VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-014-0247-1
Math. Slovaca **64** (2014), No. 4, 893–912

THE BOUNDARY VALUE PROBLEMS OF QUADRATIC MIXED TYPE OF DELAY DIFFERENTIAL EQUATIONS WITH EIGENVALUES

Zhimin He* — Jianhua Shen**

(*Communicated by Michal Fečkan*)

ABSTRACT. In this paper, by using a fixed-point theorem in cones to study the boundary value problem for a class of quadratic mixed type of delay differential equations with eigenvalue, the sufficient condition of existence of their solutions is derived. The main results in this paper are the generalization and improvement of those existing ones.

1. Introduction

In the past years, the research work upon delay differential equations becomes more and more popular [3,5–7,11–13,15,17–19], because this kind of differential equations also have broad applied backgrounds in the fields of physics, biology and control principle [1,4]. The relative differential equations include two kinds, ie, differential equations with eigenvalues (see [3,7,14,18]) or without eigenvalues [13,13,15]. The main research schemes to the equations are to apply all kinds of fixed-point theorems (see [13,16,19]), a fixed-point index theorem in cones [4,15] and the method of upper and lower solution [16]. The fixed-point theorem in cone have become a main tool for studying differential equations with eigenvalues since Henderson and Wang [12] firstly applied it to discuss the existence of positive solutions for the second-order non-delay differential with eigenvalues in

2010 Mathematics Subject Classification: Primary 34B15, 34B13; Secondary 34B13.
Keywords: boundary value problem, mixed type of delay differential equations, eigenvalue, positive solution, fixed point.
This work is supported by the NSFC of China (11171085).

1997. Recently, Bai and Xu [7] studied the following second-order differential equations with eigenvalues by using Guo-Krasnoselskii fixed point theorem

$$\begin{cases} -u''(t)+\lambda g(t,u(t-\tau))=0, & t\in[0,1] \\ u(t)=0, & -\tau\le t\le 0, \\ u(1)=0, \end{cases} \tag{1.1}$$

where

(1) $0<\tau<\frac{1}{2}$,
(2) $g(t,u)=a(t)f(t,u)$, $a(t)\colon(0,1)\to[0,+\infty)$, $f\colon[0,1]\times[0,+\infty)\to[0,+\infty)$ are continuous functions.

Wang and Shen [18] studied the existence of solutions for a class of more general delay second order differential equations with eigenvalue, which extended the domain of delay τ to $(0,1)$ and whose boundary value was more general.

The boundary value problems for integro-differential equations of mixed type were studied by some authors (see [8, 9]). In this paper, we also consider the existence of the positive solutions for following quadratic mixed type of delay differential equations with eigenvalue

$$\begin{cases} -u''(t)=\lambda p(t)f[t,u(t-\tau),\int_0^t k(t,s)u(s)ds], & t\in(0,1) \\ u(t)=0, & -\tau\le t\le 0 \\ u(1)=au(\eta) \end{cases} \tag{1.2}$$

Let $v(t)=\int_0^s k(t,s)u(s)ds$, we suppose that $f(t,u,v)$ is neither superlinear or sublinear, $p(t)$ is allowed that has some suitable singularity at the ends of $(0,1)$.

It should be also mentioned that similar (same) problems have been studied in [8, 9, 14]. In fact, when $v(t)\equiv 0$ the problem (1.2) was studied in [7, 14, 18] and when $\tau=0$, the problem (1.2) was studied in [8]. Thus, the problem (1.2) which we will study is more general.

2. Some preliminaries

For convenience of the reader, in the section, we present some definitions and some lemmas. At first, we give the necessary conditions which the differential equation (1.2).

(H_1) $f\in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $J=(0,1)$, $0<a\le 1$, $0<\eta<1, 0<\tau<1$.
(H_2) $p(s)\in C(J_1,\mathbb{R}^+)$, $J_1=[0,1]$.

(H_3) $D = \{(t,s) \in J_1 \times J_1 : t > s\}, \quad k(t,s) \in C(D, \mathbb{R}^+)$,
$k_0 = \min\{k(s,t) : (s,t) \in D\}, \quad k_1 = \max\{k(s,t) : (s,t) \in D\}$.

(H_4) $u \in C^2[-\tau, 1], \quad u(t) \geq 0, \quad t \in [-\tau, 1]$

and we assume that

(1) $L_1 = \dfrac{\gamma\gamma_1}{2} \displaystyle\int_{\gamma+\tau}^{1-\gamma} G(s,s)p(s)(1+k_0 s)\,\mathrm{d}s, \quad 0 < \gamma < \frac{1-\tau}{2} \leq 1-\gamma \leq 1-\tau,$

where $G(s,t)$ is defined as equation (2.1).

(2) $L_2 = \dfrac{1+a-a\eta}{1-a\eta} \displaystyle\int_0^1 G(s,s)p(s)(1+k_1 s)\,\mathrm{d}s.$

(3) $\gamma_1 = \dfrac{\gamma(1-a\eta)}{1+a-a\eta} \leq \gamma.$

In this paper, we need the fixed point theorem in cones to prove our conclusion, so we give now relative the definitions of cones and the definitions of positive solution of BVP (1.2).

Definition 2.1. Let X be a real Banach space and $K \in X$ be a closed, convex set. K is a cone if only and if the following conditions are satisfied

(i) $\lambda u \in K$, if $\lambda > 0$ and $u \in K$.

(ii) If $u \in K$ and $-u \in K$ then $u = 0$.

Definition 2.2. $u(t)$ is the positive solution of BVP (1.2) if and only if it satisfies the following conditions:

(i) For an arbitrary $t \in [-\tau, 1]$, $u(t) \geq 0$ and $u(t)$ is continuous.

(ii) When $t \in [-\tau, 0]$, $u(t) = 0$ and $u(1) = au(\eta)$ $(0 < \eta < 1)$.

(iii) $u''(t) = -\lambda p(s) f(t, u(t-\tau), v(t))$, for all $t \in [0,1]$.

