Research Article

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Existence and stability of fourth-order nonlinear plate problem

<https://doi.org/10.1515/msds-2019-0006>

Received September 15, 2017; accepted November 21, 2019

Abstract: In this paper, we study a fourth-order plate problem as a model for a suspension bridge in the presence of a nonlinear frictional damping and a hanger restoring force. We establish the existence of a global weak solution and prove a stability result.

Keywords: Nonlinear, plate, fourth-order, stability, well-posedness

MSC: 35A01, 35L51, 35L71, 35B35, 35B41

1 Introduction

In this paper, we consider the following fourth-order nonlinear plate problem

$$\begin{cases} u_{tt} + \delta |u_t|^{p-2} u_t + \Delta^2 u + h(x, y, u) = f(x, y, t), & \text{in } \Omega \times (0, T), \\ u(0, y, t) = u_{xx}(0, y, t) = 0, & \text{for } (y, t) \in (-\ell, \ell) \times (0, T), \\ u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, & \text{for } (y, t) \in (-\ell, \ell) \times (0, T), \\ u_{yy}(x, \pm\ell, t) + \sigma u_{xx}(x, \pm\ell, t) = 0, & \text{for } (x, t) \in (0, \pi) \times (0, T), \\ u_{yyy}(x, \pm\ell, t) + (2 - \sigma) u_{xxy}(x, \pm\ell, t) = 0, & \text{for } (x, t) \in (0, \pi) \times (0, T), \\ u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = u_1(x, y), & \text{in } \Omega. \end{cases} \quad (1.1)$$

where $\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2$, $u(x, y, t)$ represents the displacement of a suspension bridge in the downward direction, $2 \leq p < \infty$, $0 < \sigma < \frac{1}{2}$ is the Poisson ratio, h is the hangers restoring force, $f \in L^2(\Omega \times (0, T))$ is the forcing term and $\delta > 0$ is a frictional constant. This model is due to the ground work of Ferrero and Gazzola [5], where a suspension bridge is modelled through a rectangular thin plate Ω assumed to be hinged on the vertical edges

$$u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y), \quad \forall y \in (-\ell, \ell)$$

and free on the horizontal edges (see [3, 21])

$$u_{yy}(x, \pm\ell) + \sigma u_{xx}(x, \pm\ell) = 0, \quad u_{yyy}(x, \pm\ell) + (2 - \sigma) u_{xxy}(x, \pm\ell) = 0, \quad \forall x \in (0, \pi).$$

For $p = 2$, in (1.1), Ferrero and Gazzola [5] established the well-posedness and discussed many stationary problems. Al-Gwaiz et al. [1] also carried some analysis on the model suggested in [5] and the bending and stretching energies of a rectangular plate were derived. In 2014, Wang [22] considered the following fourth-order equation

$$u_{tt} + \delta u_t + \Delta^2 u + au = |u|^{m-2} u, \quad (1.2)$$

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where $a = a(x, y, t)$ is a bounded and measurable sign-changing function and $2 < m < +\infty$. He supplement (1.2) with the boundary conditions of (1.1) and initial data and established the existence and uniqueness of a local solution and a finite time blow up result. Recently, Gazzola and Wang [7] modelled suspension bridges through the Von Karman quasilinear plate equations. Berchio et al. [2] studied the structural instability of nonlinear plates modelling suspension bridges. For recently established plate problems, we refer the reader to Messaoudi and co-authors [13–16, 18–20] and to the new book [6] on mathematical models for suspension bridges by Gazzola.

Messaoudi in [17] considered the so-called Petrovsky system

$$\begin{cases} u_{tt} + \Delta^2 u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, & \text{in } \Omega \times (0, T), \\ u = \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega \times [0, T), \\ u(x, 0) = v_0(x), \quad u_t(x, 0) = v_1(x), & \text{in } \Omega, \end{cases} \quad (1.3)$$

where $a, b > 0$ are constants and $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with a smooth boundary $\partial\Omega$. For $p > m$, he proved the existence of a weak local solution. In addition, he showed that for negative initial energy, the local solution blows up in finite time. He also established the existence of global solution when $m \geq p$. The result in [17] has been improved by Chen and Zhou in [4]. Yang in [23] studied the following plate problem

$$\begin{cases} u_{tt} + \Delta^2 u + \delta u_t = \sum_{i=1}^n \alpha_i \left(\frac{\partial u}{\partial x_i} \right), & \text{in } \Omega \times (0, T), \\ u = \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega \times [0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \end{cases} \quad (1.4)$$

where $\delta \geq 0$ is a constant, $\alpha_i = \alpha_i(s)$, $i = 1, 2, \dots, N$, are known nonlinear functions and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. He established the existence of a global weak solution and an exponential decay result with some suitable conditions on the nonlinear terms and the initial data.

This paper is organized as follows: In Section 2, we state some useful assumptions and preliminary results needed in establishing our main result. In Section 3, we state and prove a global existence result for problem (1.1). Finally, in Section 4, we discuss the stability of problem (1.1).

2 Preliminaries

In this section, we state our assumptions and some preliminary materials needed for this work. For this, we assume the following hypotheses on h :

(A₁) $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing and locally lipschitz function with $h(\cdot, \cdot, 0) = 0$.

(A₂) $H(x, y, s) = \int_0^s h(x, y, \tau) d\tau$ is positive such that

$$sh(x, y, s) - H(x, y, s) \geq 0, \quad \forall s \in \mathbb{R}.$$

Example 2.1. The function

$$h(x, y, s) = a(x, y)|s|^{p-2}s, \quad p \geq 2,$$

such that $a(x, y) > 0$ and locally lipschitz satisfies conditions (A₁) and (A₂).

As in [5], let us consider the following space

$$H_*^2(\Omega) = \left\{ w \in H^2(\Omega) : w(0, y) = w(\pi, y) = 0, \quad \forall y \in (-\ell, \ell) \right\}, \quad (2.1)$$

together with the inner product

$$(u, v)_{H_*^2} = \int_{\Omega} [(\Delta u \Delta v + (1 - \sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})] dx dy, \quad (2.2)$$

and denote by $\mathcal{H}(\Omega)$ the dual of $H_*^2(\Omega)$. For the completeness of the space $H_*^2(\Omega)$, we recall some results by Ferrero and Gazzola in [5].

Lemma 2.1. [5] Assume $0 < \sigma < \frac{1}{2}$. Then, the norm $\|\cdot\|_{H_*^2(\Omega)}$ given by $\|\cdot\|_{H_*^2(\Omega)}^2 = (\cdot, \cdot)_{H_*^2(\Omega)}$ is equivalent to the usual $H^2(\Omega)$ -norm. Moreover, $H_*^2(\Omega)$, endowed with the scalar product $(\cdot, \cdot)_{H_*^2(\Omega)}$, is a Hilbert space.

