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Swirling flow of couple stress fluid due to a rotating disk

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Abstract: The main objective of the present investigation is to examine the couple stress fluid flow occurring as a result of rotation of a disk. On implementing a suitable transformation, the governing system of partial differential equations (PDEs) is converted into nonlinear differential equations of a single independent variable. These equations are solved analytically by virtue of the Homotopy Analysis Method (HAM) which gives solutions in the form of a series. The solution of most of the governing problems is determined in terms of the absolute exponential and decaying functions by means of this powerful technique. To support analytic results, some graphs are plotted for determining the convergence of the solution. Also the graphical interpretation of velocity profiles corresponding to the effects of pertinent parameters are shown and discussed in detail. The numerical results are calculated for evaluation of the influence of fluid parameter. It can also be anticipated that the radial and axial velocity components show decreasing behavior due to rise in the values of couple stress parameter which conflict the behavior of the tangential component of velocity.

Keywords: Disk Rotation; Swirling Flow; Couple Stress Fluid

1 Introduction

The flow which appears due to a rotating disk has gained a significant amount of attention in many mathematical models and has become an intriguing topic in recent in-

vestigations. These types of flows have received practical importance in many engineering processes, such as spin-coating, revolving appliances, medical apparatus, computer storage instruments, turning viscometer, machinery radiating from a central point, draining of liquid metals, growth processes of a crystal, turbo devices and in many essentially aerodynamic operations.

The pioneering work related to the Newtonian fluid flow as a result of an infinite rotating disk was examined by Karman [1] in latent ambient. The Navier-Stokes equations which govern the flow diminished to a self-similar form by utilizing Karman's transformation. Joshi et al. [2] considered the influences of porous medium on the Bode-wadt flow of a magnetic nanofluid past through a uniformly heated stationary disk in the occurrence of geothermal viscosity. To analyze the important features for the unsteady convective flow, Ram et al. [3] presented the time-dependent three-dimensional boundary layer flow of nano-suspension by considering the effects of heat transfer. A comprehensive study related to the time-dependent boundary layer flow of a magnetic nanofluid over a rotating disk with the combined effects of thermal radiation and variable viscosity has been encountered by Joshi et al. [4]. An inclusive effort to examine the solution of the problem governing the time dependent boundary layer flow of magnetic nanofluids due to rotation of stretchable plate has been conferred by Ram et al. [5]. This flow has been discussed in the presence of a porous medium and inspected the effects of geothermal viscosity with viscous dissipation. Further, for inspecting the influence of porosity of the medium, the viscous incompressible fluid flow in the presence of temperature-dependent viscosity has been presented by Attia [6]. Ram and Kumar [7] has analyzed the ferrohydrodynamic laminar boundary flow of electrically non-conducting magnetic fluid over a disk surface which is radially stretchable. To examine the behavior of Bingham plastic fluid flow, Rashaida [8] has considered the problem which deals with the flow of a non-Newtonian Bingham plastic fluid due to the rotation of a disk. Andersson et al. [9] have investigated the axi-symmetric power-law fluid flow in the boundary layer consistently. The solution of the steady laminar flow of an incompressible viscous electrically conducting fluid on account of the ro-

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tation of a disk in the presence of a magnetic field has been accomplished by Turkyilmazoglu [10]. Sibanda and Makinde [11] have investigated the influences of magnetic field, electrical heating, viscous dissipation effect, Hall effect generated by the electric potential, and current due to ion slip in the steady flow occurring due to a rotating disk. The solution of the steady flow of an incompressible fluid past through a rotating disk with the influence of an external uniform magnetic field has been provided by Ariel [12]. Attia [13] has focused on the unsteady flow of a viscous fluid in the presence of a magnetic field. Moreover, the influences of suction or injection on the flow have also been explored. Khan and Riaz [14] have established an interesting analysis for the couple stress fluid flow near a stagnation point due to the rotation of a disk which is non-aligned. A study concerning the boundary layer flow of a couple stress fluid past through a continuous moving surface with the effect of heat transfer in the fluid flow has been attempted by Hayat et al. [15]. Furthermore, to find the exact solutions of different types of flows such as Couette, Poiseuille and generalized Couette flows for the couple stress fluid model, Devakar et al. [16] presented the analytical solutions for determining the flow behavior of the fluid along with slip boundary conditions. In general, less academic attention has been paid to observe the influence of couple stress fluid flow.