If $u(t)$ is the solution of BVP (1.2) then $u(t)$ can be represented as

$$u(t) = \begin{cases} 0, & \tau \leq t \leq 0, \\ \lambda \int_0^1 G(t,s)p(s)f(s,u,v)\,\mathrm{d}s & \\ + \frac{a\lambda t}{1-a\eta} \int_0^1 G(\eta,s)p(s)f(s,u,v)\,\mathrm{d}s, & 0 \leq t \leq 1 \end{cases}$$

where

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ (1-t)s, & 0 \leq s \leq t \leq 1 \end{cases} \tag{2.1}$$

and $G(t,s)$ satisfies:

(i) $G(t,s) \le G(s,s)$, $\int\limits_0^1 G(s,s)\,\mathrm{d}s = \dfrac{1}{6}$.

(ii) Let $J_\gamma = [\gamma, 1-\gamma]$ then for any $t \in J_\gamma$, $s \in [0,1]$ we have $G(t,s) \ge \min\{t, 1-t\} G(s,s) \ge \gamma G(s,s)$.

We define the operator $\Phi\colon C[-\tau,1] \to C[-\tau,1]$ by

$$\Phi u(t) = \begin{cases} 0, & -\tau \le t \le 0, \\ \lambda \int\limits_0^1 G(t,s)p(s)f(s,u,v)\,\mathrm{d}s \\ +\frac{a\lambda t}{1-a\eta} \int\limits_0^1 G(\eta,s)p(s)f(s,u,v)\,\mathrm{d}s, & 0 \le t \le 1. \end{cases}$$

We also denote the space X as

$$X = \{u \in C[-\tau,1] : \ u(t) = 0 \ \text{ when } \ t \in [-\tau,0], \quad u(1) = au(\eta)\}$$

for any $u \in X$,the norm $\|u\| = \sup\{u(t) : -\tau \le t \le 1\}$ then X is a Banach space and for all $u \in X$, we have $\|u\| = \|u\|_{[0,1]}$.

Additionally, we define a cone K in the space X where

$$K = \{u \in X : u(t) \ge 0, \quad t \in [-\tau,0], \ \min_{\gamma \le t \le 1-\gamma} u(t) \ge \gamma_1 \|u\|\}.$$

Based on the above, we derive the following lemmas.

Lemma 2.1. *The fixed-point of the map Φ is the solution of equation* (1.2).

Proof. From

$$\begin{aligned} \Phi u(t) &= \lambda \int\limits_0^1 G(t,s)p(t)f(s,u,v)\,\mathrm{d}s + \frac{a\lambda t}{1-a\eta} \int\limits_0^1 G(\eta,s)p(t)f(s,u,v)\,\mathrm{d}s \\ &= \lambda \int\limits_0^t s(1-t)p(t)f(s,u,v)\,\mathrm{d}s + \lambda \int\limits_t^1 t(1-s)p(t)f(s,u,v)\,\mathrm{d}s \\ &\quad + \frac{a\lambda t}{1-a\eta} \int\limits_0^1 G(\eta,s)p(t)f(s,u,v)\,\mathrm{d}s. \end{aligned}$$

It is get easy to get

$$\Phi'' u(t) = -\lambda p(t) f(t, u, v).$$
$$\Phi u(t) = 0, \qquad -\tau \le t \le 0.$$
$$\Phi u(\eta) = \lambda \int_0^1 G(\eta, s) p(t) f(s, u, v)\,\mathrm{d}s + \frac{a\lambda\eta}{1 - a\eta} \int_0^1 G(\eta, s) p(t) f(s, u, v)\,\mathrm{d}s$$
$$= \frac{\lambda}{1 - a\eta} \int_0^1 G(\eta, s) f(s, u, v)\,\mathrm{d}s.$$

Then $\Phi u(1) = a\Phi u(\eta)$. Thus, the fixed point of Φ is the solution of the equation (1.2). The proof is complete. □

Lemma 2.2. *$\Phi\colon K \to K$ is a completely continuous operator.*

Proof. Clearly, we have $\|\Phi u\| = \|\Phi u\|_{[0,1]}$, $\Phi u(t) \ge 0$ for all $u(t) \in K$, and

$$\|\Phi u\| = \|\Phi u\|_{[0,1]} \le \lambda \int_0^1 G(s, s) p(s) f(s, u, v)\,\mathrm{d}s$$
$$+ \frac{a\lambda}{1 - a\eta} \int_0^1 G(s, s) p(s) f(s, u, v)\,\mathrm{d}s$$
$$= \frac{\lambda(1 + a - a\eta)}{1 - a\eta} \int_0^1 G(s, s) p(s) f(s, u, v)\,\mathrm{d}s.$$
$$\Phi u(t) \ge \lambda \int_0^1 G(t, s) p(s) f(s, u, v)\,\mathrm{d}s \ge \lambda\gamma \int_0^1 G(s, s) p(s) f(t, u, v)\,\mathrm{d}s$$
$$= \frac{\gamma(1 - a\eta)}{1 + a - a\eta} \|\Phi u\|_{[0,1]} \ge \lambda_1 \|\Phi u\|.$$

Then $\Phi\colon K \to K$. Because Φ is a sequential compact set, we can conclude that Φ is a completely continuous operator by Arzela-Ascoli Theorem. □

Lemma 2.3 ([4]). *Let X be a Banach space, K a conic in X, Ω_1, Ω_2 two open subsets in X and $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. If $\Phi\colon K \cap (\Omega_2 \setminus \Omega_1) \to K$ is a completely continuous operator and satisfies*

(i) $\|\Phi u\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|\Phi u\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|\Phi u\| \le \|u\|$, $u \in K \cap \partial\Omega_2$ *and* $\|\Phi u\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$

then Ω has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let α, β, φ and ϕ be non-negative continuous concave functional on K. Then for positive real numbers b, c, d and m, we define the following convex sets:

$$P(\beta, m) = \{x \in K : \ \beta(x) < m\}.$$

$$P(\beta, \alpha, b, m) = \{x \in K : \ b \le \alpha(x), \ \beta(x) \le m\}.$$

$$P(\beta, \varphi, \alpha, b, d, m) = \{x \in K : \ b \le \alpha(x), \ \varphi(x) \le d, \ \beta(x) \le m\}.$$

and a closed set

$$R(\beta, \phi, c, m) = \{x \in K : c \le \phi(x), \ \beta(x) \le m\}.$$

Lemma 2.4 ([1]). *Let K be a cone in a real Banach space X. Let β and φ be non-negative continuous convex functionals on K, α be a non-negative continuous concave functional on K and ϕ be a non-negative continuous functional on K satisfying $\phi(\lambda u) \le \lambda \phi(u)$ for $0 \le \lambda \le 1$ such that for some positive numbers M and m,*

$$\alpha(u) \le \phi(u), \quad \|u\| \le M\beta(u), \qquad \text{for all} \quad u \in \overline{P(\beta, m)}.$$