Theorem 2.2. [5] Assume $0 < \sigma < \frac{1}{2}$ and let $f \in L^2(\Omega)$. Then, there exists a unique $u \in H_*^2(\Omega)$ such that

$$(u, v)_{H_*^2(\Omega)} = \int_{\Omega} f v, \quad \forall v \in H_*^2(\Omega). \quad (2.3)$$

Remark 2.1. The function $u \in H_*^2(\Omega)$ satisfying (2.3) is called the weak solution of the stationary problem

$$\begin{cases} \Delta^2 u = f, \\ u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0, \\ u_{yy}(x, \pm \ell) + \sigma u_{xx}(x, \pm \ell) = 0 \\ u_{yyy}(x, \pm \ell) + (2 - \sigma)u_{xyy}(x, \pm \ell) = 0. \end{cases} \quad (2.4)$$

Lemma 2.2 (Compacity Lemma [11]). Let V, X and W be Banach spaces such that $V \subset X \subset W$, V and W are reflexive and $V \subset W$ is compact. If (u^m) is bounded in $L^p((0, T); V)$ and (u_t^m) is bounded in $L^q((0, T); W)$, $1 < p, q < +\infty$. Then there exists a subsequence (u^l) such that

$$(u^l) \longrightarrow u \text{ strongly in } L^p((0, T); X).$$

Lemma 2.3. [11] Let $V \subset H \subset V^*$, with V a reflexive Banach space, H a Hilbert space and V^* the dual of V such that the embeddings are continuous. If $u \in L^p((0, T), V)$ and $u_t \in L^q((0, T), V^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$, then

$$u \in C([0, T], H).$$

Lemma 2.4. [8] Suppose X and Y are two Banach spaces with X continuously embedded in Y . If $u \in L^\infty((0, T), X)$ and $u \in C_w([0, T], Y)$, then $u \in C_w([0, T], X)$.

Lemma 2.5. For any $a, b \in \mathbb{R}$ and $p \geq 2$, we have

$$(|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq 2^{2-p}|a - b|^p. \quad (2.5)$$

Lemma 2.6. [5] Let $u \in H_*^2(\Omega)$ and assume that $1 \leq p < +\infty$. Then, there exists a constant $C = C(\Omega, p) > 0$ such that

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{H_*^2(\Omega)}.$$

□

Lemma 2.7 (Komornik [10]). Let $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be non-increasing function. Assume that there exist $\gamma > 0$ and $\lambda > 0$ such that

$$\int_s^\infty E^{1+\gamma}(t)dt \leq \lambda E(s), \quad 0 \leq s < \infty.$$

Then, there exists $C, \omega > 0$ such that for all $t \geq 0$, we have

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{\gamma}}}, \quad \text{if } \gamma > 0,$$

$$E(t) \leq Ce^{-\omega t}, \quad \text{if } \gamma = 0.$$

3 Well posedness

In this section, we state and prove a well-posedness result for problem (1.1). First, we give the definition of a weak solution of problem (1.1).

Definition 3.1. A function

$$u \in C([0, T], H_*^2(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap C^2([0, T], \mathcal{H}^*(\Omega))$$

and

$$u_t \in L^p(\Omega \times (0, T))$$

is called a weak solution of problem (1.1), if

$$\begin{cases} \frac{d}{dt} \int_{\Omega} u_t w + \delta \int_{\Omega} |u_t|^{p-2} u_t w + (u, w)_{H_*^2} + \int_{\Omega} h(x, y, u) w = \int_{\Omega} f w, & \forall w \in H_*^2(\Omega), \\ u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = u_1(x, y), \end{cases} \quad (3.1)$$

for a.e. $t \in [0, T]$.

Now, we give the existence and uniqueness result for problem (1.1).

Theorem 3.1. Suppose $f \in L^2(\Omega \times (0, T))$ and that the assumptions (A_1) and (A_2) hold.

Let $(u_0, u_1) \in H_*^2(\Omega) \times L^2(\Omega)$ be given. Then problem (1.1) has a unique global weak solution.

Proof. We exploit the Galerkin approximation method. Note that $H_*^2(\Omega)$ is separable, so it has a countable basis. Let $\{w_j\}_{j=1}^{\infty}$ be an orthonormal basis of $H_*^2(\Omega)$ and define the finite-dimensional subspace

$$W_m = \text{span}\{w_1, \dots, w_m\}.$$

Let

$$u_0^m(x, y) = \sum_{j=1}^m \alpha_j w_j(x, y) \quad \text{and} \quad u_1^m(x, y) = \sum_{j=1}^m \beta_j w_j(x, y)$$

be sequences in $H_*^2(\Omega)$ and $L^2(\Omega)$ respectively, such that

$$u_0^m \longrightarrow u_0 \text{ in } H_*^2(\Omega), \quad u_1^m \longrightarrow u_1 \text{ in } L^2(\Omega). \quad (3.2)$$

We look for a solution of the form

$$u^m(x, y, t) = \sum_{j=1}^m a_j(t) w_j(x, y)$$

which satisfies the following approximation problem

$$\begin{cases} \frac{d}{dt} \int_{\Omega} u_t^m w_j + \delta \int_{\Omega} |u_t^m|^{p-2} u_t^m w_j + (u^m, w_j)_{H_*^2} + \int_{\Omega} h(x, y, u^m) w_j = \int_{\Omega} f w_j, \\ j = 1, 2, \dots, m \\ u^m(x, y, 0) = u_0^m(x, y), \quad u_t^m(x, y, 0) = u_1^m(x, y) \end{cases} \quad (3.3)$$

We note that (3.3) is equivalent to a finite-dimensional system of ODE. Thus, standard ODE theory guarantees the existence of differentiable functions a_j , $j = 1, 2, \dots, m$ such that

$$a_j : [0, t_m) \rightarrow \mathbb{R}, \quad 0 < t_m \leq T,$$

for which (3.3) is satisfied for a.e. $0 < t < t_m$.

