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Oscillation of nonlinear third order perturbed functional difference equations

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Abstract: This paper deals with oscillatory and asymptotic behavior of all solutions of perturbed nonlinear third order functional difference equation

$$\Delta(b_n \Delta(a_n (\Delta x_n)^\alpha)) + p_n f(x_{\sigma(n)}) = g(n, x_n, x_{\sigma(n)}, \Delta x_n), \quad n \geq n_0.$$

By using comparison techniques we present some new sufficient conditions for the oscillation of all solutions of the studied equation. Examples illustrating the main results are included.

Keywords: Comparison method, oscillation, perturbed equation, asymptotic behavior

MSC: 39A10

1 Introduction

Consider the nonlinear third order functional difference equation with perturbation term

$$\Delta(b_n \Delta(a_n (\Delta x_n)^\alpha)) + p_n f(x_{\sigma(n)}) = g(n, x_n, x_{\sigma(n)}, \Delta x_n), \quad n \in \mathbb{N}(n_0), \quad (1.1)$$

where $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots\}$, n_0 is a nonnegative integer, subject to the following conditions:

- (H₁) $\{b_n\}$, $\{a_n\}$ and $\{p_n\}$ are positive real sequences for all $n \in \mathbb{N}(n_0)$;
- (H₂) $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{N}(n_0) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, $uf(u) > 0$ for $u \neq 0$, and f is nondecreasing;
- (H₃) $f(uv) \geq f(u)f(v)$ for $uv > 0$;
- (H₄) $\{\sigma(n)\}$ is a sequence of integers with $\sigma(n) \leq n$ and $\lim_{n \rightarrow \infty} \sigma(n) = \infty$;
- (H₅) $\alpha \geq 1$ is a ratio of odd positive integers;
- (H₆) there is a positive real sequence $\{q_n\}$ such that $|g(n, u, v, w)| \leq q_n f(v)$ for all $(n, u, v, w) \in \mathbb{N}(n_0) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

By a solution of equation (1.1), we mean a nontrivial real sequence $\{x_n\}$ which is defined for all $n \in \mathbb{N}(n_0)$ and satisfying equation (1.1). We assume that equation (1.1) possesses such solutions for all $n \in \mathbb{N}(n_0)$. As usual a nontrivial solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and it is nonoscillatory otherwise. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Recently in [1–6, 9–12, 14–16], the authors studied the oscillatory and asymptotic behavior of solutions of some third and higher order nonlinear delay difference equations. In [11, 15], the authors studied the oscillatory behavior of the equation

$$\Delta(b_n \Delta(a_n (\Delta x_n)^\alpha)) + q_n f(x_{\sigma(n)}) = 0,$$

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under the conditions $(H_1) - (H_4)$ and also assumed that

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n} = \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty.$$

In [16], the authors investigated the oscillatory behavior of the equation

$$\Delta(b_n \Delta(a_n \Delta x_n)) + p_n f(x_{\tau(n)}) - q_n g(x_{\sigma(n)}) = 0,$$

under the conditions $(H_1) - (H_5)$,

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n} = \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty, \quad (1.2)$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n}^{\infty} \frac{1}{b_s} \sum_{t=s}^{\infty} q_t < \infty. \quad (1.3)$$

Employing a new method, they obtained some interesting results on the oscillatory and asymptotic behavior of solutions, see [[16], Theorem 2.1]. Also they obtained another oscillatory result, see [[16], Theorem 2.5] by replacing condition (1.3) with

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n_0}^{n-1} \frac{1}{b_s} \sum_{t=n_0}^{s-1} q_t < \infty. \quad (1.4)$$

Clearly the condition (1.3) implies that $\{q_n\}$ is small and we must have

$$\sum_{n=n_0}^{\infty} q_n < \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{b_n} \sum_{s=n}^{\infty} q_s < \infty.$$

Moreover condition (1.4) requires $\{q_n\}$ to be small in some sense relative to $\{b_n\}$ and $\{a_n\}$.

Our purpose in this paper is to establish oscillation results for equation (1.1) without imposing a “smallness” condition on the perturbation term. Also, we present some results on the boundedness of nonoscillatory solutions and oscillatory behavior of a particular case of equation (1.1), namely,

$$\Delta(b_n \Delta(a_n (\Delta x_n)^{\alpha})) + p_n x_n^{\beta} = e_n + q_n x_n^{\gamma}, \quad n \in \mathbb{N}(n_0), \quad (1.5)$$

where β and γ are ratios of odd positive integers with $\beta > \gamma$ and $\{e_n\}$ is a real sequence. Examples are provided to illustrate the importance of the main results.