Suppose $\Phi\colon \overline{P(\beta, m)} \longrightarrow \overline{P(\beta, m)}$ is completely continuous and there exist positive numbers b, c and m with $b > c$ such that

- (S_1) *$\{u \in P(\beta, \varphi, \alpha, b, d, m) | \alpha(u) > b\} \neq 0$, and $\alpha(\Phi u) > b$ for $u \in P(\beta, \varphi, \alpha, b, d, m)$.*
- (S_2) *$\alpha(\Phi(u)) > b$ for $P(\beta, \alpha, b, m)$ with $\varphi(\Phi(u)) > d$.*
- (S_3) *$0 \notin R(\beta, \phi, c, m)$ and $\phi(\Phi(u)) < c$ for $u \in R(\beta, \phi, b, m)$ with $\phi(u) = c$, then Φ has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\beta, m)}$ such that $\beta(u_i) \le m$, for $i = 1, 2, 3$; $b < \alpha(u_1)$; $c < \phi(u_2)$, with $\alpha(u_2) < b$; $\phi(u_3) < c$.*

3. The case of no less than one solution

Let

$$M_0 = \liminf_{\substack{u \to 0 \\ v \to 0}} \min_{t \in [-\tau, 1]} \frac{f(t, u, v)}{\sqrt{u^2 + v^2}}, \qquad M_\infty = \liminf_{\substack{u \to \infty \\ v \to \infty}} \min_{t \in [-\tau, 1]} \frac{f(t, u, v)}{\sqrt{u^2 + v^2}}.$$

$$M^0 = \limsup_{\substack{u \to 0 \\ v \to 0}} \max_{t \in [-\tau, 1]} \frac{f(t, u, v)}{\sqrt{u^2 + v^2}}, \qquad M^\infty = \limsup_{\substack{u \to \infty \\ v \to \infty}} \max_{t \in [-\tau, 1]} \frac{f(t, u, v)}{\sqrt{u^2 + v^2}}.$$

In the following, we discuss the existence of at least the positive solution for all kinds of values and compositions of M_0, M_∞, M^0 and M^∞. In the Theorem 3.1 and Theorem 3.2, we take $\varepsilon > 0$ such that satisfy $(M_0 - \varepsilon) > 0$, $(M_\infty - \varepsilon) > 0$.

Theorem 3.1. *If the conditions* (H_1)–(H_4) *and the following conditions*

$$0 < M_\infty < +\infty, \tag{3.1}$$

$$0 < M^0 < +\infty, \tag{3.2}$$

$$\frac{1}{L_1(M_\infty - \varepsilon)} \le \lambda \le \frac{1}{L_2(M^0 + \varepsilon)}, \tag{3.3}$$

hold then the equation (1.2) *has at least one solution.*

Proof. By (3.2), (3.3) for a given $\varepsilon > 0$, there exists $r_1 > 0$ when $0 < \sqrt{u^2 + v^2} \le r_1$ and $f(t, u, v) \le (M^0 + \varepsilon)\sqrt{u^2 + v^2}$. Let

$$\Omega_1 = \Big\{ t \in [-\tau, 1] : \ \|\sqrt{u^2 + v^2}\| < r_1 \Big\}$$

for $u, v \in K \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|\Phi u\| &\le \lambda \int_0^1 G(s,s)p(s)f(s,u,v)\,\mathrm{d}s + \frac{\lambda a}{1 - a\eta} \int_0^1 G(s,s)p(s)f(s,u,v)\,\mathrm{d}s \\
&\le \frac{\lambda(M^0 + \varepsilon)(1 + a - a\eta)}{1 - a\eta} \int_0^1 G(s,s)p(s)\sqrt{u^2 + v^2}\,\mathrm{d}s \\
&\le \frac{\lambda(M^0 + \varepsilon)(1 + a - a\eta)}{1 - a\eta} \int_0^1 G(s,s)p(s)(u + v)\,\mathrm{d}s \\
&= \frac{\lambda(M^0 + \varepsilon)(1 + a - a\eta)}{1 - a\eta} \left[\int_0^1 G(s,s)p(s)u(s - \tau)\,\mathrm{d}s \right. \\
&\qquad \left. + \int_0^1 G(s,s)p(s) \int_0^s K(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s \right] \\
&\le \frac{\lambda(M^0 + \varepsilon)(1 + a - a\eta)}{1 - a\eta} \left[\int_{-\tau}^{1-\tau} G(s + \tau, s + \tau)p(s + \tau)u(s)\,\mathrm{d}s \right. \\
&\qquad \left. + k_1 \int_0^1 G(s,s)p(s) \int_0^s u(t)\,\mathrm{d}t\,\mathrm{d}s \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda(M^0+\varepsilon)(1+a-a\eta)}{1-a\eta}\Bigg[\int_0^{1-\tau} G(s+\tau,s+\tau)p(s+\tau)u(s)\,\mathrm{d}s \\
&\qquad\qquad + k_1\int_0^1 G(s,s)p(s)\int_0^s u(t)\,\mathrm{d}t\,\mathrm{d}s\Bigg] \\
&\le \frac{\lambda(M^0+\varepsilon)(1+a-a\eta)}{1-a\eta}\int_0^1 G(s,s)p(s)(1+k_1 s\,\mathrm{d}s\|u\| \\
&= \lambda(M^0+\varepsilon)L_2\|u\| \le \|u\|.
\end{aligned}
$$

