



Computer Modeling in Philosophy

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A Computationally Assisted Reconstruction of an Ontological Argument in Spinoza's *The Ethics*

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Abstract: The comments accompanying Proposition (Prop.) 11 (“God ... necessarily exists”) in Part I of Spinoza’s *The Ethics* contain sketches of what appear to be at least three more or less distinct ontological arguments. The first of these is problematic even on its own terms. More is true: even the proposition “God exists” (GE), a consequence of Prop. 11, cannot be derived from the definitions and axioms of Part I (the “DAPI”) of *The Ethics*; thus, Prop. 11 cannot be derived from the DAPI, either. To prove these claims, I use an automated deduction system (ADS) to show that Prop. 11 is independent of the DAPI. I then augment the DAPI with some auxiliary assumptions I believe Spinoza would accept and that sustain an automated derivation of (GE). The results illustrate how an ADS can facilitate the analysis of arguments and yield an apparently novel argument cast in the style of Spinoza.

Keywords: automated deduction, Spinoza, ontological argument

1 Introduction

This paper has two objectives. The first is to show, from the point of view of modern logic, that one of Spinoza’s ontological arguments for Prop. 11 (“God ... necessarily exists”)¹ is not valid.² The second is to overcome some deficiencies in that argument, yielding a valid (and I believe novel) argument that is arguably sympathetic to Spinoza’s intentions. I take no position on whether any proposition in *The Ethics*³ is true.

Ontological arguments are arguments for the claim that (GE) God exists based on premises that purport to derive from some source other than observation of the world,⁴ that is, they purport to derive from reason alone.⁵ They have a long history in the philosophical literature, extending to at least Anselm.⁶

In what follows, I render “exists” in (GE) as a *predicate*. It could be argued that “exists” should instead be rendered in terms of the existential quantifier of a first-order language (FOL). From the perspective of

1 Spinoza, *The Ethics*.

2 In the sense that the conclusion of the argument follows from the premises by virtue of inference rules alone, i.e., without regard to the truth of the premises (Church, *Introduction to Mathematical Logic*, 228, Note 407).

3 Spinoza, *The Ethics*.

4 Oppy, “Ontological Arguments”.

5 What various philosophers would regard as “derivable from reason alone” varies widely. Descartes’ (Descartes, *Meditations*) and Locke’s (*An Essay*) views of this issue, for example, are rejected by Kant (*Critique*).

6 Anselm, *Proslogium*.

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modern logic, rendering “exists” in terms of the existential quantifier of FOL would seem to be “natural”. If one adopts this rendering, furthermore, then on a widely held view of the ontological commitments of FOL, (GE) inherits a “natural” interpretation of the ontological commitment of (GE).

The question of how to render “exists” in the context of (ontological) arguments that were formulated well before the era of modern logic, however, remains a subject of debate, for at least two reasons. First, Spinoza’s text does not allow us to unproblematically interpret “Exists” as the existential quantifier of FOL. Second, at least prior to Kant many philosophers regarded “existence” as a predicate, especially when “existence” was invoked in ontological arguments.⁷ Given this difficulty, rendering “exists” as a predicate is more ontologically circumspect than reading “exists” in terms of the existential quantifier.

The expository model of Part I of *The Ethics*, at least on the surface, *appears* to be Euclidean: definitions and axioms are first stated, then propositions/theorems are derived using only these definitions and axioms together with a set of inference rules. For the purpose of this paper, I assume as a working hypothesis that Spinoza intended to conform to Euclid’s model.

In what follows, furthermore, I adopt a logistic⁸ approach to Spinoza’s text. Taking the logistic approach helps to reveal, among other things, those aspects of an argument (its bare logical form) that do not trade on meanings a concrete argument might assume. Among other things, the logistic approach considers a logic to be a system of uninterpreted symbols, variously combinable and derivable according to a set of formation and derivation/inference rules.

Important features of the logistic method appear explicitly even in Aristotle’s theory of syllogism.⁹ That account abstracts the bare logical form of an argument from concrete instantiations of that form. For example,

All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal.

has the logical form

All A are B.
C is A.
Therefore, C is B.

A logistic abstraction of an argument evidently has no warrant to infer meanings as such, because it does not deal with meanings. For example, if in an argument the predicate “in itself” is attributed to an entity, *x*, from the logistic point of view we have no warrant on the basis of that attribution to infer that *x* is “self-caused”, because the inference from “in itself” to “self-caused” involves some consideration of meaning.

It might be objected, of course, that a logistic approach to Spinoza’s text could hardly claim to yield a *complete* picture of what Spinoza intended. I readily agree. That objection, however, is not about whether the logistic method can provide insight, but only about whether the logistic method can capture all insights of interest, into Spinoza’s view.

⁷ In “A computationally-discovered simplification of the ontological argument”, for example, Oppenheimer and Zalta render “exists” in Anselm’s *Proslogium* as a predicate.

⁸ Church, *Introduction to Mathematical Logic*, Section 07.

⁹ Aristotle, *Prior Analytics*.