2 Oscillation of Equation (1.1)

In this section, we investigate the oscillatory behavior of all solutions of equation (1.1). For any integer $N \in \mathbb{N}(n_0)$, we set

$$I(n) = \sum_{s=N}^{\sigma(n)-1} \left(\frac{1}{a_s} \sum_{t=N}^{s-1} \frac{1}{b_t} \right)^{\frac{1}{\alpha}}. \quad (2.1)$$

We also assume that $\{\tau(n)\}$ and $\{\eta(n)\}$ are sequences of integers satisfying

$$\sigma(n) \leq \tau(n) \leq \eta(n) \leq n \quad \text{for all large } n. \quad (2.2)$$

Set

$$B(n) = \left(\sum_{s=\sigma(n)}^{\tau(n)-1} \frac{1}{a_s^{\frac{1}{\alpha}}} \right) \left(\sum_{s=\tau(n)}^{\eta(n)} \frac{1}{b_s} \right)^{\frac{1}{\alpha}}, \quad (2.3)$$

and

$$Q_n = p_n - q_n, \quad (2.4)$$

where $\{q_n\}$ is as defined in (H_6) . In some of our results, we require that

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n} = \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty. \quad (2.5)$$

We begin with the following oscillation result.

Theorem 2.1. *Let $Q_n > 0$ for all large n , conditions $(H_1) - (H_6)$, (2.2) and (2.5) hold. If the first order delay difference equations*

$$\Delta y_n + Q_n f(I(n)) f\left(y_{\sigma(n)}^{\frac{1}{\alpha}}\right) = 0, \quad (2.6)$$

and

$$\Delta z_n + Q_n f(B(n)) f\left(z_{\eta(n)}^{\frac{1}{\alpha}}\right) = 0, \quad (2.7)$$

are oscillatory, then equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1) for all $n \in \mathbb{N}(n_0)$. Without loss of generality, we may assume that $\{x_n\}$ and $\{x_{\sigma(n)}\}$ are positive and condition (2.2) holds for all $n \geq N \in \mathbb{N}(n_0)$. If $\{x_n\}$ is eventually negative, then a similar proof holds. From equation (1.1) and the assumptions, it follows that

$$\Delta(b_n \Delta(a_n (\Delta x_n)^\alpha)) \leq -Q_n f(x_{\sigma(n)}) < 0, \quad (2.8)$$

for all $n \geq N$. Therefore, condition (2.5) together with discrete Kneser's theorem [2] implies that

$$(I) \ a_n (\Delta x_n)^\alpha > 0 \text{ and } b_n \Delta(a_n (\Delta x_n)^\alpha) > 0,$$

or

$$(II) \ a_n (\Delta x_n)^\alpha < 0 \text{ and } b_n \Delta(a_n (\Delta x_n)^\alpha) > 0, \\ \text{eventually for all } n \geq N.$$

First assume that Case (I) holds. For $n \geq N$, we see that

$$a_n (\Delta x_n)^\alpha \geq \sum_{s=N}^{n-1} \frac{b_s}{b_s} \Delta(a_s (\Delta x_s)^\alpha) \geq \left(\sum_{s=N}^{n-1} \frac{1}{b_s} \right) b_n \Delta(a_n (\Delta x_n)^\alpha),$$

or

$$\Delta x_n \geq \left(\frac{1}{a_n} \sum_{s=N}^{n-1} \frac{1}{b_s} \right)^{\frac{1}{\alpha}} \left(b_n \Delta(a_n (\Delta x_n)^\alpha) \right)^{\frac{1}{\alpha}}.$$

Summing the last inequality from N to $\sigma(n) - 1 > N$, we have

$$\begin{aligned} x_{\sigma(n)} &\geq \sum_{s=N}^{\sigma(n)-1} \left(\frac{1}{a_s} \sum_{t=N}^{s-1} \frac{1}{b_t} \right)^{\frac{1}{\alpha}} \left(b_s \Delta(a_s (\Delta x_s)^\alpha) \right)^{\frac{1}{\alpha}} \\ &= I(n) y_n^{\frac{1}{\alpha}} \end{aligned} \quad (2.9)$$

where $y_n = b_n \Delta(a_n (\Delta x_n)^\alpha) > 0$. Using (2.9) in (2.8) and applying (H_3) , the monotonicity of f we see that y_n is a positive solution of the inequality

$$\Delta y_n + Q_n f(I(n)) f\left(y_{\sigma(n)}^{\frac{1}{\alpha}}\right) \leq 0, \quad n \geq N.$$

It follows from Lemma 5 of [8] that the corresponding difference equation (2.6) also has a positive solution, which is a contradiction.