For the same above $\varepsilon > 0$, from (3.1) and (3.3), there exists $R_1 > r$ when $\sqrt{u^2+v^2} \ge R_1$ and $f(t,u,v) > (M_\infty-\varepsilon)\sqrt{u^2+v^2}$. Since $\gamma < \frac{1-\tau}{2}$ then $\gamma+\tau < 1-\gamma$. Let

$$
\Omega_2 = \Big\{t \in [-\tau,1] : \ \|\sqrt{u^2+v^2}\| < R_1\Big\}
$$

for $u, v \in K \cap \partial\Omega_2$, we get

$$
\begin{aligned}
\|\Phi u\| &\ge \lambda \sup_{t\in J}\int_0^1 G(t,s)P(s)f(s,u,v)\,\mathrm{d}s \\
&\ge \lambda(M_\infty-\varepsilon)\sup_{t\in J}\int_0^1 G(t,s)P(s)\sqrt{u^2+v^2}\,\mathrm{d}s \\
&\ge \lambda(M_\infty-\varepsilon)\sup_{t\in J}\int_0^1 G(t,s)P(s)\frac{u+v}{2}\,\mathrm{d}s \\
&= \frac{\lambda(M_\infty-\varepsilon)}{2}\sup_{t\in J}\Bigg[\int_0^1 G(t,s)P(s)u(s-\tau)\,\mathrm{d}s \\
&\qquad + \int_0^1 G(t,s)p(s)\int_0^s K(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s\Bigg] \\
&\ge \frac{\lambda(M_\infty-\varepsilon)}{2}\sup_{t\in J}\Bigg[\int_{-\tau}^{1-\tau} G(t,s+\tau)p(s+\tau)u(s)\,\mathrm{d}s \\
&\qquad + k_0\int_0^1 G(t,s)p(s)\int_0^s k_0 u(t)\,\mathrm{d}t\,\mathrm{d}s\Bigg]
\end{aligned}
$$

$$\geq \frac{\lambda(M_\infty-\varepsilon)}{2}\sup_{t\in J}\Bigg[\int_{\gamma}^{1-\gamma} G(t,s+\tau)p(s+\tau)\gamma_1\|u\|\,\mathrm{d}s + k_0\int_{\gamma}^{1-\gamma} G(t,s)p(s)s\gamma_1\|u\|\,\mathrm{d}t\,\mathrm{d}s\Bigg]$$

$$\geq \frac{\lambda\gamma_1(M_\infty-\varepsilon)}{2}\sup_{t\in J}\Bigg[\int_{\gamma+\tau}^{1-\gamma+\tau} G(t,s)p(s)\|u\|\,\mathrm{d}s + k_0\|u\|\int_{\gamma}^{1-\gamma} G(t,s)p(s)s\,\mathrm{d}s\Bigg]$$

$$\geq \frac{(M_\infty-\varepsilon)\lambda\gamma\gamma_1}{2}\Bigg[\int_{\gamma+\tau}^{1-\gamma} G(s,s)p(s)(1+k_0s)\,\mathrm{d}s\Bigg]\|u\|$$

$$= \lambda(M_\infty-\varepsilon)L_1 \geq \|u\|.$$

Therefore, by Lemma 2.3, Φ has a fixed point $u\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$ and $u(t)$ is a positive solution of equation (1.2), completing the proof of Theorem 3.1. $\square$

Theorem 3.2. *If the conditions* (H_1)–(H_4) *and the following conditions*

$$0 < M_0 < +\infty, \tag{3.4}$$

$$0 < M^\infty < +\infty, \tag{3.5}$$

$$\frac{1}{L_1(M_0-\varepsilon)} \leq \lambda \leq \frac{1}{L_2(M^\infty+\varepsilon)}, \tag{3.6}$$

hold then the equation (1.2) *has at least one solution.*

Proof. By (3.4), (3.6) for a given $\varepsilon>0$, there exists $r_2>0$ when $\sqrt{u^2+v^2}\leq r_2$ and $f(t,u,v)\geq(M_0-\varepsilon)\sqrt{u^2+v^2}$. Let

$$\Omega_1=\Big\{t\in[-\tau,1]:\ \|\sqrt{u^2+v^2}\|<r_2\Big\}$$

for $u,v\in K\cap\partial\Omega_1$, we have

$$\|\Phi u\| \geq \lambda\sup_{t\in J}\int_0^1 G(t,s)P(s)f(s,u,v)\,\mathrm{d}s$$

$$\geq \lambda(M_0-\varepsilon)\sup_{t\in J}\int_0^1 G(t,s)P(s)\sqrt{u^2+v^2}\,\mathrm{d}s$$

$$
\begin{aligned}
&\geq \lambda(M_0-\varepsilon)\sup_{t\in J}\int_0^1 G(t,s)P(s)\frac{u+v}{2}\,\mathrm{d}s\\
&= \frac{\lambda(M_0-\varepsilon)}{2}\sup_{t\in J}\bigg[\int_0^1 G(t,s)P(s)u(s-\tau)\,\mathrm{d}s\\
&\qquad\qquad + \int_0^1 G(t,s)p(s)\int_0^s K(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s\bigg]\\
&\geq \frac{\lambda(M_0-\varepsilon)}{2}\sup_{t\in J}\bigg[\int_{-\tau}^{1-\tau} G(t,s+\tau)p(s+\tau)u(s)\,\mathrm{d}s\\
&\qquad\qquad + k_0\int_0^1 G(t,s)p(s)\int_0^s k_0u(t)\,\mathrm{d}t\,\mathrm{d}s\bigg]\\
&\geq \frac{\lambda(M_0-\varepsilon)}{2}\sup_{t\in J}\bigg[\int_{\gamma}^{1-\gamma} G(t,s+\tau)p(s+\tau)\gamma_1\|u\|\,\mathrm{d}s\\
&\qquad\qquad + k_0\int_{\gamma}^{1-\gamma} G(t,s)p(s)s\gamma_1\|u\|\,\mathrm{d}t\,\mathrm{d}s\bigg]\\
&\geq \frac{\lambda\gamma_1(M_0-\varepsilon)}{2}\sup_{t\in J}\int_{\gamma+\tau}^{1-\gamma+\tau} G(t,s)p(s)\|u\|\,\mathrm{d}s\\
&\geq \frac{(M_0-\varepsilon)\lambda\gamma\gamma_1}{2}\int_{\gamma+\tau}^{1-\gamma} G(s,s)p(s)(1+k_0s)\,\mathrm{d}s]\|u\|\\
&= \lambda(M_0-\varepsilon)L_1\geq\|u\|.
\end{aligned}
$$

