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Positive Solution to Singular Elliptic Problems with Subcritical nonlinearities

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Abstract: In this paper, we study the existence of a non-trivial weak solution to the following singular elliptic equations with subcritical nonlinearities:

$$\begin{cases} -\operatorname{div}(|x|^{-2\beta}\nabla u) - \mu \frac{f(x)u}{|x|^{2(\beta+1)}} = \frac{\lambda g(x)}{u^\theta} + h(x)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary, $\theta, \lambda > 0$, $0 < \beta < \frac{N-2}{2}$, $0 < p < 1$, $0 < \mu < \left(\frac{N-2(\beta+1)}{2}\right)^2$, $N \geq 3$, $0 \in \Omega$ and $0 \leq f, g, h \in L^\infty(\Omega)$. We show that there exists a solution $u \in H_0^1(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega)$ to this problem.

Keywords: Nonlinear elliptic equation, Hardy potential, positive solution

MSC: Primary 35J05, 35J25; Secondary 46E35

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with C^1 boundary, $0 \in \Omega$. We consider the following singular elliptic equations with subcritical nonlinearities:

$$\begin{cases} -\operatorname{div}(|x|^{-2\beta}\nabla u) - \mu \frac{f(x)u}{|x|^{2(\beta+1)}} = \frac{\lambda g(x)}{u^\theta} + h(x)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

under assumptions $\theta, \lambda > 0$, $0 < \beta < \frac{N-2}{2}$, $0 < p < 1$, $0 < \mu < \left(\frac{N-2(\beta+1)}{2}\right)^2$, $N \geq 3$, and $0 \leq f, g, h \in L^\infty(\Omega)$. We prove existence of (1.1) in $H_0^1(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega)$. Furthermore, a solution of (1.1) is a function u in $W_0^{1,1}(\Omega, |x|^{-2\beta})$ with the following:

$$\forall \Omega' \subset\subset \Omega, \exists c_{\Omega'} : u \geq c_{\Omega'} > 0 \text{ in } \Omega', \quad (1.2)$$

and

$$\int_{\Omega} |x|^{-2\beta} \nabla u \nabla \varphi - \mu \int_{\Omega} f(x) \frac{u\varphi}{|x|^{2(\beta+1)}} = \int_{\Omega} \frac{g(x)\varphi}{u^\theta} + \int_{\Omega} h(x)u^p \varphi, \quad \forall \varphi \in C_0^1(\Omega). \quad (1.3)$$

We pointed out that the right hand side is well defined by (1.2) since φ has compact support.

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The problem of the form (1.1) arise in certain problems in fluid mechanics, chemical heterogenous catalysts, pseudo plastic flow and non-Newtonian fluids (see [3, 9, 13, 14, 17]). Simpler form of the problem (1.1) is of the type

$$-\operatorname{div}(M(x)\nabla u) = h(x, u), \quad (1.4)$$

where $M(x) = |x|^{-2\beta}$ and $h(x, s)$ is singular for $s = 0$ and $x = 0$. Equation (1.4) with $M(x)$ is identity matrix, has been widely studied by many authors in the past, see, for instance [2, 5–7, 12, 16] and the reference therein. When $h(x, s)$ that does not depend on x , Crandall, Rabinowitz and Tartar in [7], showed the existence and continuity properties of the solution of (1.4). In [12], it has been shown that the existence of the solution to (1.4), for $h(x, u) = \frac{f(x)}{u^\gamma}$, where f is a continuous function. They showed that the solution $u \in H_0^1(\Omega)$ if and only if $\gamma < 3$ and that, if $\gamma > 1$, solution $u \notin C^1(\bar{\Omega})$. For, $h(x, u) = f(x)g(u)$, Lair and Shaker [10, 11] studied the problem (1.4), and by Zhang and Cheng, see [19], using the first eigenfunction of the Laplacian in Ω .

