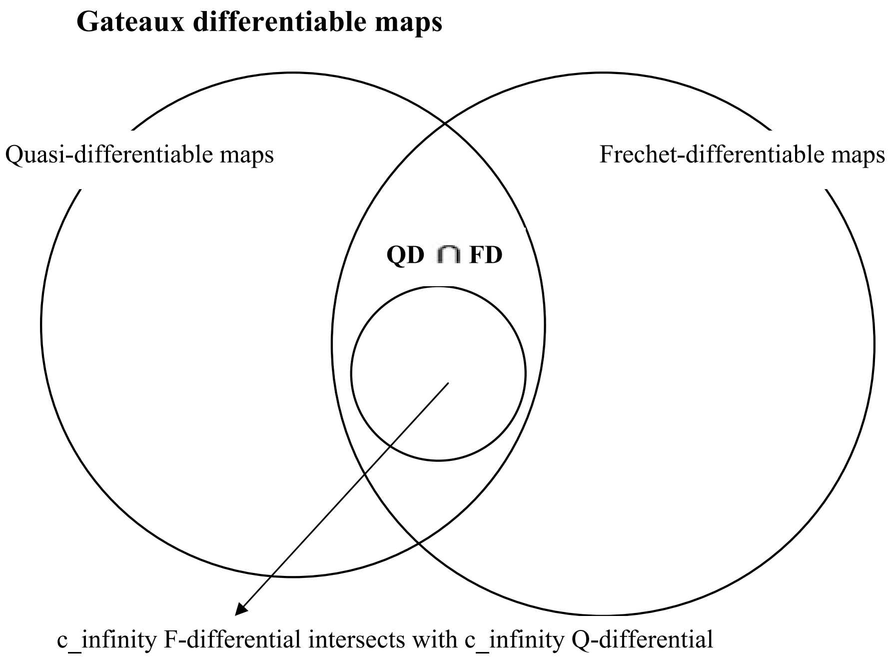

Fig. 1 Relation between the different classes of differentibility's.

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