Next, we show that $t_m = T$, $\forall m \geq 1$. For this, we multiply (3.3) by a_j' and sum over $j = 1, 2, \dots, m$ to get

$$\frac{d}{dt} \left[\frac{1}{2} \int_{\Omega} |u_t^m|^2 + \frac{1}{2} \|u^m\|_{H_*^2}^2 + \int_{\Omega} H(x, y, u^m) \right] + \delta \int_{\Omega} |u_t^m|^p = \int_{\Omega} f u_t^m. \quad (3.4)$$

Integrating equation (3.4) over $(0, t)$, $0 < t < t_m$, we obtain

$$E^m(t) + \delta \int_0^t \int_{\Omega} |u_t^m|^p = E^m(0) + \int_0^t \int_{\Omega} f u_t^m,$$

where

$$E^m(t) = \frac{1}{2} \int_{\Omega} |u_t^m|^2 + \frac{1}{2} \|u^m\|_{H_*^2}^2 + \int_{\Omega} H(x, y, u^m).$$

Using Young's inequality, we get, $\forall t \in (0, t_m)$,

$$E^m(t) + \delta \int_0^t \int_{\Omega} |u_t^m|^p \leq E^m(0) + \epsilon \int_0^t \int_{\Omega} |u_t^m|^2 + C_{\epsilon} \int_0^T \int_{\Omega} |f|^2. \quad (3.5)$$

We know that

$$H(x, y, s) = \int_0^s h(x, y, \tau) d\tau.$$

Thus, on the account of assumption (A_2) and Lemma 2.6, we get

$$\int_{\Omega} |H(x, y, u_0^m)| dx dy \leq \int_{\Omega} |u_0^m(x, y)| |h(x, y, u_0^m)| dx dy \leq C \|u_0^m\|_{H^2(\Omega)}^2. \quad (3.6)$$

Now, combining (3.5) and (3.6), with the fact that (u_0^m) and (u_1^m) are bounded sequences (as convergent sequences) in $H_*^2(\Omega)$ and $L^2(\Omega)$ respectively, we obtain

$$\frac{1}{2} \int_{\Omega} |u_t^m|^2 + \frac{1}{2} \|u^m\|_{H_*^2}^2 + \int_{\Omega} H(x, y, u^m) + \delta \int_0^t \int_{\Omega} |u_t^m|^p \leq C_{\epsilon} + \epsilon T \sup_{(0, t_m)} \int_{\Omega} |u_t^m|^2. \quad (3.7)$$

Observe that each term in (3.7) is positive, so we can take supremum over

$t \in (0, t_m)$ to get

$$\begin{aligned} & \frac{1}{2} \sup_{(0, t_m)} \int_{\Omega} |u_t^m|^2 + \frac{1}{2} \sup_{(0, t_m)} \|u^m\|_{H_*^2}^2 + \sup_{(0, t_m)} \int_{\Omega} H(x, y, u^m) \\ & + \delta \int_0^T \int_{\Omega} |u_t^m|^p \leq 4C_{\epsilon} + 4\epsilon T \sup_{(0, t_m)} \int_{\Omega} |u_t^m|^2. \end{aligned}$$

We choose $\epsilon = \frac{1}{16T}$ and obtain

$$\begin{aligned} & \frac{1}{4} \sup_{(0, t_m)} \int_{\Omega} |u_t^m|^2 + \frac{1}{2} \sup_{(0, t_m)} \|u^m\|_{H_*^2}^2 \\ & + \sup_{(0, t_m)} \int_{\Omega} H(x, y, u^m) + \delta \int_0^T \int_{\Omega} |u_t^m|^p \leq C, \end{aligned} \quad (3.8)$$

where C is a constant independent of m . So, we can extend the interval of solution of (3.3) to $[0, T]$. Moreover, we deduce from (3.8) that

$$\begin{cases} (u^m) \text{ is bounded in } L^\infty((0, T), H_*^2(\Omega)), \\ (u_t^m) \text{ is bounded in } L^\infty((0, T), L^2(\Omega)) \cap L^p(\Omega \times (0, T)). \end{cases} \quad (3.9)$$

Therefore, we can extract a subsequence (u^k) of (u^m) such that

$$\begin{cases} u^k \rightharpoonup u \text{ weakly star in } L^\infty((0, T), H_*^2(\Omega)) \text{ and weakly in } L^2((0, T), H_*^2(\Omega)), \\ u_t^k \rightharpoonup u_t \text{ weakly star in } L^\infty((0, T), L^2(\Omega)) \text{ and weakly in } L^2((0, T), L^2(\Omega)), \\ u_t^k \rightharpoonup u_t \text{ weakly in } L^p((0, T), L^p(\Omega)). \end{cases} \quad (3.10)$$

Since u_t^k is bounded in $L^p(\Omega \times (0, T))$, we easily see that

$$\int_0^T \int_\Omega |u_t^k|^{p-2} u_t^k \frac{p}{p-1} = \int_0^T \int_\Omega |u_t^k|^{p-1} \frac{p}{p-1} = \int_0^T \int_\Omega |u_t^k|^p \leq C.$$

So,

$$(|u_t^k|^{p-2} u_t^k)_{k \geq 1} \text{ is a bounded sequence in } L^{\frac{p}{p-1}}(\Omega \times (0, T)).$$

Thus, we can extract a subsequence denoted by $(|u_t^r|^{p-2} u_t^r)_{r \geq 1}$ such that

$$|u_t^r|^{p-2} u_t^r \rightharpoonup \psi \text{ weakly in } L^{\frac{p}{p-1}}(\Omega \times (0, T)), \quad (3.11)$$

as $r \rightarrow \infty$. Now, we have that

$$\begin{cases} u^r \rightharpoonup u \text{ weakly in } L^2(0, T), H_*^2(\Omega), \\ u_t^r \rightharpoonup u_t \text{ weakly in } L^2(0, T), L^2(\Omega). \end{cases} \quad (3.12)$$

Hence due to the compact embedding of $H_*^2(\Omega)$ in $L^2(\Omega)$ and Lemma 2.2, we can extract a subsequence, denoted by (u^l) such that

$$u^l \rightarrow u \text{ strongly in } L^2(0, T), L^2(\Omega) \quad (3.13)$$

and

$$u^l \rightarrow u \text{ a.e in } \Omega \times (0, T).$$

Thus, using the Lipschitz continuity of h , we obtain

$$h(x, y, u^l(x, y, t)) \rightarrow h(x, y, u(x, y, t)) \text{ a.e in } \Omega \times (0, T).$$

Now, we replace (u^m) by (u^l) in (3.3) and integrate over $(0, t)$ to get

$$\begin{aligned} & \int_\Omega u_t^l w_j + \delta \int_0^t \int_\Omega |u_t^l|^{p-2} u_t^l w_j + \int_0^t (u^l, w_j)_{H_*^2} \\ & + \int_0^t \int_\Omega h(x, y, u^l) w_j = \int_\Omega u_1^l w_j + \int_0^t \int_\Omega f w_j, \quad \forall j \leq l. \end{aligned} \quad (3.14)$$

By using (3.9) and the embedding of $H_*^2(\Omega)$ in $L^\infty(\Omega)$ (since $\Omega \subset \mathbb{R}^2$), we get that (u^l) is bounded in $L^\infty(\Omega \times (0, T))$. Thus, there exists a constant $C > 0$ such that

$$\|u^l\|_{L^\infty(\Omega \times (0, T))} \leq C, \quad \forall l \geq 1. \quad (3.15)$$