For the same above $\varepsilon > 0$, from (3.5) and (3.6), there exists $R_2 > r_2$ when $\sqrt{u^2+v^2}\geq R_2$ and $f(t,u,v) > (M^\infty+\varepsilon)\sqrt{u^2+v^2}$. Let

$$\Omega_2 = \Big\{t\in[-\tau,1]:\ \|\sqrt{u^2+v^2}\| < R_2\Big\}$$

for $u,v\in K\cap\partial\Omega_2$, we get

$$
\begin{aligned}
\|\Phi u\| &\leq \lambda\int_0^1 G(s,s)p(s)f(s,u,v)\,\mathrm{d}s + \frac{\lambda a}{1-a\eta}\int_0^1 G(s,s)p(s)f(s,u,v)\,\mathrm{d}s\\
&\leq \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta)}{1-a\eta}\int_0^1 G(s,s)p(s)\sqrt{u^2+v^2}\,\mathrm{d}s
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta)}{1-a\eta}\int_0^1 G(s,s)p(s)(u+v)\,\mathrm{d}s\\
&= \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta)}{1-a\eta}\Bigg[\int_0^1 G(s,s)p(s)u(s-\tau)\,\mathrm{d}s\\
&\qquad\qquad + \int_0^1 G(s,s)p(s)\int_0^s K(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s\Bigg]\\
&\le \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta)}{1-a\eta}\Bigg[\int_{-\tau}^{1-\tau} G(s+\tau,s+\tau)p(s+\tau)u(s)\,\mathrm{d}s\\
&\qquad\qquad + k_1\int_0^1 G(s,s)p(s)\int_0^s u(t)\,\mathrm{d}t\,\mathrm{d}s\Bigg]\\
&= \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta)}{1-a\eta}\Bigg[\int_0^{1-\tau} G(s+\tau,s+\tau)p(s+\tau)u(s)\,\mathrm{d}s\\
&\qquad\qquad + k_1\int_0^1 G(s,s)p(s)\int_0^s u(t)\,\mathrm{d}t\,\mathrm{d}s\Bigg]\\
&\le \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta)}{1-a\eta}\int_0^1 G(s,s)p(s)(1+k_1)s\,\mathrm{d}s\|u\|\\
&= \lambda(M^\infty+\varepsilon)L_2\|u\| \le \|u\|.
\end{aligned}$$

Therefore, by Lemma 2.3, Φ has a fixed point $u \in K\cap(\overline{\Omega}_2\backslash\Omega_1)$ and $u(t)$ is a positive solution of equation (1.2), completing the proof of Theorem 3.2. □

Theorem 3.3. *If the conditions* (H_1)–(H_4) *are satisfied and* $M_\infty=\infty, M^0=0$*. Then there exists two positive numbers* λ_1,λ_2 *when* $\lambda_1\le\lambda\le\lambda_2$*, BVP* (1.2) *has at least a positive solution.*

Proof. Since $M_\infty=\infty$, we can choose a positive constant $M>0$ such that $f(t,u,v)\ge M=\alpha R_3(\alpha>0)$ for any $\sqrt{u^2+v^2}\ge R_3$, $t\in J$. Let

$$\lambda_1=\left[\alpha\gamma\int_\gamma^{1-\gamma} G(s,s)p(s)\,\mathrm{d}s\right]^{-1},\quad \Omega_2=\left\{t\in[-\tau,1]:\ \|\sqrt{u^2+v^2}\|<R_3\right\}$$

for $u, v \in K \cap \partial\Omega_1, \lambda \geq \lambda_1$, we have

$$\begin{aligned}
\|\Phi u\| &\geq \lambda \sup_{t\in J} \int_0^1 G(t,s)p(s)f(s,u,v)\,\mathrm{d}s \geq \lambda M \sup_{t\in J} \int_0^1 G(t,s)p(s)\,\mathrm{d}s \\
&\geq \lambda M \sup_{t\in J} \int_\gamma^{1-\gamma} G(t,s)p(s)\,\mathrm{d}s \geq \lambda M\gamma \int_\gamma^{1-\gamma} G(s,s)p(s)\,\mathrm{d}s \\
&\geq \lambda\alpha R_3\gamma \int_\gamma^{1-\gamma} G(s,s)p(s)\,\mathrm{d}s = \frac{\lambda}{\lambda_1} R_3 \\
&= \|\sqrt{u^2+v^2}\| \geq \|u\|.
\end{aligned}$$

Because $M^0 = 0$, we choose a value small enough for $\varepsilon > 0$, so that

$$\lambda_2 = \left[\frac{\lambda\varepsilon(1+a-a\eta)}{1-a\eta} \int_0^1 G(s,s)(1+k_1 s)p(s)\,\mathrm{d}s\right]^{-1} > \lambda_1$$

and there exists $0 < r_3 < R_3$ such that $f(t,u,v) \leq \varepsilon\sqrt{u^2+v^2}$ for any $\sqrt{u^2+v^2}$ $\leq r_3$. Let

$$\Omega_1 = \Big\{t \in [-\tau, 1] :\ \|\sqrt{u^2+v^2}\| < r_3\Big\}$$

for $u, v \in K \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|\Phi u\| &\leq \lambda \int_0^1 G(s,s)p(s)f(s,u,v)\,\mathrm{d}s + \frac{\lambda a}{1-a\eta} \int_0^1 G(s,s)p(s)f(s,u,v)\,\mathrm{d}s \\
&\leq \frac{\lambda\varepsilon(1+a-a\eta)}{1-a\eta} \int_0^1 G(s,s)p(s)\sqrt{u^2+v^2}\,\mathrm{d}s \\
&\leq \frac{\lambda\varepsilon(1+a-a\eta)}{1-a\eta} \int_0^1 G(s,s)p(s)(u+v)\,\mathrm{d}s \\
&= \frac{\lambda\varepsilon(1+a-a\eta)}{1-a\eta} \bigg[\int_0^1 G(s,s)p(s)u(s-\tau)\,\mathrm{d}s \\
&\qquad + \int_0^1 G(s,s)p(s) \int_0^s K(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s\bigg]
\end{aligned}$$