The present work is more close to results in [1, 8], although we generalised the results for sublinearities, i.e. for the case $0 < p < 1$ [1, Theorem 2.4]. The aim of this paper is to show the existence to (1.1) for sublinearities. To obtain a solution to (1.1), we use the similar techniques as used in [1, 2]. The organisation of this paper is as follows. Section 2 deals with some preliminaries, basic facts which are used in the main result. Section 3 is devoted to the proof of the main theorem.

The main result of this paper is the following theorem, which we will prove in the next section.

Theorem 1.1. Assume (1.1) with $0 < p < 1$, $\theta, \lambda > 0$, $0 < \beta < \frac{N-2}{2}$, $0 < \mu < \left(\frac{N-2(\beta+1)}{2}\right)^2$, and $0 \leq f, g, h \in L^\infty(\Omega)$. Then $\exists \Lambda \in (0, \infty)$ such that for $\lambda \in (0, \Lambda)$, $\exists$ a solution u in $H_0^1(\Omega, |x|^{-2\beta})$ to (1.1).

Moreover

– If $0 < \theta < 1$, then $u \in H_0^1(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega)$.

2 Auxiliary results

Let $n \in \mathbb{N}$, let $g_n(x) = \min\{g(x), n\}$. To prove the existence results we consider the following approximate problem:

$$\begin{cases} -\operatorname{div}(|x|^{-2\beta}\nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{\lambda g_n(x)}{(u_n + \frac{1}{n})^\theta} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Let us recall the following Caffarelli-Kohn-Nirenberg inequality and a compact embedding theorem (which is an extension of the classical Rellich-Kondrachov compactness theorem).

Proposition 2.1. Caffarelli-Kohn-Nirenberg inequality [4]: There is a constant $C_{\gamma,\beta} > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-2\beta} |u|^{2^*} \right)^{2/2^*} \leq C_{\gamma,\beta} \int_{\mathbb{R}^N} |x|^{-2\gamma} |\nabla u|^2, \quad (2.2)$$

for all $u \in C_0^\infty(\mathbb{R}^N)$, where

$$-\infty < \gamma < \frac{N-2}{2}, \quad \beta \leq \gamma < \beta + 1, \quad 2^* = \frac{2N}{N-2d}, \quad d = \gamma + 1 - \beta. \quad (2.3)$$

Let $H_0^1(\Omega, |x|^{-2\beta})$ be the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the following weighted norm $\|\cdot\|$ defined by

$$\|u\| = \left(\int_{\Omega} |x|^{-2\beta} |\nabla u|^2 \right)^{1/2}. \quad (2.4)$$

From the boundedness of Ω and the standard approximation arguments, it can be shown that (2.2) holds for any $u \in H_0^1(\Omega, |x|^{-2\beta})$ in the sense:

$$\left(\int_{\Omega} |x|^{-\beta q} |u|^q \right)^{2/q} \leq C_{\beta} \int_{\Omega} |x|^{-2\beta} |\nabla u|^2, \quad (2.5)$$

for $1 \leq q \leq 2^*$.

Proposition 2.2. *General Hardy-Sobolev inequality: There is a constant $C_{\beta,N,2} > 0$ such that*

$$\left(\int_{\Omega} |x|^{-2(\beta+1)} |u|^2 \right) \leq C_{\beta,N,2}^{-1} \int_{\Omega} |x|^{-2\beta} |\nabla u|^2, \quad (2.6)$$

for all $u \in H_0^1(\Omega, |x|^{-2\beta})$, where

$$-\infty < \beta < \frac{N-2}{2}, \quad C_{\beta,N,2} = \left(\frac{N-2(\beta+1)}{2} \right)^2. \quad (2.7)$$

Moreover, $C_{\beta,N,2}^{-1}$ is optimal and it is not achieved.

Let $L^q(\Omega, |x|^{-\beta q})$ be the weighted L^q space with weighted norm defined by

$$\|u\|_{\beta,q} := \left(\int_{\Omega} |x|^{-\beta q} |u|^q \right)^{1/q}. \quad (2.8)$$

then (2.5) can be written as

$$\|u\|_{\beta,q} \leq C \|u\|, \quad \forall u \in H_0^1(\Omega, |x|^{-2\beta}), \quad (2.9)$$

where $C > 0$ denotes a universal constant, which may change its value from line to line.