It follows, from assumption (A_1) , Hölder's inequality and (3.13), that

$$\begin{aligned} \left| \int_0^t \int_{\Omega} h(x, y, u^l) w_j - \int_0^t \int_{\Omega} h(x, y, u) w_j \right| &\leq \int_0^t \int_{\Omega} |h(x, y, u^l) - h(x, y, u)| |w_j| \\ &\leq C \|w_j\|_{L^2(\Omega)} \int_0^t \|u^l - u\|_{L^2(\Omega)} \\ &\leq CT \sup_{t \in (0, T)} \|u^l - u\|_{L^2(\Omega)} \rightarrow 0, \text{ as } l \rightarrow +\infty. \end{aligned} \quad (3.16)$$

By letting $l \rightarrow +\infty$ in equation (3.14), we obtain

$$\begin{aligned} \int_{\Omega} u_t w_j + \delta \int_0^t \int_{\Omega} \psi w_j + \int_0^t (u, w_j)_{H_*^2} \\ + \int_0^t \int_{\Omega} h(x, y, u) w_j = \int_{\Omega} u_1 w_j + \int_0^t \int_{\Omega} f w_j, \quad \forall j \geq 1, \end{aligned}$$

which gives

$$\begin{aligned} \int_{\Omega} u_t w &= -\delta \int_0^t \int_{\Omega} \psi w - \int_0^t (u, w)_{H_*^2} - \int_0^t \int_{\Omega} h(x, y, u) w \\ &+ \int_{\Omega} u_1 w + \int_0^t \int_{\Omega} f w, \quad \forall w \in H_*^2(\Omega). \end{aligned} \quad (3.17)$$

Observe that each term on the right-hand side of (3.17) is absolutely continuous as functions of t defined by integrals over $(0, t)$, hence differentiable almost everywhere. Thus, differentiating equation (3.17), we obtain, for almost every $t \in (0, T)$,

$$\frac{d}{dt} \int_{\Omega} u_t w + \delta \int_{\Omega} \psi w + (u, w)_{H_*^2} + \int_{\Omega} h(x, y, u) w = \int_{\Omega} f w, \quad \forall w \in H_*^2(\Omega). \quad (3.18)$$

Next, we show that

$$|u_t|^{p-2} u_t = \psi.$$

Let $A(w) = |w|^{p-2} w$ and define χ^l as follows

$$\chi^l = \int_0^T \int_{\Omega} (A(u_t^l) - A(w))(u_t^l - w), \quad \forall w \in L^p((0, T), H_*^2(\Omega)). \quad (3.19)$$

Then $\chi^l \geq 0$ by Lemma 2.4. We have

$$\begin{aligned} \chi^l &= \int_0^T \int_{\Omega} A(u_t^l) u_t^l - \int_0^T \int_{\Omega} A(u_t^l) w - \int_0^T \int_{\Omega} A(w) (u_t^l - w) \\ &= \int_0^T \int_{\Omega} |u_t^l|^p - \int_0^T \int_{\Omega} A(u_t^l) w - \int_0^T \int_{\Omega} A(w) (u_t^l - w). \end{aligned} \quad (3.20)$$

By replacing (u^m) by (u^l) in (3.4), integrating over $(0, T)$ and replacing in (3.20), we get

$$\begin{aligned} \chi^l &= -\frac{1}{2\delta} \left[\int_{\Omega} |u_t^l|^2 + \|u^l\|_{H_x^2}^2 + 2 \int_{\Omega} H(., ., u^l) \right] \\ &+ \frac{1}{2\delta} \left[\int_{\Omega} |u_1^l|^2 + \|u_0^l\|_{H_x^2}^2 + 2 \int_{\Omega} H(., ., u_0^l) \right] \\ &+ \frac{1}{\delta} \int_0^T \int_{\Omega} f u_t^l - \int_0^T \int_{\Omega} A(u_t^l) w - \int_0^T \int_{\Omega} A(w)(u_t^l - w). \end{aligned} \quad (3.21)$$

We know that, if $x_n \rightharpoonup x$ in X (X a Banach space), then

$$\|x\|_X \leq \liminf \|x_n\|_X. \quad (3.22)$$

Also, using the same reasoning as in (3.15)-(3.16), Hölder's inequality and the fact that $u_0^l \rightarrow u_0$ strongly in $H_x^2(\Omega)$, we have

$$\begin{aligned} \left| \int_{\Omega} (H(., ., u_0^l) - H(., ., u_0)) \right| &\leq \int_{\Omega} |h(x, y, \tilde{u}_0)| |u_0^l - u_0| \\ &\leq C \int_{\Omega} |\tilde{u}_0| |u_0^l - u_0| \leq \tilde{C} \|u_0^l - u_0\|_{H_x^2(\Omega)} \rightarrow 0, \text{ as } l \rightarrow \infty, \end{aligned} \quad (3.23)$$

where $\tilde{u}_0 = \alpha u_0^l + (1 - \alpha)u_0$, $\alpha \in (0, 1)$. Similarly, using (3.13) we can show that

$$\int_{\Omega} H(., ., u^l) \rightarrow \int_{\Omega} H(., ., u) \text{ as } l \rightarrow \infty. \quad (3.24)$$

Combining (3.22)-(3.24) and letting $l \rightarrow +\infty$, we obtain

$$\begin{aligned}
 0 &\leq \limsup \chi^l \\
 &\leq \limsup -\frac{1}{2\delta} \left[\int_{\Omega} |u_t^l|^2 + \|u^l\|_{H_x^2}^2 + 2 \int_{\Omega} H(.,., u^l) \right] \\
 &+ \limsup \frac{1}{2\delta} \left[\int_{\Omega} |u_1^l|^2 + \|u_0^l\|_{H_x^2}^2 + 2 \int_{\Omega} H(.,., u_0^l) \right] \\
 &+ \limsup \frac{1}{\delta} \int_0^T \int_{\Omega} f u_t^l - \int_0^T \int_{\Omega} A(u_t^l) w - \int_0^T \int_{\Omega} A(w)(u_t^l - w) \\
 &\leq -\liminf \frac{1}{2\delta} \left[\int_{\Omega} |u_t^l|^2 + \|u^l\|_{H_x^2}^2 + 2 \int_{\Omega} H(.,., u^l) \right] \\
 &+ \frac{1}{2\delta} \left[\int_{\Omega} |u_1|^2 + \|u_0\|_{H_x^2}^2 + 2 \int_{\Omega} H(.,., u_0) \right] \\
 &+ \frac{1}{\delta} \int_0^T \int_{\Omega} f u_t - \int_0^T \int_{\Omega} A(u_t) w - \int_0^T \int_{\Omega} A(w)(u_t - w) \\
 &\leq -\frac{1}{2\delta} \left[\int_{\Omega} |u_t|^2 + \|u\|_{H_x^2}^2 + 2 \int_{\Omega} H(.,., u) \right] \\
 &+ \frac{1}{2\delta} \left[\int_{\Omega} |u_1|^2 + \|u_0\|_{H_x^2}^2 + 2 \int_{\Omega} H(.,., u_0) \right] \\
 &+ \frac{1}{\delta} \int_0^T \int_{\Omega} f u_t - \int_0^T \int_{\Omega} \psi w - \int_0^T \int_{\Omega} A(w)(u_t - w). \tag{3.25}
 \end{aligned}$$