To facilitate the logistic approach in this paper, I use an automated deduction system (ADS)¹⁰ that implements the logistic approach. In particular, I use two closely related ADS tools, *prover9* and *mace4*.¹¹ *mace4* is a software framework that searches for finite models¹² of a finite set of propositions expressed in a first-order language.¹³ *prover9* is a software framework that searches for derivations of a set of first-order propositions from another set of first-order propositions, given a set of derivation rules. *prover9* shares much of the syntax of *mace4*.¹⁴

Section 2.0 of this paper shows that the first argument sketch following the statement of Prop. 11¹⁵, which I will call “Spinoza’s first ontological argument” (SFOA), is not, when considered from a logistic point of view, valid. Section 3.0 then tackles the more speculative task of overcoming the deficiencies identified in Section 2.0. More specifically, Section 3.0 shows that Prop. 7¹⁶ (“Existence belongs to the nature of substances”), which Spinoza invokes in SFOA, can be derived from the definitions and axioms of Part I of *The Ethics* (the “DAPI”) conjoined with some auxiliary assumptions that I believe Spinoza would likely accept. Section 3.0 also shows that, without using/implying Prop. 7, (GE) can be derived from the DAPI conjoined with a set of auxiliary assumptions that are subset of those required to derive Prop. 7.

Much of the low-level detail of the arguments in this paper is relegated to Appendices. There are two good reasons for this structure. First, understanding the detail in the Appendices is not essential for understanding the general structure and flow of the arguments in this paper. Second, the low-level detail is too large to fit in a normal journal article: it spans ~270 pages. The Appendices are available in Supplement to this article.¹⁷

The scripts used in this work were executed on a Dell Inspiron 545 containing an Intel Core2 Quad CPU Q8200 (clocked @ 2.33 GHz) and 8.00 GB RAM, running under the *Windows Vista Home Premium* /*Cygwin* and *Windows 10/Cygwin* operating environments. Each script mentioned in this paper, when executed on these platforms, produced a solution in less than 0.1 second.

2 Prop. 11 cannot be derived from the definitions and axioms of Part I of *The Ethics*

The comments accompanying Prop. 11 (“God, or substance, consisting of infinite attributes, of which each expresses eternal and infinite essentiality, necessarily exists”) in Part I of Spinoza’s *The Ethics*¹⁸ contain sketches of what appear to be at least three more or less distinct ontological arguments.¹⁹ SFOA begins by asserting that if we deny Proposition 11, then we are claiming that God does not exist. But to deny God’s

¹⁰ This approach is in some ways similar to the method of Oppenheimer and Zalta, “A computationally-discovered simplification of the ontological argument”, which is likely to be the first published use of an ADS in “conventional” philosophical literature. *prover9*, *mace4* and their predecessors have been widely used in mathematical logic applications (see, for example: *Journal of Automated Reasoning*; Quaipe, *Automated Development*) for about three decades.

¹¹ McCune, *prover 9 and mace4*.

¹² Chang and Keisler, *Model Theory*.

¹³ Church, *Introduction to Mathematical Logic*.

¹⁴ Kalman, *Automated Reasoning*, contains good account of the automated deduction theory underlying *prover9* and *mace4*. See especially Chaps. 1, 2, 4-6, 7, 9, 10, 13, and 14.

¹⁵ Spinoza, *The Ethics*.

¹⁶ Spinoza, *The Ethics*, 48.

¹⁷ <https://doi.org/10.1515/opphil-2019-0012>.

¹⁸ Spinoza, *The Ethics*.

¹⁹ It’s not clear how many ontological arguments Spinoza intended to subsume under the discussion of Prop. 11 in *The Ethics*. In those passages, he suggests that he has articulated three. But on closer inspection, those arguments variously and nontrivially share at least one premise, so are not logically independent. Moreover, one could argue that there are fragments of two other ontological arguments in those passages; at least one of the latter seems to have little relation to the DAPI.

existence, Spinoza continues, is to deny that God's essence involves God's existence. That, however, is absurd, he argues, by virtue of Prop. 7²⁰ ("Existence belongs to the nature of substances").

Can the promise of SFOA (to derive Prop. 11) be realized within the resources of the the DAPI? In this section, I argue that Prop. 11 cannot be derived from the DAPI, for the following reason. A consequence of Prop. 11, (GE), cannot be derived from the DAPI; thus, by the Deduction Theorem, neither, in any of the logics discussed in Church's *Introduction to Mathematical Logic*²¹ and Cocchiarella's and Freund's *Modal Logic* (hereafter called "CFML"),²² can Prop. 11.

2.1 Rendering the DAPI in an automated deduction framework

The contents of the DAPI were rendered in the *mace4* framework (see Figure 1). Whether any rendering of a given set of natural-language sentences is captured in a set of first-order language expressions – in this case, in the *mace4* language – can be assessed only through informed inspection.²³

That said, at least one issue in Figure 1 merits special attention: What role, if any, does modality play in the DAPI? To make that question well posed, we must first be clear what we mean by "modality". At least some terms etymologically related to modal qualifiers (e.g., "necessarily P", "possibly P", where P is a proposition, a la CFML) appear in the DAPI – in particular, in Definitions VII and VIII, and Axiom III. Moreover, several philosophers²⁴ have taken at least some occurrences of "necessarily" in *Anselm's* ontological argument(s)²⁵ to be modal qualifiers in the sense of at least one of the modal logics discussed in CFML. Thus, it would seem that an adequate rendition of the DAPI would have to consider whether modal qualifiers in the sense of CFML are in play in the DAPI.