For Case (II), it is easy to see that

$$\begin{aligned} -a_{\tau(n)} (\Delta x_{\tau(n)})^\alpha &\geq \left(\sum_{s=\tau(n)}^{\eta(n)} \frac{1}{b_s} \right) b_{\eta(n)} \Delta(a_{\eta(n)} (\Delta x_{\eta(n)})^\alpha) \\ &= \left(\sum_{s=\tau(n)}^{\eta(n)} \frac{1}{b_s} \right) z_{\eta(n)}. \end{aligned} \quad (2.10)$$

for $n \geq N$, where $z_n = b_n \Delta(a_n(\Delta x_n)^\alpha) > 0$. Now

$$\Delta x_n = \frac{1}{a_n^{\frac{1}{\alpha}}} (a_n(\Delta x_n)^\alpha)^{\frac{1}{\alpha}}.$$

So summing for $j-1 \geq i \geq N$, we have

$$x_j - x_i = \sum_{s=i}^{j-1} \frac{1}{a_s^{\frac{1}{\alpha}}} (a_s(\Delta x_s)^\alpha)^{\frac{1}{\alpha}}$$

or

$$x_i - x_j = \sum_{s=i}^{j-1} \frac{1}{a_s^{\frac{1}{\alpha}}} (-a_s(\Delta x_s)^\alpha)^{\frac{1}{\alpha}} \geq \sum_{s=i}^{j-1} \frac{1}{a_s^{\frac{1}{\alpha}}} (-a_j(\Delta x_j)^\alpha)^{\frac{1}{\alpha}}.$$

Hence

$$x_i \geq \left(\sum_{s=i}^{j-1} \frac{1}{a_s^{\frac{1}{\alpha}}} \right) \left(- (a_j(\Delta x_j)^\alpha) \right)^{\frac{1}{\alpha}}.$$

Setting $i = \sigma(n)$ and $j = \tau(n)$ in the above inequality, we obtain

$$x_{\sigma(n)} \geq \left(\sum_{s=\sigma(n)}^{\tau(n)-1} \frac{1}{a_s^{\frac{1}{\alpha}}} \right) \left(- (a_{\tau(n)}(\Delta x_{\tau(n)})^\alpha) \right)^{\frac{1}{\alpha}}. \quad (2.11)$$

From (2.10) and (2.11), we see that

$$x_{\sigma(n)} \geq \left(\sum_{s=\sigma(n)}^{\tau(n)-1} \frac{1}{a_s^{\frac{1}{\alpha}}} \right) \left(\sum_{s=\tau(n)}^{\eta(n)} \frac{1}{b_s} \right)^{\frac{1}{\alpha}} z_{\eta(n)}^{\frac{1}{\alpha}} = B(n) z_{\eta(n)}^{\frac{1}{\alpha}}. \quad (2.12)$$

Substituting (2.12) in (2.8), one has z_n is a positive solution of the inequality

$$\Delta z_n + Q_n f(B(n)) f(z_{\eta(n)}^{\frac{1}{\alpha}}) \leq 0.$$

It follows from Lemma 5 of [8] that the corresponding difference equation (2.7) also has a positive solution, which is a contradiction. This completes the proof of the theorem. $\square$

The next two corollaries follow immediately from known oscillation criteria for first order delay difference equations; for example, see [7] and [13].

Now assume that $\sigma(n) = n - k$, $\tau(n) = n - l$ and $\eta(n) = n - m$ where k, l, m are positive integers such that $k \geq l \geq m$.

Corollary 2.2. Let $f(u) = u^\alpha$, $Q_n > 0$ for all large $n \in \mathbb{N}(n_0)$, and conditions (H_1) , (H_4) , (2.2) and (2.5) hold. If

$$\liminf_{n \rightarrow \infty} \sum_{s=n-k}^{n-1} Q_s I^\alpha(s) > \left(\frac{k}{k+1} \right)^{k+1} \quad \text{and} \quad \liminf_{n \rightarrow \infty} \sum_{s=n-m}^{n-1} Q_s B^\alpha(s) > \left(\frac{m}{m+1} \right)^{m+1}, \quad (2.13)$$

then equation (1.1) is oscillatory.