We note that (3.18) remains true for any $w \in C_0^\infty(\Omega \times (0, T))$, hence valid for any $w \in L^2(\Omega \times (0, T))$ by density. Thus, replacing w by u_t in (3.18) and integrating over $(0, T)$ leads to

$$\begin{aligned}
 \frac{1}{\delta} \int_0^T \int_{\Omega} f u_t &= \frac{1}{2\delta} \left[\int_{\Omega} |u_t(.,., T)|^2 + \|u(.,., T)\|_{H_x^2}^2 \right] - \frac{1}{2\delta} \left[\int_{\Omega} |u_1(.,., T)|^2 + \|u_0\|_{H_x^2}^2 \right] \\
 &+ \int_0^T \int_{\Omega} \psi u_t + \frac{1}{\delta} \left[\int_{\Omega} (H(.,., u(.,., T)) - H(.,., u_0)) \right] \tag{3.26}
 \end{aligned}$$

Substituting (3.26) into (3.25), we obtain

$$\begin{aligned}
 0 &\leq \limsup \chi^l \\
 &\leq \int_0^T \int_{\Omega} \psi u_t - \int_0^T \int_{\Omega} \psi w - \int_0^T \int_{\Omega} A(w)(u_t - w) \\
 &= \int_0^T \int_{\Omega} (\psi - A(w))(u_t - w).
 \end{aligned}$$

So,

$$\int_0^T \int_{\Omega} (\psi - A(w))(u_t - w) \geq 0, \quad \forall w \in L^p((0, T), H_x^2(\Omega)).$$

Now, let $w = \lambda v + u_t$ for $\lambda \neq 0$ and $v \in L^p(\Omega \times (0, T))$ (by density of $H_*^2(\Omega)$ in $L^p(\Omega)$). Then we get

$$-\lambda \int_0^T \int_{\Omega} (\psi - A(\lambda v + u_t)) v \geq 0, \quad \forall \lambda \neq 0, \quad \forall v \in L^p(\Omega \times (0, T)).$$

For $\lambda > 0$, we get

$$\int_0^T \int_{\Omega} (\psi - A(\lambda v + u_t)) v \leq 0. \quad (3.27)$$

For $\lambda < 0$, we obtain

$$\int_0^T \int_{\Omega} (\psi - A(\lambda v + u_t)) v \geq 0. \quad (3.28)$$

Claim: As $\lambda \searrow 0$, we obtain

$$\int_0^T \int_{\Omega} (\psi - A(u_t)) v \leq 0, \quad \forall v \in L^p(\Omega \times (0, T)). \quad (3.29)$$

To see this, we let (λ_n) be any sequence such that $\lambda_n \searrow 0$ as $n \rightarrow \infty$ and consider the sequence of functions $J(\lambda_n) = A(\lambda_n w + u_t)$. By using the continuity of A , $J(\lambda_n) \rightarrow A(u_t)$ as $n \rightarrow \infty$ (pointwise). Also, since (λ_n) is bounded (as a convergent sequence), we have

$$\begin{aligned} |J(\lambda_n)| &= |\lambda_n w + u_t|^{p-1} |v| \leq 2^{p-2} (|\lambda_n|^{p-1} |w|^{p-1} + |u_t|^{p-1}) |v| \\ &\leq C(|w|^{p-1} + |u_t|^{p-1}) |v| \in L^1(\Omega \times (0, T)). \end{aligned}$$

By applying the dominated convergent theorem to (3.27), we obtain the claim.

Similarly, as $\lambda \nearrow 0$, we get

$$\int_0^T \int_{\Omega} (\psi - A(u_t)) v \geq 0. \quad (3.30)$$

Combining (3.29) and (3.30), it follows that

$$\psi = A(u_t) = |u_t|^{p-2} u_t.$$

Therefore equation (3.18) becomes

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u_t w + \delta \int_{\Omega} |u_t|^{p-2} u_t w + (u, w)_{H_*^2} \\ + \int_{\Omega} h(x, y, u) w = \int_{\Omega} f w, \quad \forall w \in L^p((0, T), H_*^2(\Omega)). \end{aligned} \quad (3.31)$$

Now, from (2.13), we have

$$\begin{cases} u^l \rightharpoonup u \text{ weakly in } L^2((0, T), H_*^2(\Omega)), \\ u_t^l \rightharpoonup u_t \text{ weakly in } L^2((0, T), L^2(\Omega)). \end{cases} \quad (3.32)$$

Thus by Lions' lemma 2.3, we conclude that, up to a subsequence,

$$u^l \rightarrow u \text{ in } C([0, T], L^2(\Omega)). \quad (3.33)$$

As in [5] and [12], let $\phi \in C_0^\infty(0, T)$ and replace u^m by u^l in (3.3) to get for any $j \leq l$

$$\begin{aligned} - \int_0^T (u_t^l(t), w_j)_{L^2(\Omega)} \phi'(t) dt &= -\delta \int_0^T (|u_t^l(t)|^{p-2} u_t^l(t), w_j)_{L^2(\Omega)} \phi(t) dt \\ &\quad - \int_0^T (u^l(t), w_j)_{H_*^2(\Omega)} \phi(t) dt \\ &\quad - \int_0^T (h(\cdot, \cdot, u^l(t)), w_j)_{L^2(\Omega)} \phi(t) dt \\ &\quad + \int_0^T (f(t), w_j)_{L^2(\Omega)} \phi(t) dt. \end{aligned} \quad (3.34)$$

On the account of (3.16) and letting $l \rightarrow +\infty$, we obtain that

$$\begin{aligned} - \int_0^T (u_t(t), w_j)_{L^2(\Omega)} \phi'(t) dt &= -\delta \int_0^T (|u_t(t)|^{p-2} u_t(t), w_j)_{L^2(\Omega)} \phi(t) dt \\ &\quad - \int_0^T (u(t), w_j)_{H_*^2(\Omega)} \phi(t) dt \\ &\quad - \int_0^T (h(\cdot, \cdot, u(t)), w_j)_{L^2(\Omega)} \phi(t) dt \\ &\quad + \int_0^T (f(t), w_j)_{L^2(\Omega)} \phi(t) dt, \quad \forall j \geq 1, \end{aligned}$$

which gives

$$\begin{aligned} - \int_0^T (u_t(t), w)_{L^2(\Omega)} \phi'(t) dt &= -\delta \int_0^T (|u_t(t)|^{p-2} u_t(t), w)_{L^2(\Omega)} \phi(t) dt \\ &\quad - \int_0^T (u(t), w)_{H_*^2(\Omega)} \phi(t) dt \\ &\quad - \int_0^T (h(\cdot, \cdot, u(t)), w)_{L^2(\Omega)} \phi(t) dt \\ &\quad + \int_0^T (f(t), w)_{L^2(\Omega)} \phi(t) dt, \quad \forall w \in H_*^2(\Omega). \end{aligned}$$