Proposition 2.3. *Compact imbedding theorem[18]: Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, $N > 2$, $-\infty < \gamma < \frac{N-2}{2}$, $\beta \leq \gamma \leq (1+\beta)$.*

For $1 \leq q \leq 2^$, the imbedding $H_0^1(\Omega, |x|^{-2\beta}) \hookrightarrow L^q(\Omega, |x|^{-\gamma q})$ is continuous.*

For $1 \leq q < 2^$, the imbedding $H_0^1(\Omega, |x|^{-2\beta}) \hookrightarrow L^q(\Omega, |x|^{-\gamma q})$ is compact.*

Lemma 2.4. *For $C_{\beta,N,2}^{-1} \|f\|_{L^\infty(\Omega)} < \frac{1}{\mu}$ where $0 < \beta < \frac{N-2}{2}$, $C_{\beta,N,2} = \left(\frac{N-2(\beta+1)}{2} \right)^2$. Then Problem (2.1) has a nonnegative solution in $H_0^1(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega)$.*

Proof. Let $n \in \mathbb{N}$ be a fixed natural no and ω be a function in $L^2(\Omega, |x|^{-2\beta})$. By the Lax-Milgram Theorem, the following problem has a unique solution u in $H_0^1(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega)$:

$$\begin{cases} -\operatorname{div}(|x|^{-2\beta} \nabla u) - \mu \frac{f(x)u}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{g_n(x)}{(|\omega| + \frac{1}{n})^\theta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.10)$$

because the operator is coercive due to the assumptions on μ and Proposition 2.2. Now, for any $u \in L^2(\Omega, |x|^{-2(\beta+1)})$, we define the mapping

$$T : L^2(\Omega, |x|^{-2(\beta+1)}) \rightarrow H_0^1(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega),$$

as follows:

$$T(u) = \omega. \quad (2.11)$$

Let us take u as a test function, we have

$$\int_{\Omega} |x|^{-2\beta} |\nabla u|^2 - \mu \int_{\Omega} \frac{f(x)u^2}{|x|^{2(\beta+1)} + \frac{1}{n}} = \int_{\Omega} \frac{g_n(x)u}{(|\omega| + \frac{1}{n})^\theta}. \quad (2.12)$$

and so

$$\int_{\Omega} |x|^{-2\beta} |\nabla u|^2 - \mu \int_{\Omega} \frac{f(x)u^2}{|x|^{2(\beta+1)} + \frac{1}{n}} \leq n^{\theta+1} \int_{\Omega} u, \quad (2.13)$$

Since $g_n(x) = \min\{g(x), n\}$, so $g_n(x) \leq n$, this implies that

$$\frac{g_n(x)u}{(|\omega| + \frac{1}{n})^\theta} \leq \frac{nu}{(n|\omega| + 1)^\theta} \times n^\theta \leq n^{\theta+1}u,$$

The General Hardy - Sobolev inequality on the left-hand side and Hölder inequality on the right-hand side implies that

$$(C_{\beta,N,2} - \mu\|f\|_\infty) \int_{\Omega} |x|^{-2(\beta+1)} |u|^2 \leq Cn^{\theta+1} \left(\int_{\Omega} u^2 \right)^{1/2}, \quad (2.14)$$

for some constant C independent of u .

This is equivalent to

$$(C_{\beta,N,2} - \mu\|f\|_\infty) \int_{\Omega} |x|^{-2(\beta+1)} |u|^2 \leq Cn^{\theta+1} |\Omega|^{2(\beta+1)} \left(\int_{\Omega} |x|^{-2(\beta+1)} u^2 \right)^{1/2}, \quad (2.15)$$

where $|\Omega|$ denotes the diameter of Ω .