In recent years, many nonlinear problems have been extensively solved by utilizing the Homotopy Analysis Method (HAM) developed and refined by Liao [17]. This method has drawn the exclusive attention of many researchers, therefore, to analyze the convergence of the solution obtained by HAM, Hussain et al. [18] considered the impact of linear operator on the fluid flow problem by adopting a modified operator approach. The flow through stretching surface is of great significance due to its various utilizations in different fields. As a result, the unsteady flow induced due to a stretching surface with the effect of magnetic field has been inspected by Hayat et al. [19]. In addition, many analysts have served a comprehensive effort to acquire the solutions of equations as mentioned in [20] via Homotopy Analysis Method, implementation of HAM for solving the nonlinear problem [21] and analysis of the flow and heat transfer in the direction of a stretching sheet [22]. These studies enhance the efficiency and adaptability of HAM. In literature, many other new techniques can be found for solving ODEs and PDEs. Recently, a new integral transform method has been proposed and successfully applied on many physical problems [23–25].

Stokes has originated the couple stress fluid theory [26], which demonstrates the classical viscous fluid theory in generalized form that maintains couple stresses

and the body couples. The present work focuses on the flow of couple stress fluid occurring due to the rotation of a disk and obtains the analytical solution of the problem under consideration. Our concern in the present paper is to find a highly accurate analytic solution of the considered problem followed by the implementation of HAM. Here, by utilizing HAM, the solution in the form of a series is first calculated and then the convergence region of the solution is observed. The graphical and tabular forms for the numerical results have been presented and discussed.

2 Mathematical formulation of the problem

Our interest here is to explore the axi-symmetric steady couple stress fluid flow as a result of the rotation of an infinite disk about its z axis as associated with the coordinates (r, ψ, z) in the cylindrical system along ω as an angular velocity. The r -axis is assumed to be parallel to the disk surface and the z -axis is taken perpendicularly to the direction of disk surface. Figure 1 shows the physical model of the considered problem, where v_r , v_ψ and v_z denote the velocity components in the radial (r), tangential (ψ) and axial (z) directions, respectively. Also ρ , ν , p , and λ are the fluid density, kinematic viscosity coefficient, pressure, and couple stress parameter, respectively.

The constitutive partial differential equations can be given as:

$$\nabla \cdot (v_r, v_\psi, v_z) = 0. \quad (1)$$

Fig. 1: Physical model and coordinate system

$$\begin{aligned} v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\psi^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} \\ &+ v \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} \right) \\ &+ \frac{\lambda}{\rho} \left(\frac{\partial^2 \mathcal{L}}{\partial r^2} + \frac{\partial^2 \mathcal{L}}{\partial z^2} + \frac{1}{r} \frac{\partial \mathcal{L}}{\partial r} - \frac{\mathcal{L}}{r^2} \right) \end{aligned} \quad (2)$$

$$\begin{aligned} v_r \frac{\partial v_\psi}{\partial r} + v_z \frac{\partial v_\psi}{\partial z} + \frac{1}{r} v_r v_\psi \\ = v \left(\frac{\partial^2 v_\psi}{\partial r^2} + \frac{1}{r} \frac{\partial v_\psi}{\partial r} + \frac{\partial^2 v_\psi}{\partial z^2} - \frac{v_\psi}{r^2} \right) \\ + \frac{\lambda}{\rho} \left(\frac{\partial^2 \mathcal{M}}{\partial r^2} + \frac{\partial^2 \mathcal{M}}{\partial z^2} + \frac{1}{r} \frac{\partial \mathcal{M}}{\partial r} - \frac{\mathcal{M}}{r} \right) \end{aligned} \quad (3)$$

$$\begin{aligned} v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + v \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right) \\ &+ \frac{\lambda}{\rho} \left(\frac{\partial^2 \mathcal{N}}{\partial r^2} + \frac{\partial^2 \mathcal{N}}{\partial z^2} + \frac{1}{r} \frac{\partial \mathcal{N}}{\partial r} \right) \end{aligned} \quad (4)$$