There is no question that modal qualifiers are in play in at least *some* ontological arguments, and Anselm's formulation(s) may be the paradigms of such. How to capture, in those cases, how modal qualifiers play in those contexts is an important and very interesting question.

Do all ontological arguments require the use of modal qualifiers in the sense of CFML? At least *some* don't.²⁶ We therefore have to assess, for any given ontological argument, whether it actually depends modal qualifiers in the sense of CFML.

In this paper, I assume as a working hypothesis that modal qualifiers in the sense of CFML play no role in the DAPI proper, i.e., as the DAPI are articulated in pages 45-46 of *The Ethics*, for two reasons. First, one could provisionally adopt any working hypothesis in order to investigate what effect it might have on an analysis of "Spinoza's" views. All other suffering being the same, under that provision we suspend judgment, for the immediate purposes of the analysis, about whether the hypothesis "faithfully reflects Spinoza's views". Second, nowhere do the occurrences of nominally "modal" language in Spinoza's formulation of the DAPI proper unambiguously imply that "modal" language is being used as a modal qualifier in the sense of CFML. One could, of course, *choose* to interpret some of Spinoza's use of nominally modal language in a way that uses modal qualifiers in the sense of CFML. Such a choice is itself a working hypothesis. In any case, for the remainder of this paper I reserve the term "DAPI" to mean the axioms and definitions rendered in the *mace4* statements in Figure 1. The DAPI in this reserved sense are evidently non-modal in the sense of CFML.

²⁰ Spinoza, *The Ethics*, 48.

²¹ Church, *Introduction to Mathematical Logic*.

²² Cocchiarella and Freund, *Modal Logic*. These include Kr, M, Br, S1, S2, S3, S4, S4.2, S4.3, and S5.

²³ Comprehensively justifying each of the renderings in Figure 1 is beyond the scope of this paper, and, at some point, would devolve to attempting to prove a negative. The reader can critically assess the mapping between the English-language statements of the definitions and axioms of the DAPI (reproduced in comments in Figure 1) and the first-order renderings of those statements contained in Figure 1.

²⁴ See, for example: Oppenheimer and Zalta, "A computationally-discovered simplification of the ontological argument"; Gödel, "Ontological proof"; Benzmüller and Paleo, "Automating"; Scott, "Notes".

²⁵ Anselm, *Prosologium*.

²⁶ See, for example, the survey of ontological arguments in Oppy, *Ontological Arguments and Belief in God*.

```

assign(iterate_up_to, 10).
set(print_models_tabular).
formulas(theory).

% BEGIN BLOCK A

% DEFINITIONS

% Definition I. Self-caused. By that which is self-caused, I mean
% that of which the essence
% involves existence, or that of which the nature is only
% conceivable as existent. Note that "or" in the phrase
% "... or that of which the nature ..."
% must be rendered as "&" to capture what Spinoza actually
% means.
SelfCaused(x) <-> ( EssenceInvExistence(x) &
                    NatureConcOnlyByExistence(x) )
                    # label("Definition I: self-caused").

% Definition II. Finite after its kind. A thing is called
% finite after its kind, when it can be limited by another
% thing of the same nature.
FiniteAfterItsKind(x) <-> ( CanBeLimitedBy(x,y) & SameKind(x,y) )
                    # label("Definition II: finite after its kind").

% Definition III. Substance. By substance, I mean that which is in
% itself, and is conceived through itself.
Substance(x) <-> InItself(x) &
                    ConceivedThruItself(x) # label("Definition III: substance").

% Definition IV. Attribute. By attribute, I mean that which the intellect
% perceives as constituting the essence of substance.
Attribute(x) <-> IntPercAsConstEssSub(x) # label("Definition IV: attribute").

% Definition V. Mode. By mode, I mean the modifications of substance,
% or that which exists in, and is conceived through, something
% something other than itself.
Mode(x) <-> ( ( Modification(x,y) & Substance(y) ) |
              ( ExistsIn(x,z) & ConceivedThru(x,z) ) ) # label("Definition V:
mode").

% Definition VI. God. By God, I mean a being absolutely infinite.
God(x) <-> ( Being(x) & AbsolutelyInfinite(x) ) # label("Definition VI: God").
% Definition VI. Absolutely infinite. ... that is, a substance consisting in
infinite
% attributes, of which each expresses eternal and infinite essentiality.
AbsolutelyInfinite(x) <-> ( Substance(x) & ConstInInfAttributes(x) &
                          ( AttributeOf(y,x) -> ( ExpressesEternalEssentiality(y) &
                                                  ExpressesInfiniteEssentiality
(y) ) ) )
                          # label("Definition VI: absolutely infinite").

% Definition VII. Free. That thing is called free, which exists solely by the
% necessity of its own nature, and of which the action is determined by itself
alone.

```



```

% Axiom VI. A true idea must correspond with its ideate or object.
TrueIdea(x) -> ( CorrespondWith(x,y) & ( IdeateOf(y,x) | ObjectOf(y,x) ) )
# label("Axiom VI").

% Axiom VII. If a thing can be conceived as non-existing, its
%           essence does not involve its existence.
CanBeConceivedAsNonExisting(x) -> -EssenceInvExistence(x)
# label("Axiom VII").

% END OF BLOCK A

end_of_list.