This implies that

$$\leq Cn^{\theta+1} |\Omega|^{(\beta+1)} \left(\int_{\Omega} |x|^{-2(\beta+1)} u^2 \right)^{1/2}, \quad (2.16)$$

i.e. the ball of radius $Cn^{\theta+1}$ is invariant under T . Using compact imbedding, it can be shown that T is both continuous and compact on $L^2(\Omega, |x|^{-2(\beta+1)})$. So by Schauder's fixed point theorem, $\exists u_n \in H_0^1(\Omega, |x|^{-2\beta})$ such that $u_n = T(u_n)$, i.e., u_n solves

$$\begin{cases} -\operatorname{div}(|x|^{-2\beta} \nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{g_n(x)}{(u_n + \frac{1}{n})^\theta} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.17)$$

Since $\frac{g_n(x)}{(\omega + \frac{1}{n})^\theta} \geq 0$ a.e., so the maximum principle implies that $u_n \geq 0$ and hence u_n be a solution of (2.1). By the result of [15], u_n belongs to $L^\infty(\Omega)$, because of the right-hand side of (2.10) belongs to $L^\infty(\Omega)$. $\square$

3 Proof of the theorem 1.1

Proof. Consider the following Dirichlet problems

$$u_n \in H_0^1(\Omega, |x|^{-2\beta}) : -\operatorname{div}(|x|^{-2\beta} \nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{\lambda g_n(x)}{(u_n + \frac{1}{n})^\theta} + h(x)u_n^p \text{ in } \Omega. \quad (3.1)$$

To prove the existence of a solution u_n of (3.1), we apply Sattinger monotone iteration.

We first obtain a subsolution:

By Lemma(2.4) there exists a weak solution $\underline{u}_n$ of the following Dirichlet problem

$$\underline{u}_n \in H_0^1(\Omega, |x|^{-2\beta}) : -\operatorname{div}(|x|^{-2\beta} \nabla \underline{u}_n) - \mu \frac{f(x)\underline{u}_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{\lambda g_n(x)}{(\underline{u}_n + \frac{1}{n})^\theta} \text{ in } \Omega, \quad (3.2)$$

so $\underline{u}_n$ solves:

$$\underline{u}_n \in H_0^1(\Omega, |x|^{-2\beta}) : -\operatorname{div}(|x|^{-2\beta} \nabla \underline{u}_n) - \mu \frac{f(x)\underline{u}_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{\lambda g_n(x)}{(\underline{u}_n + \frac{1}{n})^\theta} \text{ in } \Omega. \quad (3.3)$$

Thus

$$\underline{u}_n \in H_0^1(\Omega, |x|^{-2\beta}) : -\operatorname{div}(|x|^{-2\beta} \nabla \underline{u}_n) - \mu \frac{f(x)\underline{u}_n}{|x|^{2(\beta+1)} + \frac{1}{n}} \leq \frac{\lambda g_n(x)}{(\underline{u}_n + \frac{1}{n})^\theta} + h(x)\underline{u}_n^p \text{ in } \Omega, \quad (3.4)$$

i.e $\underline{u}_n$ is a subsolution to the problem (3.1).

Now, we construct a supersolution of (3.1):

Let $t \geq Y_\lambda$ (yet have to determine) and consider the following Dirichlet problems $\overline{u}_{n,t}$: $\overline{u}_{n,t}$ is the solution of

$$\overline{u}_{n,t} \in H_0^1(\Omega, |x|^{-2\beta}) : -\operatorname{div}(|x|^{-2\beta} \nabla \overline{u}_{n,t}) - \mu \frac{f(x)\overline{u}_{n,t}}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{t}{(\overline{u}_{n,t} + \frac{1}{n})^\theta}, \quad (3.5)$$

then (see [2]) $\exists c_0 > 0$ such that

$$\|\overline{u}_{n,t}\|_{L^\infty(\Omega)} \leq c_0 t^{\frac{1}{\theta+1}}. \quad (3.6)$$

Then the maximum principle implies that $\overline{u}_{n,t} \geq 0$ so that $\overline{u}_{n,t}$ solves

$$-\operatorname{div}(|x|^{-2\beta} \nabla \overline{u}_{n,t}) - \mu \frac{f(x)\overline{u}_{n,t}}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{t}{(\overline{u}_{n,t} + \frac{1}{n})^\theta}. \quad (3.7)$$