Subject to the boundary conditions [27] on the disk surface and the boundary conditions at infinity

$$v_\psi = r\omega, \quad v_r = v_z = 0, \quad p = \text{const} = 0, \quad z = 0 \quad (5)$$

$$v_r = v_\psi = \frac{\partial v_z}{\partial z} = 0, \quad z = \infty, \quad (6)$$

where

$$\begin{aligned} \mathcal{L} &= -\nabla^2 v_r + \frac{2}{r^2} \frac{\partial v_\psi}{\partial \psi} + \frac{v_r}{r^2}, \quad \mathcal{M} = -\nabla^2 v_\psi - \frac{2}{r^2} \frac{\partial v_r}{\partial \psi} + \frac{v_\psi}{r^2}, \\ \mathcal{N} &= -\nabla^2 v_z. \end{aligned}$$

Using similarity transformation formulated by Karman [1], we have

$$\begin{aligned} v_r &= r\omega \mathcal{F}(\Lambda), \quad v_\psi = r\omega \mathcal{G}(\Lambda), \quad v_z = \sqrt{\nu\omega} \mathcal{H}(\Lambda), \\ p &= \rho\nu\omega P(\Lambda), \quad \Lambda = z\sqrt{\frac{\omega}{\nu}} \end{aligned} \quad (7)$$

By the above transformation, Eqs. (1)-(4) reduces to:

$$\mathcal{F}'' - \mathcal{F}^2 + \mathcal{G}^2 - \mathcal{H}\mathcal{F}' - \beta\mathcal{F}^{(4)} = 0, \quad (8)$$

$$\mathcal{G}'' - \mathcal{H}\mathcal{G}' - 2\mathcal{F}\mathcal{G} - \beta\mathcal{G}^{(4)} = 0, \quad (9)$$

$$\mathcal{H}\mathcal{H}' + P' - \mathcal{H}'' + \beta\mathcal{H}^{(4)} = 0, \quad (10)$$

$$\mathcal{H}' + 2\mathcal{F} = 0, \quad (11)$$

where $\beta = \frac{\lambda\omega}{\nu^2\rho}$.

Here, prime of the functions gives differentiation with respect to Λ , subjected to the boundary conditions [27]:

$$\mathcal{F}(0) = \mathcal{F}(\infty) = 0, \quad \mathcal{G}(0) = 1, \quad \mathcal{G}(\infty) = 0, \quad \mathcal{H}(0) = 0, \quad p(0) = 0. \quad (12)$$

According to Eq. (11), we have

$$\mathcal{F} = -\frac{\mathcal{H}'}{2}. \quad (13)$$

Substituting Eq. (13) into Eqs. (8) and (9), we have

$$\beta\mathcal{H}^{(5)}(\Lambda) - (\mathcal{H}''''(\Lambda) - \mathcal{H}''(\Lambda)\mathcal{H}(\Lambda) + \frac{1}{2}(\mathcal{H}'(\Lambda))^2 - 2\mathcal{G}^2(\Lambda)) = 0, \quad (14)$$

$$\beta\mathcal{G}^{(4)}(\Lambda) - (\mathcal{G}''(\Lambda) - \mathcal{H}(\Lambda)\mathcal{G}'(\Lambda) + \mathcal{G}(\Lambda)\mathcal{H}'(\Lambda)) = 0, \quad (15)$$

with boundary conditions:

$$\mathcal{H}(0) = \mathcal{H}(\infty) = \mathcal{H}'(\infty) = \mathcal{H}'(0), \quad \mathcal{G}(0) = 1, \quad \mathcal{G}(\infty) = 0. \quad (16)$$

3 Solution of the problem

To examine the precise and entirely analytic solutions of Eqs. (14)-(15) with the boundary conditions (16), using HAM, we have selected the following initial approximation $\mathcal{H}_0(\Lambda), \mathcal{G}_0(\Lambda)$,

$$\mathcal{H}_0 = -1 + e^{-\Lambda}(1 + \Lambda), \quad \mathcal{G}_0 = e^{-\Lambda} \quad (17)$$

and the auxiliary linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ as:

$$\mathcal{L}_1(\mathcal{H}) = \mathcal{H}''' - \mathcal{H}' \quad (18)$$

$$\mathcal{L}_2(\mathcal{G}) = \mathcal{G}'' - \mathcal{G} \quad (19)$$

which fulfill the following characteristics:

$$\mathcal{L}_1[c_1 + c_2 e^\Lambda + c_3 e^{-\Lambda}] = 0 \quad (20)$$

$$\mathcal{L}_2[c_4 e^\Lambda + c_5 e^{-\Lambda}] = 0 \quad (21)$$

Here, c_1, c_2, c_3, c_4 and c_5 are known as the arbitrary constants, $p \in [0, 1]$ yields an embedding parameter, and $\hbar$ is the convergence control parameter. The zeroth-order deformation equations can be given as:

$$(1-p)\mathcal{L}_1[\tilde{\mathcal{H}}(\Lambda, p) - \tilde{\mathcal{H}}_0(\Lambda)] = p\hbar\mathcal{N}_1[\tilde{\mathcal{H}}(\Lambda, p), \tilde{\mathcal{G}}(\Lambda, p)], \quad (22)$$

$$(1-p)\mathcal{L}_2[\tilde{\mathcal{G}}(\Lambda, p) - \tilde{\mathcal{G}}_0(\Lambda)] = p\hbar\mathcal{N}_2[\tilde{\mathcal{H}}(\Lambda, p), \tilde{\mathcal{G}}(\Lambda, p)]. \quad (23)$$

The boundary conditions for this deformation are:

$$\mathcal{H}(\Lambda, 0) = \mathcal{H}_0(\Lambda), \mathcal{H}(\Lambda, 1) = \mathcal{H}(\Lambda) \quad (24) \quad \text{where}$$

$$\mathcal{G}(\Lambda, 0) = \mathcal{G}_0(\Lambda), \mathcal{G}(\Lambda, 1) = \mathcal{G}(\Lambda). \quad (25)$$

On the basis of Eqs. (14)-(15), the leading operators $\mathcal{N}_1$ and $\mathcal{N}_2$ can be signified as:

$$\mathcal{N}_1 = \frac{\partial^3 \tilde{\mathcal{H}}(\Lambda, p)}{\partial \Lambda^3} - \tilde{\mathcal{H}}(\Lambda, p) \frac{\partial^2 \tilde{\mathcal{H}}(\Lambda, p)}{\partial \Lambda^2} + \frac{1}{2} \left(\frac{\partial \tilde{\mathcal{H}}(\Lambda, p)}{\partial \Lambda} \right)^2 - 2\tilde{\mathcal{G}}(\Lambda, p)^2 + \beta \left(\frac{\partial^5 \tilde{\mathcal{H}}(\Lambda, p)}{\partial \Lambda^5} \right), \quad (26)$$

$$\mathcal{N}_2 = \frac{\partial^2 \tilde{\mathcal{G}}(\Lambda, p)}{\partial \Lambda^2} - \tilde{\mathcal{H}}(\Lambda, p) \left(\frac{\partial \tilde{\mathcal{G}}(\Lambda, p)}{\partial \Lambda} \right) + \tilde{\mathcal{G}}(\Lambda, p) \left(\frac{\partial \tilde{\mathcal{H}}(\Lambda, p)}{\partial \Lambda} \right) + \beta \left(\frac{\partial^4 \tilde{\mathcal{G}}(\Lambda, p)}{\partial \Lambda^4} \right) \quad (27)$$

By utilizing the Taylor's series when changes the values from 0 to 1, it can be shown that

$$\mathcal{H}(\Lambda, p) = \mathcal{H}_0(\Lambda) + \sum_{m=1}^{\infty} \mathcal{H}_m(\Lambda) p^m, \quad \mathcal{H}_m(\Lambda) = \frac{1}{m!} \frac{\partial^m \mathcal{H}(\Lambda, p)}{\partial p^m} \Big|_{p=0}, \quad (28)$$

$$\mathcal{G}(\Lambda, p) = \mathcal{G}_0(\Lambda) + \sum_{m=1}^{\infty} \mathcal{G}_m(\Lambda) p^m, \quad \mathcal{G}_m(\Lambda) = \frac{1}{m!} \frac{\partial^m \mathcal{G}(\Lambda, p)}{\partial p^m} \Big|_{p=0}. \quad (29)$$