Corollary 2.3. Let $f(u) = u^\beta$, $\alpha > \beta$, $Q_n > 0$ for all large n , and conditions (H_1) , (H_4) , (2.2) and (2.5) hold. If

$$\sum_{n=N}^{\infty} Q_n I^\beta(n) = \infty \quad \text{and} \quad \sum_{n=N}^{\infty} Q_n B^\beta(s) = \infty, \quad (2.14)$$

then equation (1.1) is oscillatory.

Instead of condition (2.2), let us assume that there is a sequence $\{\delta(n)\}$ of integers such that

$$\Delta\delta(n) > 0, \quad \delta(n) > n, \quad \omega(n) = \delta(\delta(\sigma(n))) < n, \quad (2.15)$$

and we set

$$B^*(n) = \sum_{s=\sigma(n)}^{\delta(\sigma(n))} \left(\frac{1}{a_s} \sum_{t=s}^{\delta(s)} \frac{1}{b_t} \right)^{\frac{1}{\alpha}}. \quad (2.16)$$

Next, we present another oscillation theorem for the equation (1.1).

Theorem 2.4. *Let $Q_n > 0$ for large n , conditions $(H_1) - (H_6)$ and (2.5) hold. If all the solutions of equations (2.6) and*

$$\Delta z_n + Q_n f(B^*(n)) f\left(z_{\omega(n)}^{\frac{1}{\alpha}}\right) = 0, \quad (2.17)$$

are oscillatory, then equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1) such that $x_n > 0$, $x_{\sigma(n)} > 0$ and condition (2.5) holds for all $n \geq N \in \mathbb{N}(n_0)$. Proceeding as in the proof of Theorem 2.1, we get (2.8). The proof of Case (I) is similar to that of in Theorem 2.1 and hence is omitted.

For Case (II), it is easy to see that

$$\begin{aligned} -a_n(\Delta x_n)^\alpha &\geq \sum_{s=n}^{\delta(n)} \frac{1}{b_s} \left(b_s \Delta(a_s(\Delta x_s)^\alpha) \right) \\ &\geq \left(\sum_{s=n}^{\delta(n)} \frac{1}{b_s} \right) \left(b_{\delta(n)} \Delta(a_{\delta(n)}(\Delta x_{\delta(n)})^\alpha) \right) \\ &= \left(\sum_{s=n}^{\delta(n)} \frac{1}{b_s} \right) z_{\delta(n)}, \end{aligned} \quad (2.18)$$

where $z_n = b_n \Delta(a_n(\Delta x_n)^\alpha) > 0$. Dividing by a_n and then summing from $\sigma(n)$ to $\delta(\sigma(n))$, we obtain

$$\begin{aligned} x_{\sigma(n)} &\geq \sum_{s=\sigma(n)}^{\delta(\sigma(n))} \left(\left(\frac{z_{\delta(s)}}{a_s} \right)^{\frac{1}{\alpha}} \left(\sum_{t=s}^{\delta(s)} \frac{1}{b_t} \right)^{\frac{1}{\alpha}} \right) \\ &\geq B^*(n) z_{\omega(n)}^{\frac{1}{\alpha}}, \end{aligned} \quad (2.19)$$

for all large n . Using (2.19) in (2.8) and then proceeding as in the proof of Case (II) of Theorem 2.1, we obtain a desired contradiction. This completes the proof of the theorem. $\square$

3 Boundedness and Oscillation of Equation (1.5)

In this section, we present conditions for the boundedness of nonoscillatory solutions and oscillation of all solutions of equation (1.5)

Theorem 3.1. *In addition to (H_1) , assume that*

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n} < \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} < \infty, \quad \sum_{n=n_0}^{\infty} q_n^{\frac{\beta}{\beta-\gamma}} p_n^{\frac{\gamma}{\gamma-\beta}} < \infty, \quad (3.1)$$

and

$$\sum_{n=n_0}^{\infty} |e_n| < \infty. \quad (3.2)$$

Then every nonoscillatory solution of equation (1.5) is bounded.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.5) with $x_n > 0$ for all $n \geq N \in \mathbb{N}(n_0)$. Using Young's inequality to $[q_n x_n^\gamma - p_n x_n^\beta]$, we obtain

$$q_n x_n^\gamma - p_n x_n^\beta \leq \frac{(\beta - \gamma)}{\gamma} \left(\frac{\gamma}{\beta} q_n \right)^{\frac{\beta}{\beta - \gamma}} p_n^{\frac{\gamma}{\gamma - \beta}}. \quad (3.3)$$