We show that for some $t > 0$,

$$\frac{t}{\left(\frac{1}{n} + \overline{u}_{n,t}\right)^\theta} \geq \frac{\lambda g(x)}{\left(\frac{1}{n} + \overline{u}_{n,t}\right)^\theta} + h(x)\overline{u}_{n,t}^p, \quad (3.8)$$

i.e;

$$t \geq \lambda g(x) + h(x)\overline{u}_{n,t}^p \left(\frac{1}{n} + \overline{u}_{n,t}\right)^\theta. \quad (3.9)$$

The above inequality is true if t satisfy the following

$$t \geq \lambda \|g\|_\infty + \|h\|_\infty \left[c_0 t^{\frac{1}{\theta+1}} \right]^p \left(\frac{1}{n} + c_0 t^{\frac{1}{\theta+1}} \right)^\theta. \quad (3.10)$$

Suppose Y_λ be such that

$$Y_\lambda = \lambda \|g\|_\infty + 2^\theta \|h\|_\infty c_0^{p+\theta} Y_\lambda^{\frac{p+\theta}{\theta+1}}. \quad (3.11)$$

Let $F(Y_\lambda) = Y_\lambda - \lambda \|g\|_\infty - 2^\theta \|h\|_\infty c_0^{p+\theta} Y_\lambda^{\frac{p+\theta}{\theta+1}}$.

Then

$$F(0) = -\lambda \|g\|_\infty < 0.$$

Now, let

$$G(Y_\lambda) = Y_\lambda^{1-\frac{p+\theta}{\theta+1}} - 2^\theta \|h\|_\infty c_0^{p+\theta} - \lambda \|g\|_\infty Y_\lambda^{-\frac{p+\theta}{\theta+1}}.$$

Then let $Y_\lambda \rightarrow \infty$,

$$\Rightarrow G(Y_\lambda) \rightarrow \infty, \text{ i.e.};$$

$$\exists N \in \mathbb{N} \text{ such that } G(Y_\lambda) > 1, \forall Y_\lambda \geq N,$$

$$\Rightarrow G(N) > 1.$$

Now,

$$F(Y_\lambda) = Y_\lambda^{\frac{p+\theta}{\theta+1}} G(Y_\lambda).$$

Hence

$$F(N) = N^{\frac{p+\theta}{\theta+1}} G(N) > 1 \cdot N^{\frac{p+\theta}{\theta+1}} > 0.$$

By Mean Value Theorem, $\exists Y_\lambda$ such that $0 < Y_\lambda < N$ and $F(Y_\lambda) = 0$. That is there exists $\Lambda > 0$ such that for $0 < \lambda < \Lambda$, there exists the solution of the equation (3.11). Therefore, we only have to prove that if n is large enough, we have

$$\lambda \|g\|_\infty + 2^\theta \|h\|_\infty c_0^{p+\theta} Y_\lambda^{\frac{p+\theta}{\theta+1}} \geq \lambda \|g\|_\infty + \|h\|_\infty c_0^p Y_\lambda^{\frac{p}{\theta+1}} \left(\frac{1}{n} + c_0 Y_\lambda^{\frac{1}{\theta+1}} \right)^\theta, \quad (3.12)$$

thus,

$$2^\theta c_0^\theta Y_\lambda^{\frac{\theta}{\theta+1}} \geq \left(\frac{1}{n} + c_0 Y_\lambda^{\frac{1}{\theta+1}} \right)^\theta, \quad (3.13)$$

this will hold if $\frac{1}{n} \leq c_0 Y_\lambda^{\frac{1}{\theta+1}}$ and this latter inequality true for n large.

Since Y_λ independent of n . Thus we proved that for $Y \geq Y_\lambda$, $\overline{u_{n,t}}$ is a super solution to the problem (3.1).