$$\leq \frac{\lambda\varepsilon(1+a-a\eta)}{1-a\eta}\left[\int_{-\tau}^{1-\tau} G(s+\tau,s+\tau)p(s+\tau)u(s)\,\mathrm{d}s + k_1\int_0^1 G(s,s)p(s)\int_0^s u(t)\,\mathrm{d}t\,\mathrm{d}s\right]$$

$$\leq \frac{\lambda\varepsilon(1+a-a\eta)}{1-a\eta}\int_0^1 G(s,s)(1+k_1 s)p(s)\,\mathrm{d}s\|u\| \leq \frac{\lambda}{\lambda_2}\|u\| \leq \|u\|.$$

Therefore, by Lemma 2.3, Φ has at least one fixed point $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $u(t)$ has at least one positive solution of equation (1.2), completing the proof of Theorem 3.3. □

Remark 1. If $M_0 = \infty, M^\infty = 0$, similarly we can verify that BVP (1.2) has at least one positive solution.

4. The case of no less than three solutions

In this section, by Lemma 2.4, we prove that BVP (1.2) has at least three solutions when $f(t,u,v)$ satisfies some certain conditions. First, we define four non-negative continuous concave functionals in K α, β, φ and ϕ as

$$\begin{aligned}\alpha(u) &= \min_{\gamma\leq t\leq 1-\gamma} |u(t)|,\\ \beta(u) &= \max_{0\leq t\leq 1} |u(t)|,\\ \varphi(u) = \phi(u) &= \max_{k-\tau\leq t\leq 1} |u(t)|\end{aligned} \tag{4.1}$$

where k is an constant and satisfies $\tau < k < 1$, set

(1) $\omega_1(t)$, $\omega_2(t)$ are two non-negative characteristic functions and $\omega_1(t) \in C[0,1]$, $\omega_2(t) \in C[\gamma, 1-\gamma]$.

(2) $L_3 = \frac{1+a-a\eta}{1-a\eta}\left[\int_0^1 G(s,s)p(s)\omega_1(s)(1+k_1 s)\,\mathrm{d}s\right]$.

(3) $N_1 = \frac{c}{\lambda L_3(m+c)}$, $N_2 = \frac{1}{\lambda\gamma\gamma_1\sqrt{k_0}\int_{\gamma+\tau}^{1-\gamma}\sqrt{s}G_2(s,s)p(s)\omega_2(s)\,\mathrm{d}s}$.

As we know, some researchers [2,14] had discussed BVPs with at least three solutions. In these papers, the relative conclusions were based on the assumption that $f(t,u(t))$ be more than or less than a given constant. In fact, it's very difficult to find such functions. In our paper, $f(t,u,v)$ is assumed to be a function which satisfies the conditions $f(t,u,v) \le N_1\omega_1(u+v)$ or $f(t,u,v) \ge N_2\omega_2\sqrt{uv}$. Meanwhile, we introduce two characteristic ω_1, ω_2. The conditions $f(t,u,v) \le N_1\omega_1(u+v)$ or $f(t,u,v) \ge N_2\omega_2\sqrt{uv}$ can be easily satisfied for some equations when we choose suitable characteristic functions. The conclusion is described in Theorem 4.1. In Section 5, we give Example 2 to illustrate our conclusion.

Theorem 4.1. *If the conditions* (H_1)–(H_4) *hold and there exist positive numbers* b, c, m *with* $m > b > c > 0$ *such that the following conditions are satisfied*

(i) $f(t,u,v) \le N_1\omega_1(t)(u+v), (t,u) \in [0,1] \times [0,m]$,

(ii) $f(t,u,v) \ge N_2\omega_2(t)\sqrt{uv}, (t,u) \in [\gamma, 1-\gamma] \times [b, \frac{(\gamma+1)^2}{\gamma_1}b]$.

Then BVP (1.2) *has at least three solutions* $u_1, u_2, u_3 \in \overline{P(\beta,m)}$ *such that* $\beta(u_i) \le m$, *for* $i = 1,2,3$;

$$b < \alpha(u_1); \quad c < \phi(u_2), \qquad \textit{with} \quad \alpha(u_2) < b; \quad \phi(u_3) < c.$$

P r o o f. By Lemma 2.1, we can derive that $\Phi\colon K \to K$ is completely continuous. It is easy to verify that $\phi(\lambda u) = \lambda\phi(u)$ for $0 \le \lambda \le 1$ and $\alpha(u) \le \phi(u)$. According to the definition of norm and (4.1) if taking $M \ge 1$ then we have that $\|u\| = \beta(u)$ and $\|u\| \le M\beta(u)$ for all $u \in \overline{P(\beta,m)}$. Therefore if $u \in \overline{P(\beta,m)}$ then $\beta(u) = \|u\| \le m$, in addition

$$\max_{t\in[0,1]} |u(t-\tau)| = \max_{t\in[-\tau,1-\tau]} |u(t)| = \max_{t\in[0,1-\tau]} |u(t)| \le \max_{t\in[0,1]} |u(t)| \le m.$$