```

Figure 1: A *mace4* script for the DAPI. A percent (%) mark at the beginning of a line delimits a comment. For the purpose of this paper, a *mace4* statement is any non-blank sequence of characters that does not begin with a % sign and which ends with a period. The *mace4* statements shown in this Figure are intended to be sentences in a first-order language. More specifically, “<->” (if and only if), “->” (implies), “&” (and), “|” (or), “=” (identity), “!=” (not identical), and “-.” (not) are standard first-order connectives.²⁷ *x* and *y* are first-order variable names.²⁸ All variables, unless otherwise specified, are (implicitly) universally quantified.²⁹ Strings of the form “# label (...)” are tag-phrases that *mace4* transmits in various ways through a *mace4* analysis. The content of such tags can be formulated in such a way that they help facilitate traceability of the use of the tagged statement in a derivation or a model construction. Each *mace4* statement in this Figure is preceded by a comment containing a literal copy of the text³⁰ that the *mace4* sentence immediately following that comment is intended to represent. The comments “% BEGIN BLOCK A” and “% END BLOCK A” delimit a sequence that is incorporated by reference in subsequent scripts in this paper.³¹

2.2 The DAPI are consistent

It’s important to determine at the outset whether a given axiomatization is consistent; if it is not, “anything” is derivable from it.³² How can we show that the DAPI are consistent? A fundamental theorem of model theory tells us that to show a theory is consistent, it suffices to show that there is a *model* of that theory.³³ Informally, a model of a set of sentences is a universe, together with a mapping of those sentences that renders the sentences “true” in that universe. What is permitted as a universe can be almost anything that is internally consistent in at least one possible world: the universe of a model need not, and in practice often does not, bear any relationship to what someone might regard as the “natural” meaning for a set of axioms and definitions. Indeed, from the perspective of model theory, there is no privileged meaning of a set of sentences expressed as a set of uninterpreted symbols.³⁴ To show that the DAPI are consistent, therefore, it suffices to show that there is a model of the DAPI. Appendix 11 documents in detail a model of the DAPI

²⁷ Church, *Introduction to Mathematical Logic*.

²⁸ The DAPI, (SE), (GE), and the Auxiliary Assumptions (described below) all fall within the Effectively Propositional fragment of FOL (see Drews and Albarghouthi, “Effectively Propositional Interpolants”); therefore, all questions about them are decidable. I thank an anonymous referee for this observation.

²⁹ An anonymous referee rightly noted that explicitly quantifying *mace4* scripts is a desirable practice.

³⁰ Spinoza, *The Ethics*, 45-46.

³¹ For further details of *mace4* syntax and semantics, see McCune, *prover9 and mace4*; Chang and Keisler, *Model Theory*, Chap. 1.

³² Chang and Keisler, *Model Theory*, 25.

³³ *Ibid.*, 18, 33.

³⁴ For details of what it means for a sentence to be true in a universe, see Chap. 1 of Chang and Keisler, *Model Theory*. The models generated by *mace4* (and by any other general-purpose model generator) are determined solely by the concepts of formal model theory (see, for example, Chang and Keisler, *Model Theory*). As such, these models, including their domains, have no topic-specific meaning outside of the concepts of model theory. Therefore, the elements of the domain of such models do not “represent” anything that is topic-specific outside that theory.

that is generated by *mace4*. Informally, here’s a sketch, using Appendix 11 as an example, of how to read the *mace4* proofs used in this paper.³⁵

Figure 2 is an excerpt from Appendix 11.

```

===== Mace4 =====
. . .
===== end of head =====

===== INPUT =====
assign(iterate_up_to,10).
  % assign(iterate_up_to, 10) -> assign(end_size, 10).
set(print_models_tabular).
  % set(print_models_tabular) -> clear(print_models).

formulas(theory).
SelfCaused(x) <-> EssenceInvExistence(x) & NatureConcOnlyByExistence(x) # label
("Definition I: self-caused").
. . .
end_of_list.
===== end of input =====

===== PROCESS NON-CLAUSAL FORMULAS =====
% Formulas that are not ordinary clauses:
1 SelfCaused(x) <-> EssenceInvExistence(x) & NatureConcOnlyByExistence(x) # label
("Definition I: self-caused") # label(non_clause). [assumption].
. . .
end_of_list.

===== CLAUSES FOR SEARCH =====
formulas(mace4_clauses).
-SelfCaused(x) | EssenceInvExistence(x) # label("Definition I: self-caused").
. . .
end_of_list.
===== end of clauses for search =====
. . .
===== DOMAIN SIZE 2 =====
. . .

SelfCaused :
      0 1
      -----
      0 0
. . .

Exiting with 1 model.
. . .

```

Figure 2: An excerpt from the *mace4* output shown in Appendix 11. “. . .” indicates that I deleted text in *mace4*’s output at this location.

³⁵ The example does not imply that one must perform a manual confirmation of all models produced by *mace4*. Such an effort would be, even for some relatively small axiom systems, intractable. Worse, this kind of issue backs into questions about whether we can comprehensively verify software (see, for example, Horner and Symons, “Software errors”). Addressing questions about the reliability of a given software system, unless we already have definite, assignable reasons for doubting that reliability, is beyond the scope of this paper.