We show that $\underline{u_n} \leq \overline{u_{n,t}}$:

Define

$$L(v) = -\operatorname{div}(|x|^{-2\beta} \nabla v) - \mu \frac{f(x)v}{|x|^{2(\beta+1)} + \frac{1}{n}}. \quad (3.14)$$

Next aim to show $\underline{u_n} \leq \overline{u_{n,t}}$. Indeed

$$L(\underline{u_n} - \overline{u_{n,t}}) = \quad (3.15)$$

$$-\operatorname{div}(|x|^{-2\beta} \nabla \underline{u_n}) - \mu \frac{f(x)\underline{u_n}}{|x|^{2(\beta+1)} + \frac{1}{n}} + \operatorname{div}(|x|^{-2\beta} \nabla \overline{u_{n,t}}) + \mu \frac{f(x)\overline{u_{n,t}}}{|x|^{2(\beta+1)} + \frac{1}{n}} \quad (3.16)$$

$$= \frac{\lambda g_n(x)}{\left(\frac{1}{n} + |\underline{u_n}| \right)^\theta} - \frac{t}{\left(\frac{1}{n} + \overline{u_{n,t}} \right)^\theta}, \quad (3.17)$$

implies

$$\langle L(\underline{u_n} - \overline{u_{n,t}}), (\underline{u_n} - \overline{u_{n,t}})^+ \rangle = \int_\Omega \left[\frac{\lambda g_n(x)}{\left(\frac{1}{n} + |\underline{u_n}| \right)^\theta} - \frac{t}{\left(\frac{1}{n} + \overline{u_{n,t}} \right)^\theta} \right] (\underline{u_n} - \overline{u_{n,t}})^+. \quad (3.18)$$

Since $g_n(x) = \min\{n, g(x)\} \leq g(x) \leq \|g(x)\|_{L^\infty(\Omega)}$, so we have

$$\langle L(\underline{u_n} - \overline{u_{n,t}}), (\underline{u_n} - \overline{u_{n,t}})^+ \rangle \leq \int_\Omega \left[\frac{\lambda \|g\|_{L^\infty}}{\left(\frac{1}{n} + |\underline{u_n}| \right)^\theta} - \frac{t}{\left(\frac{1}{n} + \overline{u_{n,t}} \right)^\theta} \right] (\underline{u_n} - \overline{u_{n,t}})^+$$

$$\begin{aligned}
&= \int_{\Omega} \left[\frac{\lambda \|g\|_{L^\infty}}{\left(\frac{1}{n} + |\underline{u}_n|\right)^\theta} - \frac{\lambda \|g\|_{L^\infty}}{\left(\frac{1}{n} + \overline{u}_{n,t}\right)^\theta} \right] (\underline{u}_n - \overline{u}_{n,t})^+ \\
&\quad + \int_{\Omega} \left[\frac{\lambda \|g\|_{L^\infty}}{\left(\frac{1}{n} + |\underline{u}_n|\right)^\theta} - \frac{t}{\left(\frac{1}{n} + \overline{u}_{n,t}\right)^\theta} \right] (\underline{u}_n - \overline{u}_{n,t})^+ \\
&= \lambda \int_{\{\underline{u}_n \geq \overline{u}_{n,t}\}} \|g\|_{L^\infty} \left[\frac{1}{\left(\frac{1}{n} + |\underline{u}_n|\right)^\theta} - \frac{1}{\left(\frac{1}{n} + \overline{u}_{n,t}\right)^\theta} \right] (\underline{u}_n - \overline{u}_{n,t}) \\
&\quad + \int_{\{\underline{u}_n \geq \overline{u}_{n,t}\}} \frac{1}{\left(\frac{1}{n} + |\underline{u}_n|\right)^\theta} [\lambda \|g\|_{L^\infty} - t] (\underline{u}_n - \overline{u}_{n,t}). \tag{3.19}
\end{aligned}$$

Note that, in the last line, the first integral is negative because the real function $\frac{1}{\left(\frac{1}{n} + l\right)^\theta}$ is decreasing if $l > 0$ and the second integral is negative because $t \geq Y_\lambda$ and by the definition (3.11), $Y_\lambda \geq \lambda \|g\|_\infty$. Next we apply Sattinger Method:

$$w(s) := \frac{\lambda g(x)}{\left(\frac{1}{n} + s\right)^\theta} + h(x)s^p + n^{\theta+1}\lambda\theta s, \quad 0 < s < \infty. \tag{3.20}$$

It can be shown that function $w(s)$ is increasing.

