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The $\exp_a$ function method and the conformable time-fractional KdV equations

<https://doi.org/10.1515/nleng-2018-0094>

Received June 26, 2018; revised September 1, 2018; accepted November 6, 2018.

Abstract: The nonlinear fractional differential equations (FDEs) are produced by mathematical modelling of some nonlinear physical systems. The study of such nonlinear physical models through wave solutions analysis corresponding to their FDEs, has a dynamic role in applied sciences. In this paper, we are going to explore the conformable time-fractional KdV equations using the $\exp_a$ function method. The way to reach explicit exact wave solutions is to transform the fractional order PDE into a nonlinear ODE of discrete order through travelling wave transforms. The subsequent equation has been explored by utilizing the $\exp_a$ function approach. Consequently, some new explicit exact wave solutions of the said equations are effectively formulated and graphically conveyed with the help of numerical simulation.

Keywords: time-fractional KdV and time-fractional modified KdV equations; Conformable derivatives; $\exp_a$ function method; Explicit exact solutions

1 Introduction

The Korteweg–de Vries (KdV) equations have been observed in a broad variety of material science phenomena as a model for the development and communication of nonlinear waves. The KdV equation was introduced by Korteweg and de Vries to describe shallow water waves of long wavelength with small amplitude. Subsequently the KdV equation has been merged into a number of other physical contexts as collision-free hydromagnetic waves, stratified internal waves, particle acoustic waves, plasma physics, and so on [1–7].

Various fields of science and engineering are influenced by Leibnitz’s work on fractional calculus and having

substantially increasing impact on these sciences during last twenty years ostensibly [8–10]. From different minds for different applications, many definitions of fractional derivatives, Like Hilfer, Riemann-Liouville, Caputo form and so on, have been introduced in the literature [11, 12], but the most fascinating definition and the geometrical explanation for complex fractional transform and fractional derivative are given in [13, 14].

The explicit exact wave solutions have always been a particular importance among the researchers in many fields of nonlinear sciences. The availability of symbolic computation softwares is a direct help to minimize the manual labor for finding problematic solutions to nonlinear evolution equations. Various significant methods have been presented to explore exact solutions in different research articles, for example, the ansatz method, modified simple equation method, the extended trial equation, the first integral, the improved $\tan(\phi(\eta)/2)$ -expansion, the $\exp$ -function and the $\exp(-\phi(\epsilon))$ methods [15–21]. Also some other reliable techniques, like homotopy method [22–25], generalized Kudryashov method [26–29], modified Kudryashov and the sine-Gordon expansion approach [30–33] have been done by different researchers. Furthermore, the auxiliary equation method, the extended $\tanh$ -function method, generalized projective Riccati equation method, functional variable method, and Jacobi elliptic function expansion method have been explored in [34–39] for integral and fractional order as well. The $\exp_a$ function method is a new and an efficient technique which has been acknowledged by the scholars rapidly. In particular, Ali and Hassan, Hosseini et al., Zayed and Al-Nowehy all have utilized the $\exp_a$ function method in [40–43] respectively.

The paper’s aim is to apply the $\exp_a$ function approach to nonlinear conformable time-fractional KdV equations. The investigation for exact solutions is as follow. The description of conformable derivative and $\exp_a$ function approach is provided in section 2. The section 3 represents the method utilization for extracting new exact solutions with their graphical representation. The conclusion is provided in the last section.

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2 A new approach for fractional derivatives

A simple but a fascinating definition of the fractional derivative called conformable fractional derivative has been introduced by Khalil et al. [13]. The Leibniz rule and the chain rule both are obeyed by this derivative. Recall the definition of the conformable derivative and with some properties.