Refer to Figure 2. Given the way *mace4* executions are configured for this paper, a *mace4* output has several sections. In the order they appear, these sections include:

1a processing header, whose beginning delimiter contains the string “Mace 4” and whose ending delimiter contains the string “end of head”. This section contains software version number information, process and platform identifiers, a run date-time, and part of the command string used to invoke *mace4*. You don't need this information to be able to informally understand the models *mace4* produces, but it is useful for configuration management purposes.

2an input section, whose beginning delimiter contains the string “INPUT” and whose ending delimiter contains the string “end of input”. This section echos the content of the file input to *mace4*. It is useful for reference and configuration management purposes, but typically you will not need it to informally understand the models *mace4* produces.

3a list of “non-clausal formulas” that occur in the “INPUT” section. The beginning delimiter of the “non-clausal formulas” section contains the string “PROCESS NON-CLAUSAL FORMULAS” is a list the formulas in the “INPUT” section that *mace4* will translate to a logically equivalent clausal form that facilitates *mace4*'s search for a model. Understanding the details of the translation from formulas of the kind that appear in the “INPUT” section to *mace4*'s clausal form is not required for an informal understanding the models *mace4* produces. *mace4* translates all the formulas in the “INPUT” section into that clausal form.

4a clausal translation section, whose beginning delimiter contains the string “CLAUSES FOR SEARCH” and whose ending delimeter contains the string “end of clauses for search”. This section contains *mace4*'s translation of the formulas described in (3) to *mace4*'s clausal form. In order to show that such a sentence is “true”, it suffices to show that at least one disjunct in that setence is true.

5a model-description section that is produced if *mace4* finds a model. The model-description section has a beginning delimiter that contains the string “DOMAIN SIZE n” (where n is an integer greater than 0; in the example, n = 2. In the example, the two domain elements are the integers 0 and 1). This model-description section contains the fundamental description of the model of interest. Further information on how to read this section are contained in the narrative below.

6a process-exit summary, only a portion of which (“Exiting with 1 model”) is shown in Figure 2. For the purposes of an informal understanding of the models that *mace4* produces, the phrase “Exiting with 1 model” is important. It confirms that *mace4* has found a model. If *mace4* executed correctly, that information is sufficient to show that there is a model of the set of sentences (in the example, the information shows that the DAPI conjoined with the negation of (GE)) has a model.

If the phrase “Exiting with 1 model” in the process-exit summary (see (6) above) is sufficient to convince you that a model of *mace4*'s input exists, you can skip the remainder of the narrative in this section (Section 2.1). Else, here's a sketch of how to “manually” confirm that *mace4* has found a model of input.

First, consider a sentence in the input to *mace4*. For illustration, let's choose the first such sentence

```
SelfCaused(x) <-> EssenceInvExistence(x) &
NatureConcOnlyByExistence(x)
```

in the PROCESS NON-CLAUSAL FORMULAS section of the *mace4* output. Note that this sentence comes directly from Figure 1.

mace4 translates this sentence into a sequence of sentences, and reports that translation in the CLAUSES FOR SEARCH section. The first sentence in that translation happens to be

```
-SelfCaused(x) | EssenceInvExistence(x)
```

In the DOMAIN SIZE 2 section, locate the subheader

SelfCaused :

Under the “SelfCaused :” subheader, the first line lists the possible values *mace4* has determined that variable *x* could be assigned in this model. Under the (*x* =) “0” and under the (*x* =) “1” that occurs above the dashed line, the value “0” appears below the dashed line. This means that *mace4* has assigned the expression “SelfCaused (0)” and “SelfCaused(1)” the value “0”, which *for the purposes of the example*, we can interpret as “SelfCaused (0)’ and ‘SelfCaused(1)’ are assigned the truth-value FALSE”. That is, under this assignment, the first disjunct of

$$\neg \text{SelfCaused}(x) \mid \text{EssenceInvExistence}(x)$$

is “TRUE” for all possible values of *x* (0 or 1). Thus, at least one disjunct in the sentence shown is TRUE, and therefore that entire sentence is TRUE. That means that under the assignment mentioned,

$$\neg \text{SelfCaused}(x) \mid \text{EssenceInvExistence}(x)$$

is satisfied by the model *mace4* produces in this example.

Repeat the procedure sketched above for all the sentences in the CLAUSES FOR SEARCH section. If all these sentences evaluate to TRUE, there is a model for the INPUT section for *mace4*, or equivalently, all the formulas in the INPUT section have a model.

Because all the *mace4* scripts in the Appendices are highly similar to the one in Appendix 11, they can all be interpreted by following the rubric sketched above.

2.3 (GE) cannot be derived from the DAPI

Could at least (GE), which is a (non-modal) implication of Prop. 11 in any of the modal logics described in CFML be derived from the DAPI? Whatever Spinoza may have had in mind, it seems highly likely that Df. VI of the DAPI (“By God, I mean ... a substance consisting in infinite attributes, of which each expresses eternal and infinite essentiality”),³⁶ would play some role in a more explicit derivation. From Df. VI, at least, we could derive that “God is substance”, and from that, together with Prop. 7, we could derive (GE). To deny that God exists, therefore, would be, by modus tollens, to deny that God is substance, which would contradict Df. VI, an “absurdity” in the sense Spinoza appears to use that term in SFOA.