From the conditions (i) of Theorem 4.1, we have

$$\begin{aligned}
\beta(\Phi x) &= \max_{t\in[0,1]} |\Phi(x)| \\
&\le \lambda \int_0^1 G(s,s)p(s)f(s,u,v)\,\mathrm{d}s + \frac{a\lambda}{1-a\eta}\int_0^1 G(s,s)p(s)f(s,u,v)\,\mathrm{d}s \\
&\le \frac{\lambda N_1(1+a-a\eta)}{1-a\eta}\int_0^1 G(s,s)p(s)\omega_1(s)(u+v)\,\mathrm{d}s \\
&= \frac{\lambda N_1(1+a-a\eta)}{1-a\eta}\Bigg[\int_0^1 G(s,s)p(s)\omega_1(s)u(s-\tau)\,\mathrm{d}s \\
&\qquad + \int_0^1 G(s,s)p(s)\omega_1(s)\int_0^s K(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s\Bigg]
\end{aligned}$$

$$\leq \frac{\lambda N_1(1+a-a\eta)}{1-a\eta}\int_0^1 G(s,s)p(s)\omega_1(s)\Big(m+k_1 m\int_0^s \mathrm{d}t\Big)\,\mathrm{d}s$$
$$= \frac{\lambda N_1(1+a-a\eta)m}{1-a\eta}\int_0^1 (1+k_1 s)G(s,s)p(s)\omega_1(s)\,\mathrm{d}s = \lambda N_1 m L_3$$
$$\leq \frac{mc}{(m+c)} < m.$$

So $\Phi\colon \overline{P(\beta,m)} \to \overline{P(\beta,m)}$. If one choose

$$u_0(t) = -\frac{4b}{\gamma_1}\Big(t-\frac{1+\gamma}{2}\Big)^2 + \frac{(1+\gamma)^2 b}{\gamma_1}, \qquad t\in[0,1].$$

We have that $\varphi(u_0) = \frac{(1+\gamma)^2 b}{\gamma_1}$ and

$$\alpha(u_0) = \min_{t\in[\gamma,1-\gamma]} |u_0(t)| = u_0(\gamma) = \frac{4b\gamma}{\gamma_1} > b$$

then $u_0 \in P(\beta,\varphi,\alpha,b,\frac{(1+\gamma)^2 b}{\gamma_1},m)$. Therefore $\Big\{u \in P\Big(\beta,\varphi,\alpha,b,\frac{(1+\gamma)^2 b}{\gamma_1},m\Big) : \alpha(u) > b\Big\} \neq 0$.

On the other hand if $u \in P(\beta,\varphi,\alpha,b,\frac{(1+\gamma)^2 b}{\gamma_1}m)$ then

$$\min_{t\in[\gamma+\tau,1-\gamma]} |u(t-\tau)| = \min_{t\in[\gamma,1-\gamma-\tau]} |u(t)| \geq \min_{t\in[\gamma,1-\gamma]} |u(t)| \geq b,$$
$$\min_{t\in[\gamma+\tau,1-\gamma]} |u(t)| \geq \min_{t\in[\gamma,1-\gamma]} |u(t)| \geq b.$$

From the conditions (ii) of Theorem 4.1 and Lemma 2.2, we have

$$\alpha(\Phi(x)) = \min_{t\in[\gamma,1-\gamma]} |\Phi(x)| \geq \gamma_1\|\Phi(x)\|,$$
$$\geq \lambda\gamma_1 \sup_{t\in J}\int_0^1 G(t,s)p(s)f(s,u,v)\,\mathrm{d}s$$
$$\geq \lambda\gamma_1 N_2 \sup_{t\in J}\int_0^1 G(t,s)p(s)\omega_2(s)\sqrt{uv}\,\mathrm{d}s$$
$$= \lambda\gamma_1 N_2\lambda \sup_{t\in J}\int_0^1 G(t,s))p(s)\omega_2(s)\sqrt{u(t-\tau)\int_0^s k(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s}$$

$$\geq \lambda\gamma_1 N_2 \lambda \sup_{t\in J} \int_{\gamma+\tau}^{1-\gamma} G(t,s)p(s)\omega_2(s)\sqrt{u(t-\tau)\int_0^s k(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s}$$
$$\geq \left[\lambda\gamma\gamma_1 N_2\sqrt{k_0}\int_{\gamma+\tau}^{1-\gamma} G(s,s)\sqrt{s}p(s)\omega_2(s)\,\mathrm{d}s\right] b \geq b.$$

So the condition (S_1) of Lemma 2.4 is satisfied.

For all $u \in P(\beta,\alpha,b,m)$ with $\varphi(\Phi u) > \frac{b(1+\gamma)^2}{\gamma_1}$ then one has $\alpha(\Phi u) \geq \gamma_1\varphi(\Phi u) > \gamma_1\frac{b(1+\gamma)^2}{\gamma_1} > b$. So the condition ($S_2$) of Lemma 2.4 is also satisfied.

Finally, we verify that the condition (S_3) of Lemma 2.4 holds. Obviously, $0 \notin R(\beta,\phi,c,m)$. As if $0 \in R(\beta,\phi,c,m)$ then it is conflicts with $\phi(0) = 0 < c$. For all $u \in R(\beta,\phi,c,m)$ with $\phi(u) = c$ then

$$\max_{t\in[0,1]} |u(t)| \leq m,$$
$$\max_{t\in[k,1]} |u(t-\tau)| = \max_{t\in[k-\tau,1-\tau]} |u(t)| \leq \max_{t\in[k-\tau,1]} |u(t)| = c.$$

From the conditions (i) of Theorem 4.1, we have

$$\phi(\Phi u) = \max_{t\in[\gamma-\tau,1]} |(\Phi u)(t)| \leq \max_{t\in[0,1]} |(\Phi u)(t)|$$
$$\leq \lambda\int_0^1 G(s,s)p(s)f(s,u,v)\,\mathrm{d}s + \frac{a\lambda}{1-a\eta}\int_0^1 G(s,s)p(s)f(s,u,v)\,\mathrm{d}s$$
$$\leq \frac{\lambda N_1(1+a-a\eta)}{1-a\eta}\int_0^1 G(s,s)p(s)\omega_1(s)(u+v)\,\mathrm{d}s$$
$$= \frac{\lambda N_1(1+a-a\eta)}{1-a\eta}\left[\int_0^1 G(s,s)p(s)\omega_1(s)u(s-\tau)\,\mathrm{d}s \right.$$
$$\left. + \int_0^1 G(s,s)p(s)\omega_1(s)\int_0^s K(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s\right].$$