So, $u_{tt} \in L^2([0, T], \mathcal{H}^*(\Omega))$. Thus,

$$u_t \in L^2((0, T), L^2(\Omega)), \quad u_{tt} \in L^2([0, T], \mathcal{H}^*(\Omega))$$

which gives

$$u_t \in C([0, T], \mathcal{H}(\Omega)). \quad (3.35)$$

Next, we show that our solution u satisfies the regularity in definition 3.1. It follows from (3.12) and (3.33) that

$$u \in L^\infty((0, T), H_*^2(\Omega)) \cap C([0, T], L^2(\Omega)).$$

Thus, we get, from Lemma 2.4,

$$u \in C_w([0, T], H_*^2(\Omega)). \quad (3.36)$$

Again, from (3.12) and (3.35), we obtain

$$u_t \in L^\infty((0, T), L^2(\Omega)) \cap C([0, T], \mathcal{H}^*(\Omega)).$$

So, we get, from Lemma 2.4,

$$u \in C_w^1([0, T], L^2(\Omega)), \quad (3.37)$$

where C_w means weak continuity. Thus combining (3.36) and (3.37), we have

$$u \in C_w([0, T], H_*^2(\Omega)) \cap C_w^1([0, T], L^2(\Omega)). \quad (3.38)$$

To have the regularity in definition 3.1 (strong continuity), we just show that the function

$$t \longmapsto g(t) = \|u_t(t)\|_{L^2}^2 + \|u(t)\|_{H_*^2}^2 \text{ is continuous in } [0, T].$$

For this, we multiply (3.31) by u_t and integrate over Ω to obtain

$$\frac{d}{dt}g(t) = -2\delta\|u_t\|_{L^p}^p - 2(h(., ., u), u_t)_{L^2} + 2(f, u_t)_{L^2}. \quad (3.39)$$

Integrating both sides over (t_1, t_2) and recalling (3.12), we see that

$$\begin{aligned} |g(t_2) - g(t_1)| &= \left| \int_{t_1}^{t_2} (-2\delta\|u_t\|_{L^p}^p - 2(h(., ., u), u_t)_{L^2} + 2(f, u_t)_{L^2}) dt \right| \\ &\leq \int_{t_1}^{t_2} \left(2\delta\|u_t\|_{L^p}^p + 2\|u_t\|_{L^2}^2 + C_e\|u\|_{H_*^2}^2 + \|f\|_{L^2}^2 \right) dt. \end{aligned} \quad (3.40)$$

Observe that, from (3.12), the integrands in the right-hand side of (3.40) are $L^1([0, T], \mathbb{R})$, thus

$$|g(t_2) - g(t_1)| \rightarrow 0 \text{ as } t_2 \rightarrow t_1.$$

Hence,

$$u \in C([0, T], H_*^2(\Omega)) \cap C^1([0, T], L^2(\Omega)). \quad (3.41)$$

For the initial conditions, we let $\phi \in C^2([0, T], H_*^2(\Omega))$ with $\phi(T) = \phi_t(T) = 0$. Then by using (3.31), integration by parts leads to

$$\begin{aligned} &\int_0^T (u, \phi_{tt}) dt + \delta \int_0^T (|u_t|^{p-2} u_t, \phi) dt + \int_0^T (u, w)_{H_*^2} dt \\ &+ \int_0^T (h(., ., u), \phi) dt = \int_\Omega (f, \phi) dt + (u_t(0), \phi(0)) - (u(0), \phi_t(0)). \end{aligned} \quad (3.42)$$

Again, replacing (u^m) by (u^l) in (3.3) and integrating by parts gives

$$\begin{aligned} &\int_0^T (u^l, \phi_{tt}) dt + \delta \int_0^T (|u_t^l|^{p-2} u_t^l, \phi) dt + \int_0^T (u^l, w)_{H_*^2} dt \\ &+ \int_0^T (h(., ., u^l), \phi) dt = \int_\Omega (f, \phi) dt + (u_t^l(0), \phi(0)) - (u^l(0), \phi_t(0)). \end{aligned}$$

As $l \rightarrow \infty$, we obtain

$$\begin{aligned} & \int_0^T (u, \phi_{tt}) dt + \delta \int_0^T (|u_t|^{p-2} u_t, \phi) dt + \int_0^T (u, w)_{H_*^2} dt \\ & + \int_0^T (h(\cdot, \cdot, u), \phi) dt = \int_\Omega (f, \phi) dt + (u_1, \phi(0)) - (u_0, \phi_t(0)), \end{aligned} \quad (3.43)$$

since $u^l(0) \rightarrow u_0$ and $u_t^l(0) \rightarrow u_1$. Since $\phi(0)$ and $\phi_t(0)$ are arbitrary, a comparison of (3.42) and (3.43) gives

$$u(x, y, 0) = u_0(x, y), \quad \text{and} \quad u_t(x, y, 0) = u_1(x, y) \quad \text{in } \Omega. \quad (3.44)$$

For the uniqueness, suppose u^1 and u^2 satisfy (3.31) and (3.44). Then $u = u^1 - u^2$ satisfies

$$\begin{cases} \frac{d}{dt} \int_\Omega u_t w + \delta \int_\Omega (|u_t^1|^{p-2} u_t^1 - |u_t^2|^{p-2} u_t^2) w + (u, w)_{H_*^2} \\ \quad + \int_\Omega [h(x, y, u^1) - h(x, y, u^2)] w = 0, \quad \forall w \in L^p((0, T), H_*^2(\Omega)), \\ u(x, y, 0) = u_t(x, y, 0) = 0. \end{cases} \quad (3.45)$$

The system (3.45) remains true for any $w \in C_0^\infty(\Omega \times (0, T))$. Hence, valid for any $w \in L^2(\Omega \times (0, T))$ by density. Thus, replacing w by u_t in (3.45), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega |u_t|^2 + \frac{1}{2} \frac{d}{dt} \|u\|_{H_*^2}^2 + \delta \int_\Omega (|u_t^1|^{p-2} u_t^1 - |u_t^2|^{p-2} u_t^2) (u_t^1 - u_t^2) \\ & + \int_\Omega [h(x, y, u^1) - h(x, y, u^2)] u_t = 0. \end{aligned}$$