The convergence of the aforementioned series given in Eqs. (28) and (29) is strictly based on $\hbar$. The $\hbar$ values are to be chosen so that the series in Eqs. (28) and (29) have convergent behavior at $p = 1$ and hence Eqs. (24) and (25) yield

$$\mathcal{H}(\Lambda) = \mathcal{H}_0(\Lambda) + \sum_{m=1}^{\infty} \mathcal{H}_m(\Lambda), \quad (30)$$

$$\mathcal{G}(\Lambda) = \mathcal{G}_0(\Lambda) + \sum_{m=1}^{\infty} \mathcal{G}_m(\Lambda). \quad (31)$$

For the m^{th} order deformation, differentiating Eqs. (26) and (27) m times with respect to the embedding parameter p and dividing them by $m!$, then set $p = 0$. The resulting deformation problems at the m^{th} order are:

$$\mathcal{L}_1[\mathcal{H}_m(\Lambda, p) - \xi_m \mathcal{H}_{m-1}(\Lambda)] = \hbar R_m(\Lambda) \quad (32)$$

$$\mathcal{L}_2[\mathcal{G}_m(\Lambda, p) - \xi_m \mathcal{G}_{m-1}(\Lambda)] = \hbar Q_m(\Lambda) \quad (33)$$

$$\mathcal{H}_m(0) = \mathcal{H}'_m(\infty) = \mathcal{H}'_m(0) = 0, \mathcal{G}_m(0) = \mathcal{G}_m(\infty) = 0, \quad (34)$$

$$R_m = \beta \mathcal{H}''''_{m-1} + \mathcal{H}'''_{m-1} - \sum_{i=0}^{m-1} \mathcal{H}''_i \mathcal{H}_{m-1-i} - 2 \sum_{i=0}^{m-1} \mathcal{G}_i \mathcal{G}_{m-1-i}$$

$$+ \frac{1}{2} \sum_{i=0}^{m-1} \mathcal{H}'_i \mathcal{H}'_{m-1-i}$$

$$Q_m = \beta \mathcal{G}''''_{m-1} + \mathcal{G}''_{m-1} - \sum_{i=0}^{m-1} \mathcal{H}_i \mathcal{G}'_{m-1-i} + \sum_{i=0}^{m-1} \mathcal{H}'_i \mathcal{G}_{m-1-i}$$

$$\xi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

We have used the symbolic computation software MATHEMATICA 10 for the solutions of Eqs. (32) and (33) with boundary conditions (34).

4 Graphical and numerical results and discussion

The present work is concerned with the study of an incompressible steady flow of couple stress fluid due to the rotation of a disk. The main interest is to examine the analytical solution of the considered problem and analyze the flow behavior of the fluid. For the sake of clarity and precision, in this section, the comparability and the graphs of velocity profiles obtained via HAM have been conferred. A family of explicit analytic expressions involving auxiliary parameter has been presented by the infinite solution series as mentioned in Eqs. (32) and (33). The convergence area of the solutions and the rate of approximation for HAM depend on the values of parameter $\hbar$. By selecting the appropriate value of $\hbar$, which increases the credibility that the series solution approaches to its unique solution. The graphs of the curves related to the flow field parameters $\mathcal{F}''(0)$, $\mathcal{G}''(0)$, and $\mathcal{H}''(0)$ are illustrated by taking the sufficient and convenient region of $\hbar$ for various values of couple stress parameter. By utilizing the converged conditions, the sufficient region for convergence of $\hbar$ relating to the line segments is nearly along the horizontal axis. Figures 2-4 show the $\hbar$ -curves obtained at the 15th order of HAM approximation and clearly expound that the range for the admissible values.

In order to demonstrate our analytical solutions of couple stress fluid flow, it is worth mentioning to verify the numerical method. The extensive part of attention has purposely given to viscous fluid [27] which not only gives

(a)

(a)

(b)

(b)

(c)

(c)

Fig. 2: The graph of $\mathcal{F}''(0)$ against h for (a) $\beta = 0.0$ (Newtonian fluid) (b) $\beta = 0.1$ (c) $\beta = 0.2$

Fig. 3: The h -curve of $\mathcal{G}''(0)$ for (a) $\beta = 0$ (Newtonian fluid) (b) $\beta = 0.1$ (c) $\beta = 0.2$

the reliability but also provide the way to go forward. Figure 5 depicts the dimensionless velocity profiles for Kármán flow of viscous fluid. The main result here is to note the influence of couple stress parameter β on the radial, tangential, axial velocity profiles, and pressure distribution, and hence we analyze these important characteris-

tics of the problem. For this purpose, the following graphs have been sketched.