Now the classical Amann-Sattinger method will give the existence of u_n solution of

$$u_n \in H_0^1(\Omega, |x|^{-2\beta}) : -\operatorname{div}(|x|^{-2\beta}\nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} + n^{\theta+1}\lambda\theta u_n \tag{3.21}$$

$$= \frac{\lambda g(x)}{\left(\frac{1}{n} + u_n\right)^\theta} + h(x)(u_n^+)^p + n^{\theta+1}\lambda\theta u_n, \tag{3.22}$$

and such that

$$\underline{u}_n \leq u_n \leq \omega_{n,t} \leq c_0 t^{\frac{1}{\theta+1}}. \tag{3.23}$$

also using [2], it can be shown that the sequence $\{u_n\}$ is increasing with respect to n , $\underline{u}_n > 0$ in Ω and for every $\Omega' \subset \subset \Omega$, $\exists c_{\Omega'}$ (independent of n) such that

$$\underline{u}_n(x) \geq \omega_1(x) \geq c_{\Omega'} > 0, \quad \forall x \in \Omega', n \in \mathbb{N}. \tag{3.24}$$

We will use u_n as a test function in the following equation:

$$u_n \in H_0^1(\Omega, |x|^{-2\beta}) : -\operatorname{div}(|x|^{-2\beta}\nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{\lambda g_n(x)}{\left(u_n + \frac{1}{n}\right)^\theta} + h(x)u_n^p. \tag{3.25}$$

Indeed,

$$\int_{\Omega} |x|^{-2\beta}\nabla u_n \nabla u_n - \frac{\mu}{2} \int_{\Omega} \frac{f(x)u_n^2}{|x|^{2(\beta+1)} + \frac{1}{n}} = \int_{\Omega} \frac{\lambda g(x)u_n}{\left(\frac{1}{n} + u_n\right)^\theta} + \int_{\Omega} h(x)(u_n^+)^p u_n, \tag{3.26}$$

$$\left(1 - \mu \|f\|_\infty C_{\beta,N,2}^{-1}\right) \int_{\Omega} |x|^{-2\beta} |\nabla u_n|^2 dx \leq \lambda \|g\|_{L^\infty(\Omega)} \int_{\Omega} u_n^{1-\theta} + \|h\|_\infty \int_{\Omega} |u_n|^{p+1}, \tag{3.27}$$

(3.23) implies that $\{u_n\}$ is bounded in $H_0^1(\Omega, |x|^{-2\beta})$. Then $\exists$ a subsequence $\{u_n\}$ (still denoted by $\{u_n\}$) of sequence $\{u_n\}$ and $u \in H_0^1(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega)$ such that

$$u_n \rightharpoonup u \tag{3.28}$$

weakly in $H_0^1(\Omega, |x|^{-2\beta})$ and strongly in $L^2(\Omega, |x|^{-2\beta})$,

$$u_n \rightarrow u \text{ a.e. in } \Omega, \tag{3.29}$$

$$\nabla u_n \rightarrow \nabla u \text{ weakly in } L^2(\Omega, |x|^{-2\beta}). \quad (3.30)$$

So we have,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |x|^{-2\beta} \nabla u_n \nabla \varphi = \int_{\Omega} |x|^{-2\beta} \nabla u \nabla \varphi, \quad \forall \varphi \in H_0^1(|x|^{-2\beta}). \quad (3.31)$$

Furthermore, we have, for $\varphi \in H_0^1(\Omega, |x|^{-2\beta})$,

$$0 \leq \frac{\lambda g(x) \varphi}{\left(\frac{1}{n} + u_n\right)^{\theta}} \leq \frac{\lambda \|g\|_{\infty} \varphi}{(c_{\Omega'})^{\theta}}, \quad (3.32)$$

since, recalling (3.23) and (3.24), $u_n \geq \underline{u}_n \geq \underline{u}_1 \geq c_{\Omega'}$ and $0 < \theta \leq 1$.