We thus might hope, by careful textual analysis, to flesh out the sketch in SFOA along the lines he suggests in order to show (GE). But no search for a derivation of (GE) from the DAPI proper, I will argue, is possible, because (GE), which is an implication of Prop. 11, cannot be derived from the DAPI.

In model theory, a sentence B is *independent* of a set of sentences A if there is a model in which

A & -B

is “true”.³⁷

To prove the independence of (GE) from the DAPI, therefore, it suffices to show that there is a model that satisfies the DAPI and also satisfies the negation of (GE). Figure 3 shows the *mace4* script used to show the independence of (GE) from the DAPI.

³⁶ Spinoza, *The Ethics*, 45.

³⁷ Chang and Keisler, *Model Theory*, 18, 33.

```

formulas(theory).
% INSERT BLOCK A (from Figure 1) here
% Let a be God.
  God(a).
% Negate "God exists".
  -Exists(a).
    # label("Negate *God exists*").
end_of_list.

```

Figure 3: A *mace4* script for generating a model that shows the independence of (GE) from the DAPI.

Executing the *mace4* script shown in Figure 2 produces the model shown in Appendix 1. Thus, (GE), and therefore by the Deduction Theorem, Prop. 11 (“God ... necessarily exists”), cannot be derived from the DAPI.

2.4 Prop. 7 cannot be derived from the DAPI

SFOA invokes Prop. 7 (“Existence is involved in the nature of substances”).³⁸ For the purposes of this paper, I assume as a working hypothesis that Prop. 7 can be captured by

(SE) If x is/has substance, x exists

If we *assume* (SE), a concise argument for (GE) follows. In particular:

- a God is absolutely infinite (Df. VI).
- b If x is absolutely infinite, then x has substance (Df. VI).
- c By (a) and (b), God has substance.
- d If x is/has substance, x exists (SE).
- e By (c) and (d), God exists (GE).

Figure 4: A derivation of (GE) that assumes (SE) and the DAPI.

Figure 4 is not literally identical to SFOA, but if we recast it as a proof by contradiction, beginning with the assumption “God does not exist” (negation of (e) in Figure 3), the recasting implies the equivalent of the SFOA. *If* (SE) can be derived from the DAPI (Spinoza thinks he has shown as much³⁹), the derivation of (GE) shown in Figure 3 would follow from the DAPI alone.

I now show that (SE) *cannot* be derived from the DAPI by showing that (SE) is independent of the DAPI. To prove the independence of (SE) from the DAPI, it suffices to show that there is a model that satisfies the DAPI and also satisfies the negation of (SE).⁴⁰ Figure 5 is a *mace4* script that can be used to show the independence of (SE) from the DAPI.

³⁸ Spinoza, *The Ethics*, 48.

³⁹ *Ibid.*

⁴⁰ Chang and Keisler, *Model Theory*, 18, 33.

```

formulas(theory).

% INSERT BLOCK A (from Figure 1) HERE

% Proposition VII. Negate "substance -> existence".
  -( Substance(x) -> Exists(x) )
      # label("Negation of (SE).").

end_of_list.

```

Figure 5: A *mace4* script for generating a model that shows the independence of the (SE) from the DAPI.

Appendix 2 depicts a model, produced by the script shown in Figure 5, that shows the independence of (SE) from the DAPI. Thus, (SE) does not follow from the DAPI. Given this result, by the Deduction Theorem, Prop. 7 does not follow from the DAPI, either. Thus Prop. 7, contrary to what Spinoza suggests, does not help us to derive (GE).

3 Can the deficiencies (of SFOA) identified in Section 2.0 be overcome?

Section 2.0 shows that, taken as formulated, SFOA is not valid. Can these deficiencies be overcome without doing violence to Spinoza's idiom?

This question is challenging in several ways. First, from a purely logistic point of view, it has no unique answer, because "Spinoza's idiom", broadly conceived, presumably concerns at least some issues of meaning, and the logistic approach contains no warrant for preferring any meaning over any other. Second, what is meant by "doing violence to Spinoza's idiom" may lie partly in the mind of the perceiver. To be sure, the discovery of ways to redeem SFOA is not a mechanical process. Third, historical texts often underdetermine interpretation.

Those challenges stipulated, at least three desiderata are in play in any selection of candidate auxiliary assumptions to be used for this purpose. The candidate auxiliary assumptions should:

- D1. when conjoined with the DAPI, imply (GE).
- D2. be formulated in such a way that they would be regarded as "nearly" definitional or tautological when considered in the context of Spinoza's philosophy.
- D3. be independent of BLOCK A (see Figure 1) conjoined with the remaining auxiliary assumptions.

Given (D1) – (D3), consider the Auxiliary Assumptions shown in Figure 6.⁴¹

```

Substance(x) -> Being(x)
  # label("Auxiliary assumption 1: if x is/has substance, x is a being").

InItself(x) -> SelfCaused(x)
  # label("Auxiliary assumption 4: if x is in itself, x is self-caused").

Being(x) -> HasEssence(x)
  # label("Auxiliary assumption 7: If x has being, then x has essence").

(EssenceInvExistence(x) & HasEssence(x)) -> Exists(x)
  # label("Auxiliary assumption 8: if the essence of x involves
  the existence of x and x has essence, then x exists").

```

Figure 6: A set of Auxiliary Assumptions that are candidates for overcoming some of the deficiencies of SFOA.