It can be observed from Figure 6 that the dimensionless velocity component $\mathcal{F}$ exhibits decreasing behavior by rising in the values of couple stress parameter β nearly to the surface of the disk. The tangential component of velocity $\mathcal{G}$ is noticed to increase with increase in couple stress

(a)

(b)

(c)

Fig. 4: The graph of $\mathcal{H}''(0)$ against h for (a) $\beta = 0$ (Newtonian fluid) (b) $\beta = 0.1$ (c) $\beta = 0.2$

parameter at any given tangential point over the disk surface as illustrated in Figure 7. As we maximize the values of couple stress parameter, it becomes the cause of decreasing the axial velocity component $\mathcal{H}$ at all the points located above the disk surface as depicted in Figure 8. It can be deduced from Figure 9 that the pressure exceeds as the couple stress parameter β increases.

Fig. 5: Dimensionless velocity and pressure profile for Karman viscous fluid (data taken from [27])

Fig. 6: Influence of couple stress parameter on $\mathcal{F}$

Fig. 7: Influence of couple stress parameter on $\mathcal{G}$

Figure 10 gives the comparison of velocity, temperature, and pressure profiles for $h = -1$ in HAM, which is the traditional homotopy perturbation method, with ND-Solve values for viscous fluid. It is clearly observed that this choice of h value is not accurate and some other value must be selected from the range discussed in Figures 2-4.

Fig. 8: Influence of couple stress parameter on $\mathcal{H}$ Fig. 9: Influence of couple stress parameter on $\mathcal{P}$ Fig. 10: Graphical comparison for $h = -1$ in HAM with NDSolve results

As mentioned afore, the thickness δ can be regarded as the point where the circumferential velocity v_ψ drops to 1 percent of its wall value or $\mathcal{G} \approx 0.01$, which appears at around the value of $\Lambda \approx 5.4$ then the layer thickness for viscous flow is $\delta \approx 5.4\sqrt{\frac{\nu}{\omega}}$. The thickness δ for couple

stress fluid occurs at $\Lambda \approx 5.9$ and its layer thickness is $\delta \approx 5.9\sqrt{\frac{\nu}{\omega}}$.

The asymptotic value for viscous flow is $\mathcal{H}(\infty) = -0.8838$ and it is clear that the fluid is drawn from the disk surface at the rate $v_r(\infty) = -0.8838\sqrt{\nu\omega}$ and for the couple stress fluid asymptotic value is $\mathcal{H}(\infty) = -0.836136$, therefore the fluid is drawn from disk towards it at the rate $v_z(\infty) = -0.836136\sqrt{\nu\omega}$. Hence, both the viscosity and angular velocity increases the disk's pumping action. The disk surface has the circumferential wall shear stress as follows:

$$\tau_{z\psi} = \mu \frac{\partial v_\psi}{\partial z} \Big|_{z=0} = \rho r \mathcal{G}'_0 \sqrt{\nu\omega^3}, \quad (35)$$

where $\mathcal{G}'_0 = -0.6159$ for viscous flow and $\mathcal{G}'_0 = -0.623153$ for couple stress fluid from the table.

5 Conclusion

In this paper, we have presented the swirling flow of couple stress fluid occurring due to the rotation of a disk. The Karman's similarity transformation has been implemented for converting the partial differential equations into the ordinary differential equations which are further solved by the analytical technique HAM. Moreover, it is also of significance to present the results for the illustration of the flow characteristics associated with the velocity and pressure profiles. The variations on velocity profiles have also been observed by taking various values of the couple stress parameter. Numerical values on which the results have been evaluated are given in the form of Table 1. The following outcomes have been extracted from the above mentioned work:

- The 15th order approximation for a series solution is obtained in Table 1.
- The inverse relation has been observed on radial and axial components of velocity, because the rising values of couple stress parameter produce the decreasing effects on the radial and axial velocity components. But in contrast, it causes to increase the pressure.
- The couple stress parameter increases the values of the tangential component of velocity.