By the Lebesgue dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{\lambda g(x) \varphi}{\left(\frac{1}{n} + u_n\right)^{\theta}} = \lambda \int_{\Omega} \frac{g(x) \varphi}{u^{\theta}}. \quad (3.33) \quad \square$$

Remark 3.1. As in [1], the existence to (1.1) still holds for $p > 1$.

Remark 3.2. The similar proof as in the Theorem 1.1, will work to treat the following semilinear elliptic problem:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) - \mu \frac{g(x)u}{|x|^2} = \frac{\lambda f(x)}{u^{\gamma}} + h(x)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.34)$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded subset, $N \geq 3$; $0 \in \Omega$, $0 \leq f, g, h \in L^{\infty}(\Omega)$. Under assumptions $\gamma, 0 < p < 1$, $\lambda > 0$ small and M be bounded elliptic matrix, i.e., $\exists 0 < \alpha \leq \beta$ such that $\alpha|\eta|^2 \leq \langle M(x)\eta, \eta \rangle$, $|M(x)| \leq \beta$. As in Theorem 1.1. $\exists \Lambda \in (0, \infty)$ such that for $\lambda \in (0, \Lambda)$ $\exists$ a solution u , strictly positive in Ω of (3.34), in the following sense that

$$\int_{\Omega} M(x) \nabla u \nabla \varphi - \mu \int_{\Omega} \frac{f(x)u\varphi}{(|x|^2)} = \lambda \int_{\Omega} \frac{g(x)\varphi}{u^{\gamma}} + \int_{\Omega} h(x)u^p \varphi, \quad \forall \varphi \in H_0^1(\Omega), \quad (3.35)$$

where Ω' is an arbitrary open subset of Ω , such that $\overline{\Omega'} \subset \Omega$.

Moreover if $0 < \gamma < 1$, then $u \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$. For the sake of brevity, we omit the proof.

Remark 3.3. [8] The similar proof as in the Theorem 1.1, will work to treat the following semilinear elliptic problem:

$$\begin{cases} -\operatorname{div}(M(\xi)\nabla_{\mathbb{H}^n} u) - \mu \frac{g(\xi)u}{(|z|^4 + t^2)^{1/2}} = \frac{\lambda f(\xi)}{u^{\gamma}} + h(\xi)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.36)$$

where $\Omega \subset \mathbb{H}^n$ is an open bounded subset, $N \geq 3$; $0 \in \Omega$, $0 \leq f, g, h \in L^{\infty}(\Omega)$. Under assumptions $\gamma, 0 < p < 1$, $\lambda > 0$ small and M be bounded elliptic matrix, i.e., $\exists 0 < \alpha \leq \beta$ such that $\alpha|\eta|^2 \leq \langle M(\xi)\eta, \eta \rangle$, $|M(\xi)| \leq \beta$. As in Theorem 1.1. $\exists \Lambda \in (0, \infty)$ such that for $\lambda \in (0, \Lambda)$ $\exists$ a solution u , strictly positive in Ω of (3.36), in the following sense that

$$\int_{\Omega} M(\xi) \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} \varphi - \mu \int_{\Omega} \frac{f(\xi)u\varphi}{(|z|^4 + t^2)^{1/2}} = \lambda \int_{\Omega} \frac{g(\xi)\varphi}{u^{\gamma}} + \int_{\Omega} h(\xi)u^p \varphi, \quad (3.37)$$

$\forall \varphi \in H_0^1(\Omega, \mathbb{H}^n)$, where Ω' is an arbitrary open subset of Ω , such that $\overline{\Omega'} \subset \Omega$.

Moreover if $0 < \gamma < 1$, then $u \in H_0^1(\Omega, \mathbb{H}^n) \cap L^{\infty}(\Omega)$. For the sake of brevity, we omit the proof.

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