Definition 1 Suppose $h : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function. Then, for all $t > 0$,

$$D_t^\gamma(h(t)) = \lim_{\varepsilon \rightarrow 0} \frac{h(t + \varepsilon t^{1-\gamma}) - h(t)}{\varepsilon}$$

is known as the conformable fractional derivative of h of order γ , $0 < \gamma \leq 1$. Some useful properties are being listed as follows:

$$D_t^\gamma(a h + b g) = a D_t^\gamma(h) + b D_t^\gamma(g), \text{ for all } a, b \in \mathbb{R}$$

$$D_t^\gamma(h g) = h D_t^\gamma(g) + g D_t^\gamma(h)$$

Let $h : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a differentiable and γ -differentiable function, g be a differentiable function defined in the range of h .

$$D_t^\gamma(h \circ g(t)) = t^{1-\gamma} g'(t) h'(g(t)).$$

On the top of that, the following rules hold.

$$D_t^\gamma(t^p) = p t^{p-\gamma}, \text{ for all } p \in \mathbb{R}$$

$$D_t^\gamma(\lambda) = 0, \text{ where } \lambda \text{ is constant.}$$

$$D_t^\gamma(h/g) = \frac{g D_t^\gamma(h) - h D_t^\gamma(g)}{g^2}.$$

Conjointly, if h is differentiable, then $D_t^\gamma(h(t)) = t^{1-\gamma} \frac{dh(t)}{dt}$.

2.1 A transitory explanation of $\exp_a$ function method

The present subsection provides a concise explanation for the $\exp_a$ [40, 42, 44] in generating new explicit exact solutions of nonlinear conformable time-fractional DE. For this purpose, suppose that we have a nonlinear conformable time-fractional DE that can be presented in the form

$$F(u, D_t^\gamma u, D_x u, D_t^{2\gamma} u, D_{xx} u, \dots) = 0 \quad (1)$$

The FDE (1) can be changed into the following nonlinear ODE of integer order

$$P(U, U', U'', \dots) = 0 \quad (2)$$

with the use of following transformation

$$u(x, t) = U(\xi), \xi = kx - t \frac{t^\gamma}{\gamma}, \quad (3)$$

where k, l are nonzero arbitrary constants.

Let us try to search a non-trivial solution for the Eq. (2) in the following form

$$U(\xi) = \frac{a_0 + a_1 a^\xi + \dots + a_N a^{N\xi}}{b_0 + b_1 a^\xi + \dots + b_N a^{N\xi}} \quad (4)$$

where a_i and b_i for $(0 \leq i \leq N)$ are found later and N is a free positive integer.

Replacing the Eq.(4) in the nonlinear Eq.(2), yields

$$P(a^\xi) = q_0 + q_1 a^\xi + \dots + q_\tau a^{\tau\xi} = 0 \quad (5)$$

Setting $l_i (0 \leq i \leq \tau)$ in Eq.(4) to be zero, results in a set of nonlinear equations as below

$$q_i = 0, \quad i = 0, \dots, \tau \quad (6)$$

which by solving the generated set (6), we approach to non-trivial solutions of the nonlinear FDE (1).

3 Explicit exact wave solutions for time fractional KdV equations

In this section, the following time fractional KdV and modified KdV equations [7], are going to be considered for solution via $\exp_a$ method.

$$D_t^\gamma u + p u^2 u_x + q u_{xxx} = 0, \quad 0 < \gamma \leq 1, \quad (7)$$

Using the transformation (3), integrating w.r.t. ξ and taking constant of integration is zero, we get

$$-lu + \frac{1}{3} p k u^3 + q k^3 u'' = 0. \quad (8)$$

Through balancing, we select $N = 1$, the nontrivial solution (4) reduces to:

$$U(\xi) = \frac{\alpha_1 a^\xi + \alpha_0}{\beta_1 a^\xi + \beta_0}, \quad \alpha \neq 1 \quad (9)$$

By setting the above solution in reduced equation Eq. (8) and equating factors of each power of a^ξ in the resulting equation, we reach a nonlinear algebraic set as

$$\begin{aligned} \alpha_0^3 k p - 3 \alpha_0 \beta_0^2 l &= 0 \\ 3 \alpha_1 \beta_0^2 k^3 q \log^2(\alpha) - 3 \alpha_0 \beta_0 \beta_1 k^3 q \log^2(\alpha) + 3 \alpha_0^2 \alpha_1 k p \\ &- 3 \alpha_1 \beta_0^2 l - 6 \alpha_0 \beta_0 \beta_1 l = 0 \\ 3 \alpha_0 \beta_1^2 k^3 q \log^2(\alpha) - 3 \alpha_1 \beta_0 \beta_1 k^3 q \log^2(\alpha) + 3 \alpha_0 \alpha_1^2 k p \\ &- 3 \alpha_0 \beta_1^2 l - 6 \alpha_1 \beta_0 \beta_1 l = 0 \\ \alpha_1^3 k p - 3 \alpha_1 \beta_1^2 l &= 0 \end{aligned}$$