Table 1: The numerical result at $\beta = 0.1$ and for viscous flow i.e., at $\beta = 0$. (Values inside the bracket evaluating at $\beta = 0$)

A	$\mathcal{F}$	$\mathcal{F}'$	$\mathcal{G}$	$\mathcal{G}'$	$\mathcal{H}$	$-P$
0.00	0.00 (0.00)	0.4380420 (0.5102300)	1.0000000 (1.0000000)	-0.6231530 (-0.6159200)	0.00 (-0.00)	0.00127331 (0.00)
0.10	0.03964650 (0.0462000)	0.3567860 (0.4163000)	0.9385710 (0.9386000)	0.6046910 (-0.6112000)	-0.004100040 (-0.0048000)	0.0955010 (0.09240005)
0.20	0.0717176 (0.0836000)	0.2863450 (0.3380000)	0.8791860 (0.8780000)	-0.5825080 (-0.5987000)	-0.01535380 (-0.0179000)	0.1714510 (0.1674000)
0.30	0.0972387 (0.1133000)	0.2256040 (0.2620000)	0.8221540 (0.1890000)	-0.5577800 (-0.5803000)	-0.03235070 (-0.0377000)	0.2314990 (0.2274000)
0.40	0.1171270 (0.1364000)	0.1735200 (0.1999000)	0.7676830 (0.7621000)	-0.5314400 (-0.5577000)	-0.0538740 (-0.0628000)	0.2780240 (0.2747000)
0.50	0.1321990 (0.1536000)	0.1291260 (0.1467000)	0.7158950 (0.7075000)	-0.5042280 (-0.5321000)	-0.07888060 (-0.0919000)	0.3132820 (0.3115000)
0.60	0.1431790 (0.1660000)	0.09152970 (0.1015000)	0.6668470 (0.6557000)	-0.4767190 (-0.5047000)	-0.1064810 (-0.1239000)	0.3393430 (0.3396000)
0.70	0.1507040 (0.1742000)	0.05991440 (0.0635000)	0.6205460 (0.6067000)	-0.4493580 (-0.4763000)	-0.1359220 (-0.1580000)	0.3580410 (0.3608000)
0.80	0.15553360 (0.1789000)	0.0335370 (0.0317000)	0.5769590 (0.5605000)	-0.4224820 (-0.4476000)	-0.1665700 (-0.1933000)	0.3709580 (0.3764000)
0.90	0.1575640 (0.1807000)	0.011172520 (0.0056000)	0.5360250 (0.5171000)	-0.3963400 (-0.4191000)	-0.1978960 (-0.2293000)	0.3794320 (0.3877000)
1.00	0.1578130 (0.1801000)	-0.00612615 (-0.0157000)	0.4976610 (0.4766000)	-0.3711120 (-0.3911000)	-0.2294640 (-0.2655000)	0.3845620 (0.3955000)
1.20	0.1537990 (0.1737000)	-0.0320536 (-0.0461000)	0.4282390 (0.4037000)	-0.3238520 (-0.3381000)	-0.2919580 (-0.3364000)	0.3881650 (0.4040000)
1.40	0.1456620 (0.1625000)	-0.0478920 (-0.0640000)	0.3678110 (0.3411000)	-0.2812170 (-0.2898000)	-0.3519560 (-0.4038000)	0.3868250 (0.4066000)
1.60	0.1351240 (0.1487000)	-0.0564812 (-0.0728000)	0.3154390 (0.2875000)	-0.2432790 (-0.2470000)	-0.4081700 (-0.4661000)	0.3348600 (0.4061000)
1.80	0.1234070 (0.1338000)	-0.0599934 (-0.0754000)	0.2701980 (0.2419000)	-0.2098530 (-0.2095000)	-0.4599000 (-0.5226000)	0.3797150 (0.4042000)
2.00	0.1135600 (0.1188000)	-0.0600606 (-0.0739000)	0.2312180 (0.2034000)	-0.1806150 (-0.1771000)	-0.5068530 (-0.5732000)	0.3762490 (0.4019000)
2.20	0.0995315 (0.1044000)	-0.0578875 (-0.0698000)	0.1976990 (0.1708000)	-0.1555760 (-0.1494000)	-0.5490160 (-0.6178000)	0.3733580 (0.3997000)