By using the assumption (A_1) and Lemma 2.6, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\int_\Omega |u_t|^2 + \|u\|_{H_*^2}^2 \right] + \delta \int_\Omega (|u_t^1|^{p-2} u_t^1 - |u_t^2|^{p-2} u_t^2) (u_t^1 - u_t^2) \\ & \leq \int_\Omega [h(x, y, u^1) - h(x, y, u^2)] |u_t| \\ & \leq L_I \int_\Omega |u| |u_t| \leq L_I \|u\|_{L^2} \|u_t\|_{L^2} \\ & \leq \frac{L_I}{2} \|u\|_{L^2}^2 + \frac{L_I}{2} \|u_t\|_{L^2}^2 \leq C \left[\|u_t\|_{L^2}^2 + \|u\|_{H_*^2}^2 \right], \end{aligned} \quad (3.46)$$

where L_I is a lipschitz constant. Thus, by using Lemma 2.5 and (3.46), we obtain

$$\frac{d}{dt} \left[\|u_t\|_{L^2}^2 + \|u\|_{H_*^2}^2 \right] \leq C \left[\|u_t\|_{L^2}^2 + \|u\|_{H_*^2}^2 \right]. \quad (3.47)$$

Applying Gronwall's Lemma to (3.47) lead to

$$\|u_t\|_{L^2}^2 + \|u\|_{H_*^2}^2 \leq C e^{ct} \left[\|u_t(0)\|_{L^2}^2 + \|u(0)\|_{H_*^2}^2 \right] = 0.$$

It follows that $u^1 = u^2$. This complete the proof of Theorem 3.1. $\square$

4 Stability

In this section we state and prove a stability result for problem (1.1) with $f \equiv 0$, i.e. the problem

$$\begin{cases} u_{tt} + \delta |u_t|^{p-2} u_t + \Delta^2 u + h(x, y, u) = 0, & \text{in } \Omega \times (0, T), \\ u(0, y, t) = u_{xx}(0, y, t) = 0, & \text{for } (y, t) \in (-\ell, \ell) \times (0, T), \\ u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, & \text{for } (y, t) \in (-\ell, \ell) \times (0, T), \\ u_{yy}(x, \pm \ell, t) + \sigma u_{xx}(x, \pm \ell, t) = 0, & \text{for } (x, t) \in (0, \pi) \times (0, T), \\ u_{yyy}(x, \pm \ell, t) + (2 - \sigma) u_{xxy}(x, \pm \ell, t) = 0, & \text{for } (x, t) \in (0, \pi) \times (0, T), \\ u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = u_1(x, y), & \text{in } \Omega. \end{cases} \quad (4.1)$$

The energy functional associated to problem (4.1) is defined by

$$E(t) = \frac{1}{2} \int_{\Omega} |u_t|^2 + \frac{1}{2} \|u\|_{H_x^2(\Omega)}^2 + \int_{\Omega} H(x, y, u). \quad (4.2)$$

Lemma 4.1. The energy functional defined in (4.2) satisfies,

$$\frac{dE(t)}{dt} = -\delta \int_{\Omega} |u_t|^p \leq 0. \quad (4.3)$$

Proof. Multiply (4.1) by u_t and integrate over Ω , we obtain

$$\frac{d}{dt} \left[\frac{1}{2} \int_{\Omega} |u_t|^2 + \frac{1}{2} \|u\|_{H_x^2(\Omega)}^2 + \int_{\Omega} H(x, y, u) \right] + \delta \int_{\Omega} |u_t|^p = 0. \quad (4.4)$$

This lead to

$$\frac{dE(t)}{dt} = -\delta \int_{\Omega} |u_t|^p \leq 0. \quad (4.5)$$

Thus, E is non-increasing. $\square$

Theorem 4.1. Let $(u_0, u_1) \in H_x^2(\Omega) \times L^2(\Omega)$ be given and assume (A_1) and (A_2) hold. Then the energy functional E satisfies, $\forall t \geq 0$,

$$E(t) \leq \frac{C}{(1+t)^{\frac{2}{p-2}}}, \quad \text{if } p > 2$$

$$E(t) \leq C e^{-\omega t}, \quad \text{if } p = 2,$$

where C, ω are positive constants.

Proof. Multiply (4.1) by $E^q(t)u$ for q to be specified later and integrate over $\Omega \times (s, T)$, $T > s$, to get

$$\int_s^T E^q(t) \left(\int_{\Omega} u_{tt} u + \|u\|_{H_x^2(\Omega)}^2 + \delta \int_{\Omega} |u_t|^{p-2} u_t u + \int_{\Omega} h(x, y, u) u \right) = 0.$$

Simple calculations lead to

$$\begin{aligned} & \int_s^T E^q(t) \left(\int_{\Omega} [(u u_t)_t - u_t^2] + \|u\|_{H_x^2(\Omega)}^2 \right) \\ & + \int_s^T E^q(t) \left(\delta \int_{\Omega} |u_t|^{p-2} u_t u + \int_{\Omega} h(., ., u) u \right) = 0, \end{aligned}$$

which gives

$$\begin{aligned} & \int_s^T \int_{\Omega} E^q(t)(uu_t)_t + \int_s^T E^q(t) \left(\frac{1}{2} \int_{\Omega} u_t^2 + \frac{1}{2} \|u\|_{H_*^2(\Omega)}^2 + \int_{\Omega} H(.,., u) \right) \\ & - \frac{3}{2} \int_s^T \int_{\Omega} E^q(t) u_t^2 + \frac{1}{2} \int_s^T E^q(t) \|u\|_{H_*^2(\Omega)}^2 + \delta \int_s^T \int_{\Omega} E^q(t) |u_t|^{p-2} u_t u \\ & + \int_s^T E^q(t) \int_{\Omega} [uh(.,., u) - H(.,., u)] = 0. \end{aligned}$$

Then taking in account (4.2), we get

$$\begin{aligned} \int_s^T E^{q+1}(t) dt &= - \int_s^T \int_{\Omega} E^q(t)(uu_t)_t + \frac{3}{2} \int_s^T \int_{\Omega} E^q(t) u_t^2 \\ &- \frac{1}{2} \int_s^T E^q(t) \|u\|_{H_*^2(\Omega)}^2 - \delta \int_s^T \int_{\Omega} E^q(t) |u_t|^{p-2} u_t u \\ &+ \int_s^T E^q(t) \int_{\Omega} [H(.,., u) - uh(.,., u)]. \end{aligned}$$