Using (3.3) in equation (1.5), we have

$$\Delta(b_n \Delta(a_n (\Delta x_n)^\alpha)) \leq |e_n| + c q_n^{\frac{\beta}{\beta - \gamma}} p_n^{\frac{\gamma}{\gamma - \beta}},$$

where $c = \left(\frac{\beta - \gamma}{\gamma} \right) \left(\frac{\gamma}{\beta} \right)^{\frac{\beta}{\beta - \gamma}}$. Summing the above inequality from N to $n - 1$ yields

$$\Delta(a_n (\Delta x_n)^\alpha) \leq \frac{c_1}{b_n} + \frac{1}{b_n} \sum_{s=N}^{n-1} |e_s| + \frac{c}{b_n} \sum_{s=N}^{n-1} q_s^{\frac{\beta}{\beta - \gamma}} p_s^{\frac{\gamma}{\gamma - \beta}},$$

where $c_1 = b_N \Delta(a_N (\Delta x_N)^\alpha)$. Another summation yields

$$a_n (\Delta x_n)^\alpha \leq c_2 + c_1 \sum_{s=N}^{n-1} \frac{1}{b_s} + c \sum_{s=N}^{n-1} \frac{1}{b_s} \sum_{t=N}^{s-1} q_t^{\frac{\beta}{\beta - \gamma}} p_t^{\frac{\gamma}{\gamma - \beta}} + \sum_{s=N}^{n-1} \frac{1}{b_s} \sum_{t=N}^{s-1} |e_t| \quad (3.4)$$

where $c_2 = a_N (\Delta x_N)^\alpha$. In view of conditions (3.1) and (3.2), there is a constant M such that

$$\Delta x_n \leq \left(\frac{M}{a_n} \right)^{\frac{1}{\alpha}}.$$

Summing the last inequality from N to $n - 1$ and using condition (3.1), we get the desired conclusion. $\square$

Our next result is concerned with the oscillation of equation (1.5).

Theorem 3.2. *In addition to (H_1) , assume that*

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} < \infty, \quad \sum_{n=n_0}^{\infty} \left(\frac{1}{a_n} \sum_{s=n_0}^{n-1} \frac{1}{b_s} \right)^{\frac{1}{\alpha}} < \infty, \quad (3.5)$$

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n} \sum_{s=n_0}^{n-1} \frac{1}{b_s} \sum_{t=n_0}^{s-1} q_t^{\frac{\beta}{\beta - \gamma}} p_t^{\frac{\gamma}{\gamma - \beta}} \right)^{\frac{1}{\alpha}} < \infty, \quad (3.6)$$

$$\liminf_{n \rightarrow \infty} \sum_{s=n_0}^{n-1} \left(\frac{1}{a_s} \sum_{t=n_0}^{s-1} \frac{1}{b_t} \sum_{j=n_0}^{t-1} e_j \right)^{\frac{1}{\alpha}} = -\infty, \quad (3.7)$$

and

$$\limsup_{n \rightarrow \infty} \sum_{s=n_0}^{n-1} \left(\frac{1}{a_s} \sum_{t=n_0}^{s-1} \frac{1}{b_t} \sum_{j=n_0}^{t-1} e_j \right)^{\frac{1}{\alpha}} = \infty, \quad (3.8)$$

then equation (1.5) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.5) such that $x_n > 0$ for all $n \geq N \in \mathbb{N}(n_0)$. Proceeding as in the proof of Theorem 3.1, we obtain (3.4). Summing again, we have

$$\Delta x_n \leq \left(\frac{c_2}{a_n} + \frac{c_1}{a_n} \sum_{s=N}^{n-1} \frac{1}{b_s} + \frac{c}{a_n} \sum_{s=N}^{n-1} \frac{1}{b_s} \sum_{t=N}^{s-1} q_t^{\frac{\beta}{\beta - \gamma}} p_t^{\frac{\gamma}{\gamma - \beta}} + \frac{1}{a_n} \sum_{s=N}^{n-1} \frac{1}{b_s} \sum_{t=N}^{s-1} e_t \right)^{\frac{1}{\alpha}}.$$