2.40	0.0882911 (0.0910000)	-0.0543470 (-0.0643000)	0.1689210 (0.1433000)	-0.1331330 (-0.1258000)	-0.5865560 (-0.6568000)	0.3710810 (0.3977000)
2.60	0.0778422 (0.0788000)	-0.0500586 (-0.0580000)	0.1442460 (0.1202000)	-0.1140900 (-0.1057000)	-0.6197540 (-0.6907000)	0.3693520 (0.3961000)
2.80	0.0682887 (0.0678000)	-0.0454514 (-0.0517000)	0.1231100 (0.1008000)	-0.0976795 (-0.0888000)	-0.6489500 (-0.7200000)	0.3680710 (0.3948000)
3.00	0.0596634 (0.0581000)	-0.0408141 (-0.0455000)	0.1050210 (0.0845000)	-0.0835632 (-0.0746000)	-0.6745090 (-0.7452000)	0.3671390 (0.3938000)
3.20	0.0519524 (0.0496000)	-0.0363328 (-0.0397000)	0.0895518 (0.0709000)	-0.0714386 (-0.0625000)	-0.6968030 (-0.7666000)	0.3664680 (0.3930000)
3.40	0.0451123 (0.0422000)	-0.0321204 (-0.0344000)	0.0763309 (0.0594000)	-0.0610373 (-0.0525000)	-0.7161870 (-0.7850000)	0.3659900 (0.3924000)
3.60	0.0390822 (0.0358000)	-0.0282387 (-0.0296000)	0.0650378 (0.0498000)	-0.0521232 (-0.0440000)	-0.7330000 (-0.8005000)	0.3656510 (0.3920000)
3.80	0.0337929 (0.0303000)	-0.0247142 (-0.0254000)	0.0553963 (0.0417000)	-0.0444901 (-0.0369000)	-0.7475220 (-0.8137000)	0.3654120 (0.3917000)
4.00	0.0291724 (0.0256000)	-0.0215499 (-0.0217000)	0.0471684 (0.0349000)	-0.0379585 (-0.0309000)	-0.7601240 (-0.8249000)	0.3652440 (0.3914000)
4.50	0.0200897 (0.0167000)	-0.0151155 (-0.0144000)	0.0315126 (0.0225000)	-0.0254795 (-0.0199000)	-0.7844870 (-0.8457000)	0.3650120 (0.3911000)
5.00	0.0137576 (0.0108000)	-0.010476 (-0.009500)	0.0210146 (0.0144000)	-0.0170660 (-0.0128000)	-0.8121800 (-0.8594000)	0.3649170 (0.3910000)
5.50	0.00938588 (0.0070000)	-0.00720543 (-0.0062000)	0.0139896 (0.0093000)	-0.0114084 (-0.0082000)	-0.8126540 (-0.8682000)	0.3648780 (0.3910000)
6.00	0.00638651 (0.0045000)	-0.00493142 (-0.0040000)	0.00929752 (0.0059000)	-0.00761233 (-0.0053000)	-0.8204450 (-0.8739000)	0.3648620 (0.3909000)
7.00	0.00294084 (0.0018000)	-0.00228921 (-0.0017000)	0.00408721 (0.0024000)	-0.00337178 (-0.0022000)	-0.8293360 (-0.8799000)	0.3648530 (0.3909000)
8.00	0.0013469 (0.0007000)	-0.00105496 (-0.0007000)	0.00178594 (0.0010000)	-0.0014838 (-0.0009000)	-0.8334200 (-0.8824000)	0.3648510 (0.3908000)
9.00	0.000614031 (0.0002000)	-0.000483721 (-0.0003000)	0.000775944 (0.0004000)	-0.000648973 (-0.0004000)	-0.8352870 (-0.8834000)	0.3648510 (0.3908000)
10.00	0.000278655 (0.0001000)	-0.000220802 (-0.0001000)	0.000335316 (0.0001000)	-0.000282202 (-0.0001000)	-0.8361360 (-0.8838000)	0.3648510 (0.3907000)
∞	0.0000000 (0.0000000)	-0.0000000 (-0.0000000)	0.0000000 (0.0000000)	-0.0000000 (-0.0000000)	-0.8361360 (-0.8838000)	0.3648500 (0.3906000)

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