By recalling assumption (A_2) , we obtain

$$\begin{aligned} \int_s^T E^{q+1}(t) dt &\leq - \int_s^T \int_{\Omega} [E^q(t)uu_t]_t + q \int_s^T \int_{\Omega} E^{q-1}(t) E'(t) uu_t \\ &- \delta \int_s^T \int_{\Omega} E^q(t) |u_t|^{p-2} u_t u + \frac{3}{2} \int_s^T \int_{\Omega} E^q(t) u_t^2. \end{aligned} \quad (4.6)$$

By using Young's inequality and Lemma 2.6, we estimate the integrals on the right-hand side of (4.6) as follows. For the first term, we have

$$\begin{aligned} \left| - \int_s^T \int_{\Omega} [E^q(t)uu_t]_t \right| &\leq \left| \int_{\Omega} E^q(T)u(T)u_t(T) \right| + \left| \int_{\Omega} E^q(s)u(s)u_t(s) \right| \\ &\leq \frac{1}{2} E^q(T) \int_{\Omega} (u^2(x, y, T) + u_t^2(x, y, T)) \\ &+ \frac{1}{2} E^q(s) \int_{\Omega} (u^2(x, y, s) + u_t^2(x, y, s)). \end{aligned}$$

Now, using the fact E is nonincreasing and

$$\frac{1}{2} \left(\|u(t)\|_{L^2(\Omega)}^2 + \|u_t(t)\|_{L^2(\Omega)}^2 \right) \leq \frac{1}{2} \left(C_e \|u(t)\|_{H_*^2(\Omega)}^2 + \|u_t(t)\|_{L^2(\Omega)}^2 \right) \leq CE(t),$$

we get

$$\left| - \int_s^T \int_{\Omega} (E^q(t)uu_t) \right| \leq C \left(E^{q+1}(T) + E^{q+1}(s) \right) \leq CE^{q+1}(s). \quad (4.7)$$

For the second term, we have

$$\begin{aligned}
 \left| q \int_s^T \int_{\Omega} E^{q-1}(t) E'(t) u u_t \right| &\leq -q \int_s^T \int_{\Omega} E^{q-1}(t) E'(t) |u u_t| \\
 &\leq -\frac{q}{2} \int_s^T E^{q-1}(t) E'(t) \int_{\Omega} (u^2(x, y, t) + u_t^2(x, y, t)) \\
 &\leq -C \int_s^T E^q(t) E'(t) = -C \int_s^T (E^{q+1}(t))' \leq C E^{q+1}(s).
 \end{aligned} \tag{4.8}$$

For the third term, we have

$$\begin{aligned}
 \left| -\delta \int_s^T \int_{\Omega} E^q(t) |u_t|^{p-2} u u_t \right| &\leq \delta \int_s^T E^q(t) \int_{\Omega} |u_t|^{p-1} |u| \\
 &\leq \delta \int_s^T E^q(t) \left(\epsilon \int_{\Omega} |u|^p + C_{\epsilon} \int_{\Omega} |u_t|^p \right) \\
 &= \delta \epsilon \int_s^T \int_{\Omega} E^q(t) |u|^p + C_{\epsilon} \int_s^T \int_{\Omega} E^q(t) (-E'(t)) \\
 &\leq C \delta \epsilon \int_s^T E^q(t) \|u\|_{H_x^2(\Omega)}^p + C_{\epsilon} E^{q+1}(s) \\
 &= C \delta \epsilon \int_s^T E^q(t) (\|u\|_{H_x^2(\Omega)}^2)^{\frac{p}{2}} + C_{\epsilon} E^{q+1}(s) \\
 &\leq C \delta \epsilon \int_s^T E^q(t) (E(t))^{\frac{p}{2}} + C_{\epsilon} E^{q+1}(s) \\
 &\leq \epsilon C \delta (E(0))^{\frac{p}{2}-1} \int_s^T E^{q+1}(t) + C_{\epsilon} E^{q+1}(s).
 \end{aligned} \tag{4.9}$$

For the fourth term, we first note that $p > 2$ and Ω is bounded, so

$$\|u_t\|_{L^2(\Omega)} \leq C \|u_t\|_{L^p(\Omega)}.$$

Thus, we obtain

$$\begin{aligned}
 \frac{3}{2} \int_s^T \int_{\Omega} E^q(t) u_t^2 &\leq \frac{3}{2} \int_s^T E^q(t) \|u_t\|_{L^2(\Omega)}^2 \leq C \int_s^T E^q(t) (\|u_t\|_{L^p(\Omega)}^p)^{\frac{2}{p}} \\
 &= C \int_s^T E^q(t) (-E'(t))^{\frac{2}{p}}.
 \end{aligned} \tag{4.10}$$

Now, using Young's inequality with $r = q + 1$ and $r' = \frac{q+1}{q}$, we arrive at

$$\frac{3}{2} \int_s^T \int_{\Omega} E^q(t) u_t^2 \leq \epsilon \int_s^T E^{q+1}(t) + C_{\epsilon} \int_s^T (-E'(t))^{\frac{2(q+1)}{p}}.$$

We choose q such that

$$\frac{2(q+1)}{p} = 1, \text{ hence } q = \frac{p-2}{2}.$$

Consequently, we get

$$\begin{aligned} \left| \frac{3}{2} \int_{\Omega} \int_s^T E^q(t) u_t^2 \right| &\leq \epsilon \int_s^T E^{q+1}(t) dt + C_{\epsilon} \int_s^T (-E'(t)) dt \\ &\leq \epsilon \int_s^T E^{q+1}(t) dt + C_{\epsilon} E(s). \end{aligned} \quad (4.11)$$

Combining (4.7), (4.8), (4.9) and (4.11), we obtain from (4.6) that

$$\begin{aligned} \int_s^T E^{q+1}(t) dt &\leq C E^{q+1}(s) + C \delta \epsilon E^{\frac{p}{2}-1}(0) \int_s^T E^{q+1}(t) \\ &\quad + C_{\epsilon} E^{q+1}(s) + \epsilon \int_s^T E^{q+1}(t) dt + C_{\epsilon} E(s). \end{aligned} \quad (4.12)$$

This leads to

$$\left(1 - \epsilon - \epsilon C \delta E^{\frac{p}{2}-1}(0)\right) \int_s^T E^{q+1}(t) dt \leq (C + C_{\epsilon}) E^{q+1}(s) + C_{\epsilon} E(s). \quad (4.13)$$

Now, we choose ϵ small enough so that $(1 - \epsilon - \epsilon C \delta E^{\frac{p}{2}-1}(0)) > 0$, to arrive at

$$\int_s^T E^{\frac{p}{2}}(t) dt \leq C(E^{\frac{p}{2}}(s) + E(s)) \leq \tilde{C} E(s) \quad (4.14)$$

By letting $T \rightarrow \infty$ and applying Lemma 2.7, we obtain the result. This completes the proof. $\square$

Acknowledgement: The authors thank KFUPM, University of Hafr Al Batin and University of Sharjah for their continuous support.

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