which its solution yields

$$\alpha_0 = \mp \sqrt{-\frac{3q}{2p}} \beta_0 k \log(\alpha), \quad \alpha_1 = \pm \sqrt{-\frac{3q}{2p}} \beta_1 k \log(\alpha),$$

$$l = -\frac{1}{2} k^3 q \log^2(\alpha)$$

Thus, the following new explicit exact solutions to the conformable time fractional third order modified KdV equation can be written as

$$U_{1,2}(\xi) = \frac{ik \log(\alpha) \sqrt{\frac{3q}{2p}} (\mp \beta_0 \pm \beta_1 \alpha^\xi)}{\beta_0 + \beta_1 \alpha^\xi} \quad (10)$$

where $\xi = kx + \frac{1}{2} k^3 q \log^2(\alpha) \frac{t^\gamma}{\gamma}$.

Fig. 1: Solution profile of $U_{1,2}$ corresponding to $p = 3, q = 2, \beta_0 = 1 = \beta_1$ and $\alpha = 3$

We now consider the time fractional conformable KdV Equation as follows.

$$D_t^\gamma u + puu_x + qu_{xxx} = 0, \quad 0 < \gamma \leq 1, \quad (11)$$

Using the transformation (3), integrating w.r.t. ξ and taking constant of integration is zero, we get

$$-lu + \frac{1}{2} pku^2 + qk^3 u'' = 0. \quad (12)$$

Through balancing the terms we select $N = 2$, the non-trivial solution (4) becomes

$$U(\xi) = \frac{\alpha_2 \alpha^{2\xi} + \alpha_1 \alpha^\xi + \alpha_0}{\beta_2 \alpha^{2\xi} + \beta_1 \alpha^\xi + \beta_0}, \quad \alpha \neq 1 \quad (13)$$

By setting the above non-trivial solution in reduced equation Eq. (12) and equating factors of each power of α^ξ in

the resulting equation, we reach a nonlinear algebraic set as

$$\begin{aligned} \alpha_0^2 \beta_0 kp - 2\alpha_0 \beta_0^2 l &= 0, \\ 2\alpha_1 \beta_0^2 k^3 q \log^2(\alpha) - 2\alpha_0 \beta_0 \beta_1 k^3 q \log^2(\alpha) \\ &+ 2\alpha_0 \alpha_1 \beta_0 kp + \alpha_0^2 \beta_1 kp - 2\alpha_1 \beta_0^2 l - 4\alpha_0 \beta_0 \beta_1 l = 0, \\ 8\alpha_2 \beta_0^2 k^3 q \log^2(\alpha) + 2\alpha_0 \beta_1^2 k^3 q \log^2(\alpha) \\ &- 2\alpha_1 \beta_0 \beta_1 k^3 q \log^2(\alpha) - 8\alpha_0 \beta_0 \beta_2 k^3 q \log^2(\alpha) \\ &+ \alpha_1^2 \beta_0 kp + 2\alpha_0 \alpha_2 \beta_0 kp + 2\alpha_0 \alpha_1 \beta_1 kp + \alpha_0^2 \beta_2 kp \\ &- 2\alpha_2 \beta_0^2 l - 2\alpha_0 \beta_1^2 l - 4\alpha_1 \beta_0 \beta_1 l - 4\alpha_0 \beta_0 \beta_2 l = 0, \\ 6\alpha_2 \beta_0 \beta_1 k^3 q \log^2(\alpha) - 12\alpha_1 \beta_0 \beta_2 k^3 q \log^2(\alpha) \\ &+ 6\alpha_0 \beta_1 \beta_2 k^3 q \log^2(\alpha) + 2\alpha_1 \alpha_2 \beta_0 kp \\ &+ \alpha_1^2 \beta_1 kp + 2\alpha_0 \alpha_2 \beta_1 kp + 2\alpha_0 \alpha_1 \beta_2 kp - 2\alpha_1 \beta_1^2 l \\ &- 4\alpha_2 \beta_0 \beta_1 l - 4\alpha_1 \beta_0 \beta_2 l - 4\alpha_0 \beta_1 \beta_2 l = 0, \\ 2\alpha_2 \beta_1^2 k^3 q \log^2(\alpha) + 8\alpha_0 \beta_2^2 k^3 q \log^2(\alpha) \\ &- 8\alpha_2 \beta_0 \beta_2 k^3 q \log^2(\alpha) - 2\alpha_1 \beta_1 \beta_2 k^3 q \log^2(\alpha) \\ &+ \alpha_2^2 \beta_0 kp + 2\alpha_1 \alpha_2 \beta_1 kp + \alpha_1^2 \beta_2 kp + 2\alpha_0 \alpha_2 \beta_2 kp \\ &- 2\alpha_2 \beta_1^2 l - 2\alpha_0 \beta_2^2 l - 4\alpha_2 \beta_0 \beta_2 l - 4\alpha_1 \beta_1 \beta_2 l = 0, \\ 2\alpha_1 \beta_2^2 k^3 q \log^2(\alpha) - 2\alpha_2 \beta_1 \beta_2 k^3 q \log^2(\alpha) + \alpha_2^2 \beta_1 kp \\ &+ 2\alpha_1 \alpha_2 \beta_2 kp - 2\alpha_1 \beta_2^2 l - 4\alpha_2 \beta_1 \beta_2 l = 0, \\ \alpha_2^2 \beta_2 kp - 2\alpha_2 \beta_2^2 l &= 0. \end{aligned}$$