Using the fact that $\left(\sum_{i=1}^n u_i\right)^\delta \leq \sum_{i=1}^n u_i^\delta$ for $u_i \geq 0$, $i = 1, 2, \dots, n$ and $0 < \delta < 1$, and summing again, we obtain

$$x_n \leq x_N + \sum_{s=N}^{n-1} \left(\frac{c_2}{a_s}\right)^{\frac{1}{\alpha}} + \sum_{s=N}^{n-1} \left(\frac{c_1}{a_s} \sum_{t=N}^{s-1} \frac{1}{b_t}\right)^{\frac{1}{\alpha}} + \sum_{s=N}^{n-1} \left(\frac{c}{a_s} \sum_{t=N}^{s-1} \frac{1}{b_t} \sum_{j=N}^{t-1} q_j^{\frac{\beta}{\beta-\gamma}} p_j^{\frac{\gamma}{\gamma-\beta}}\right)^{\frac{1}{\alpha}} + \sum_{s=N}^{n-1} \left(\frac{1}{a_s} \sum_{t=N}^{s-1} \frac{1}{b_t} \sum_{j=N}^{t-1} e_j\right)^{\frac{1}{\alpha}}.$$

Taking limit infimum on both sides of the above inequality as $n \rightarrow \infty$, and applying conditions (3.5) - (3.7), we obtain a contradiction to $\{x_n\}$ being a positive solution. The proof for the case $\{x_n\}$ is eventually negative is similar. This completes the proof of the theorem. $\square$

4 Examples

In this section, we provide some examples to illustrate the above results.

Example 4.1. Consider the third order difference equation

$$\Delta^2(n^3(\Delta x_n)^3) + 2n^3 x_{n-3}^3 = n^3 \frac{x_{n-3}^3 \Delta x_n}{(1 + x_{n-3}^2)(1 + |\Delta x_n|)}, \quad n \geq 1. \quad (4.1)$$

Comparing to equation (1.1), we have $a_n = n^3$, $b_n = 1$ and $\sigma(n) = n - 3$, $f(u) = u^3$, $p(n) = 2n^3$, and $q_n = n^3$, and $g(n, x_n, x_{n-3}, \Delta x_n) = n^3 \frac{x_{n-3}^3 \Delta x_n}{(1 + x_{n-3}^2)(1 + |\Delta x_n|)}$. Then $Q_n = n^3$ and by taking $\tau(n) = n - 2$ and $\eta(n) = n - 1$, we have

$$I(n) = \sum_{s=1}^{n-4} \left(\frac{s-1}{s^3}\right)^{\frac{1}{3}} \approx 3n^{\frac{1}{3}}$$

and

$$B(n) = 2 \left(\sum_{s=n-3}^{n-2} \frac{1}{s}\right) \approx \frac{4}{n}.$$

It is easy to see that condition (2.12) is satisfied. Hence by Corollary 2.2, the equation (4.1) is oscillatory.

Example 4.2. Consider the third order difference equation

$$\Delta(4^n \Delta(2^n(\Delta x_n)^3)) + \frac{1}{4^n} x_n^5 = \frac{1}{16^n} x_n^3, \quad n \geq 1. \quad (4.2)$$

It is easy to see that all conditions of Theorem 3.1 are satisfied with $e_n = 0$ and so every nonoscillatory solution of (4.2) is bounded. One such bounded solution is $\{x_n\} = \left\{\frac{1}{2^n}\right\}$.

Example 4.3. Consider the third order difference equation

$$\Delta(n^3 \Delta(n^4(\Delta x_n)^3)) + \frac{1}{n^4} x_n^5 = \frac{1}{n^6} x_n^3 + (-1)^n n^6, \quad n \geq 1. \quad (4.3)$$

It is easy to check all the assumptions of Theorem 3.2 are satisfied and hence all solutions of equation (4.3) are oscillatory.

5 Conclusion

In this paper, we have derived some new sufficient conditions by using comparison techniques for the oscillation of all solutions of equations (1.1) and (1.5) which improve and extend that of in [16]. Finally, we provided three examples that illustrates the significance of the main results. It would be interesting to obtain results similar to this paper when $0 < \alpha < 1$.

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