⁴¹ Other Auxiliary Assumptions candidates may be possible. I formulated the Assumptions shown in Figure 6 by conjecture and testing. To say more is to say less. (For the purpose of this paper, the labels "Auxiliary assumption N", where N = 1, 4, 7, 8, are arbitrary unique labels. There is no significance to the fact that N has just these values and no others, such as 2, 3, 5, or 6.)

I think it likely that Spinoza would have accepted Auxiliary Assumptions 1, 4, 7, and 8, for the following reasons:

- i. Auxiliary Assumptions 1 and 7 were widely held by the Aristotelian Scholastics⁴² and Spinoza was deeply aware of (but hardly committed to) the Scholastic tradition.⁴³
- ii. The predicates of the hypotheses of Auxiliary Assumptions 4 and 8 are arguably what Spinoza would have regarded as involving the meaning of the respective predicates of the consequents of those Assumptions.

Before proceeding, it's worth asking whether there is a more succinct representation of the DAPI and Auxiliary Assumptions 1, 4, 7, 8. (If a more succinct representation were within easy reach, we would likely want to use it.) In its most general form, however, this question is not well-posed, because it is always "possible" that some more general representation of the DAPI conjoined with Auxiliary Assumptions 1, 4, 7, and 8 might be found in the future. We can replace this ill-formed question with a more modest one that is well-posed by asking instead whether the members of a theory T (here, the DAPI conjoined with Auxiliary Assumptions 1, 4, 7, and 8) are independent. If they are, none can be derived from rest.⁴⁴ Such a result would imply that *given these sentences as is*, there can be no more concise representation of T than that given by the sentences of T (i.e., none of the sentences can be deleted from T without loss). Jointly, Appendices 3, 4, 5, 6, and 12-28 imply that the individual formulae in the DAPI and each of Auxiliary Assumptions 1, 4, 7, and 8, as formulated, are independent. Thus, there is no more concise representation of *this* collection of propositions, i.e., none can be discarded without loss of implicational content.

3.1 (SE) can be derived from the DAPI conjoined with the Auxiliary Assumptions 1, 4, 7, and 8

Figure 7 shows a *prover9* script that can derive (SE) from the DAPI conjoined with Auxiliary Assumptions 1, 4, 7, and 8.

```
formulas(assumptions).
%   Insert BLOCK A (from Figure 1) here
%   Insert Auxiliary Assumptions 1,4,7, and 8 here
end_of_list.

formulas(goals).
%   Proposition 7. "Substance -> existence".
    Substance(x) -> Exists(x)
        # label("Proposition 7: Substance exists").
end_of_list.
```

Figure 7: A *prover9* script⁴⁵ for generating a derivation of (SE) from the DAPI conjoined with Auxiliary Assumptions 1, 4, 7, and 8.

Appendix 7 shows the derivation of (SE) from the script in Figure 7.

⁴² See, for example, Aquinas, *On Being and Essence*, Chapter 1; Aristotle, *Metaphysics*, Book IV, Chap. 2.

⁴³ Nadler, *Spinoza's Ethics*.

⁴⁴ Chang and Keisler, *Model Theory*, 33.

⁴⁵ McCune, *prover9 and mace4*.

3.2 (SE) is independent of the DAPI conjoined with Auxiliary Assumptions 4, 7, and 8

Figure 8 shows a *mace4* script that can be used to derive the independence of (SE) from the DAPI conjoined with Auxiliary Assumptions 4, 7, and 8.

```
formulas(theory).
    % INSERT BLOCK A here.
    % Insert Auxiliary Assumptions 4, 7, and 8 here.
% Negate Prop. 7.
-(Substance(x) -> Exists(x)).
end_of_list.
```

Figure 8: A *mace4* script that shows (SE) is independent of the DAPI conjoined with Auxiliary Assumptions 4,7, and 8 (i.e., (SE) is independent of the assumptions used to derive (GE)).

3.3 Without using/implying (SE), the DAPI conjoined with Auxiliary Assumptions 4, 7, and 8 imply (GE)

It turns out that if we conjoin to the DAPI with Auxiliary Assumptions 4,7, and 8, (GE) follows without implying (SE). Figure 9 shows a *prover9* script that generates such a derivation. Appendix 8 shows the resulting derivation of (GE).

```
formulas(assumptions).
% Insert BLOCK A here
% Insert Auxiliary Assumptions 4, 7, and 8 here
God(a).
end_of_list.

formulas(goals).

% Prop 11 (GE)
Exists(a).

end_of_list.
```

Figure 9: A *prover9* script using the DAPI and Auxiliary Assumptions 4, 7, and 8 that implies (GE).