which its solution yields

$$\begin{aligned} \alpha_0 &= -\frac{2\beta_0 k^2 q \log^2(\alpha)}{p}, \quad \alpha_1 = \frac{4\beta_1 k^2 q \log^2(\alpha)}{p}, \\ \alpha_2 &= -\frac{\beta_1^2 k^2 q \log^2(\alpha)}{2\beta_0 p}, \quad \beta_2 = \frac{\beta_1^2}{4\beta_0}, \\ l &= -k^3 q \log^2(\alpha), \\ \alpha_0 &= 0, \quad \alpha_1 = \frac{6\beta_1 k^2 q \log^2(\alpha)}{p}, \quad \alpha_2 = 0, \\ \beta_2 &= \frac{\beta_1^2}{4\beta_0}, \quad l = k^3 q \log^2(\alpha) \end{aligned}$$

Thus, the following new explicit exact solution to the conformable time fractional third order KdV equation can be written as

$$U_1(\xi) = \frac{-\frac{\beta_1^2 k^2 q \log^2(\alpha)}{2\beta_0 p} \alpha^{2\xi} + \frac{4\beta_1 k^2 q \log^2(\alpha)}{p} \alpha^\xi - \frac{2\beta_0 k^2 q \log^2(\alpha)}{p}}{\frac{\beta_1^2}{4\beta_0} \alpha^{2\xi} + \beta_1 \alpha^\xi + \beta_0} \quad (14)$$

where $\xi = kx + k^3 q \log^2(\alpha) \frac{t^\gamma}{\gamma}$.

$$U_2(\xi) = \frac{\frac{6\beta_1 k^2 q \log^2(\alpha)}{p} \alpha^\xi}{\frac{\beta_1^2}{4\beta_0} \alpha^{2\xi} + \beta_1 \alpha^\xi + \beta_0} \quad (15)$$

where $\xi = kx - k^3 q \log^2(\alpha) \frac{t^\gamma}{\gamma}$.

Fig. 2: Solution profile of U_1 corresponding to $p = 3$, $q = 2$, $\beta_0 = 1 = \beta_1$ and $\alpha = 3$

Fig. 3: Solution profile of U_2 corresponding to $p = 3$, $q = 2$, $\beta_0 = 1 = \beta_1$ and $\alpha = 3$

4 Conclusion

We have succeeded in exploring the explicit exact wave solutions of conformable time fractional KdV equations via $\exp_a$ function method. These solutions are given explic-

itly and verified by setting back in the reduced equations with the aid of Mathematica. Furthermore, the numerical simulation of some but different kind of solutions through MATLAB has been given for the reader to visualize the basic phenomena.

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