$$
\begin{aligned}
&= \frac{\lambda N_1(1+a-a\eta)}{1-a\eta}\Bigg[\int_0^k G(s,s)p(s)\omega_1(s)u(s-\tau)\,\mathrm{d}s \\
&\qquad + \int_k^1 G(s,s)p(s)\omega_1(s)u(s-\tau)\,\mathrm{d}s \\
&\qquad + \int_0^{k-\tau} G(s,s)p(s)\omega_1(s)\int_0^s K(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s \\
&\qquad + \int_{k-\tau}^1 G(s,s)p(s)\omega_1(s)\int_0^s K(t,s)u(t)\,\mathrm{d}t\,\mathrm{d}s\Bigg] \\
&\le \frac{\lambda N_1(1+a-a\eta)}{1-a\eta}\Bigg[m\int_0^k G(s,s)p(s)\omega_1(s)\,\mathrm{d}s \\
&\qquad + c\int_k^1 G(s,s)p(s)\omega_1(s)\,\mathrm{d}s \\
&\qquad + mk_1\int_0^{k-\tau} G(s,s)p(s)\omega_1(s)s\,\mathrm{d}s \\
&\qquad + k_1 c\int_{k-\tau}^1 G(s,s)p(s)\omega_1(s)s\,\mathrm{d}s\Bigg] \\
&\le \frac{\lambda N_1(m+c)(1+a-a\eta)}{1-a\eta}\int_0^1 G(s,s)p(s)\omega_1(s)(1+k_1 s)\,\mathrm{d}s \\
&= \lambda N_1(m+c)L_3 \le c.
\end{aligned}
$$

Therefore, by Lemma 2.4, Φ has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\beta, m)}$ then u_1, u_2, u_3 are three positive solutions of equation (1.2) and u_1, u_2, u_3 satisfy that $\beta(u_i) \le m$ for $i = 1, 2, 3$;

$$
b < \alpha(u_1); \quad c < \phi(u_2), \qquad \text{with} \quad \alpha(u_2) < b; \quad \phi(u_3) < c.
$$

□

5. Example

***Example* 1.** Consider the equation

$$\begin{cases} -u''(t) = \dfrac{1}{t}\sqrt{t^2+1} \\ \qquad\qquad \times \dfrac{[u^2(t-\frac{1}{6})+v^2(t)][1+u(t-\frac{1}{6})+v(t)]}{2+u(t-\frac{1}{6})+v(t)}, & 0<t<1, \\ u(t)=0, & -\tau \le t \le 0, \\ u(1) = \dfrac{1}{2}u\left(\dfrac{1}{2}\right), \end{cases} \tag{5.1}$$

where

$$f(t,u,v) = \sqrt{t^2+1}\frac{\left[u^2\left(t-\frac{1}{6}\right)+v^2(t)\right]\left[1+u\left(t-\frac{1}{6}\right)+v(t)\right]}{2+u\left(t-\frac{1}{6}\right)+v(t)},$$

$$v(t) = \int_0^1 (t+s+1)u(s)\,\mathrm{d}s; \qquad \text{and} \qquad k(s,t) = t+s+1.$$

Then $k_1 = 3$; $p(t) = \frac{1}{t}$ and $t=0$ is its singularity. Here we have $M_\infty = \infty$, $M^0 = 0$. If we choose $M = 100$, $\gamma = \frac{1}{4}$, $\alpha = \frac{100}{11}$ then when $\sqrt{u^2+v^2} \ge 11$ and $f(t,u,v) \ge M$, we can calculate that $\lambda_1 = \frac{44}{25}$. If we choose $\varepsilon = 0.01$, calculations show that $\lambda_2 = 240$.

From Theorem 3.3, we have that if $\frac{44}{25} = \lambda_1 \le \lambda \le \lambda_2 = 240$, the problem (5.1) has at least one positive solution.

***Example* 2.** Consider the boundary value problem

$$\begin{cases} -u''(t) = \dfrac{300}{1-t}\dfrac{1}{1+2t}\left[\sqrt{u^2\left(t-\dfrac{1}{9}\right)+v^2(t)+u\left(t-\dfrac{1}{9}\right)v(t)}\right. \\ \qquad\qquad \left. +\ 999\min(1-t)(u+v)\right], & t\in J \\ u(t)=0, & -\tau\le t\le 0, \\ u(1) = \dfrac{1}{2}u\left(\dfrac{1}{2}\right), \end{cases} \tag{5.2}$$

where

$$\lambda = 1; \qquad \tau = \frac{1}{9}; \qquad k(t,s) = t^2s^2+1;$$

$$v(t) = \int_0^1 k(t,s)u(s)\,\mathrm{d}s, \qquad (t,s)\in[0,1]\times[0,1];$$

$$f(t,u,v)=\frac{1}{1+2t}\left[\sqrt{u^2\left(t-\frac{1}{9}\right)+v^2(t)+u\left(t-\frac{1}{9}\right)v(t)}+999\min(1-t)(u+v)\right],$$

$$t\in(0,1);\qquad p(t)=\frac{300}{1-t}$$

and $t=1$ is its singularity; $u(t)$ satisfies the definition of (4.1). If we choose $m=2,\ c=1,\ \gamma=\frac{1}{3},\ \eta=\frac{1}{2}$ such that

$$\max_{0\le t\le 1}|u(t)|\le m,\qquad \phi(u)=\max_{k-\tau\le t\le 1}|u(t)|>c,$$

calculates easily that

$$L_3=25,\qquad N_1=0.0133,\qquad N_2=0.225.$$

Obviously, we have that

$$f(t,u,v)\le 0.001(u+v)<N_1(u+v),\qquad t\in[0,1],$$

$$f(t,u,v)\ge 0.25\sqrt{uv}p(t)>\frac{N_2\sqrt{uv}}{p(t)},\qquad t\in\left[\frac{1}{3},\frac{2}{3}\right].$$

From Theorem 4.1, the problem (5.2) has at least three positive solutions.

Acknowledgement. The author is very grateful to the referees for their helpful suggestions.

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Received 6. 2. 2012
Accepted 21. 4. 2012

* *Tianmu College*
Zhejiang A and F University
311300 Hangzhou
P. R. CHINA
E-mail: hzm2007@aliyun.com

** *Department of Mathematics*
Hangzhou Normal University
Zhejiang
310036 Hangzhou
P. R. CHINA