Because the proof generated by the script shown in Figure 9 is a major result of this paper, let's unpack at least some of the details of that proof. Refer to Figure 10, which contains an excerpt from the *prover9* output corresponding to the input file listed in Figure 9.

```
===== Prover9 =====
Prover9 (32) version 2009-11A, November 2009.
Process 1752 was started by #AUTHOR on DESKTOP-AM4IKPU,
Tue May 7 08:51:53 2019
The command was "../bin/prover9".
===== end of head =====
```

```

===== INPUT =====

formulas(assumptions).
SelfCaused(x) <-> EssenceInvExistence(x) & NatureConcOnlyByExistence(x) #
label("Definition I: self-caused").
FiniteAfterItsKind(x) <-> CanBeLimitedBy(x,y) & SameKind(x,y) # label("Definition II:
finite after its kind").
Substance(x) <-> InItself(x) & ConceivedThruItself(x) # label("Definition III:
substance").
Attribute(x) <-> IntPercAsConstEssSub(x) # label("Definition IV: attribute").
Mode(x) <-> Modification(x,y) & Substance(y) | ExistsIn(x,z) & ConceivedThru(x,z) #
label("Definition V: mode").
God(x) <-> Being(x) & AbsolutelyInfinite(x) # label("Definition VI: God").
AbsolutelyInfinite(x) <-> Substance(x) & ConstInInfAttributes(x) & (AttributeOf(y,x) ->
ExpressesEternalEssentiality(y) & ExpressesInfiniteEssentiality(y)) # label("Definition
VI: absolutely infinite").
Free(x) <-> ExistsOnlyByNecessityOfOwnNature(x) & (ActionOf(y,x) ->
DeterminedByItselfAlone(y,x)) # label("Definition VII: free").
Necessary(x) <-> ExternalTo(y,x) & DeterminedByFixedMethod(x,y) &
DeterminedByDefiniteMethod(x,y) & (IsMethodAction(y) | IsMethodExistence(y)) #
label("Definition VII: necessary").
Eternity(x) <-> ExistConcFollowFromDefEternal(x) # label("Definition VIII: eternity").
Exists(x) <-> ExistsIn(x,x) | ExistsIn(x,y) & x != y # label("Axiom I").
-ConceivedThru(x,x) -> ConceivedThru(x,y) & x != y # label("Axiom II").
DefiniteCause(x) -> EffectNecessarilyFollowsFrom(y,x) & (-DefiniteCause(x) -> -Effect
NecessarilyFollowsFrom(y,x)) # label("Axiom III").
KnowledgeOfEffect(x,y) <-> KnowledgeOfACause(x) # label("Axiom IV: The knowledge of an
effect depends on and involves the knowledge of a cause").
HaveNothingInCommon(x,y) -> -CanBeUnderstoodInTermsOf(x,y) &
-CannotBeUnderstoodInTermsOf(y,x) & -ConceptionInvolves(x,y) & -ConceptionInvolves(y,x)
# label("Axiom V: Things which have nothing in common cannot be understood, the one by
means of the other.").
TrueIdea(x) -> CorrespondWith(x,y) & (IdeateOf(y,x) | ObjectOf(y,x)) # label("Axiom VI").
CanBeConceivedAsNonExisting(x) -> -EssenceInvExistence(x) # label("Axiom VII").
InItself(x) -> SelfCaused(x) # label("Auxiliary assumption 4: if x is in itself, x is
self-caused").
Being(x) -> HasEssence(x) # label("Auxiliary assumption 7: If x has being, then x has
essence").
EssenceInvExistence(x) & HasEssence(x) -> Exists(x) # label("Auxiliary assumption 8:
if the essence of x involves the existence of x and x has essence, then x exists").
God(a).
end_of_list.

formulas(goals).
Exists(a) # label("Prop. XI: God exists").
end_of_list.
===== end of input =====

```

. . .

```

===== PROOF =====
% Proof 1 at 0.03 (+ 0.05) seconds.
% Length of proof is 27.
% Level of proof is 5.

. . .

1  SelfCaused(x) <-> EssenceInvExistence(x) & NatureConcOnlyByExistence(x) #
label("Definition I: self-caused") # label(non_clause). [assumption].
3  Substance(x) <-> InItself(x) & ConceivedThruItself(x) # label("Definition III:
substance") # label(non_clause). [assumption].
6  God(x) <-> Being(x) & AbsolutelyInfinite(x) # label("Definition VI: God") # label(non_
clause). [assumption].
7  AbsolutelyInfinite(x) <-> Substance(x) & ConstInInfAttributes(x) & (AttributeOf(y,x) ->
ExpressesEternalEssentiality(y) & ExpressesInfiniteEssentiality(y)) # label("Definition
VI: absolutely infinite") # label(non_clause). [assumption].
18 InItself(x) -> SelfCaused(x) # label("Auxiliary assumption 4: if x is in itself, x
is self-caused") # label(non_clause). [assumption].
19 Being(x) -> HasEssence(x) # label("Auxiliary assumption 7: If x has being, then x
has essence") # label(non_clause). [assumption].
20 EssenceInvExistence(x) & HasEssence(x) -> Exists(x) # label("Auxiliary assumption
8: if the essence of x involves the existence of x and x has essence, then x exists")
# label(non_clause). [assumption].
21 Exists(a) # label("Prop. XI: God exists") # label(non_clause) # label(goal). [goal].
23 -SelfCaused(x) | EssenceInvExistence(x) # label("Definition I: self-caused").
[clausify(1)].
25 -InItself(x) | SelfCaused(x) # label("Auxiliary assumption 4: if x is in itself, x
is self-caused"). [clausify(18)].
30 -Substance(x) | InItself(x) # label("Definition III: substance"). [clausify(3)].
35 -AbsolutelyInfinite(x) | Substance(x) # label("Definition VI: absolutely infinite").
[clausify(7)].
56 -God(x) | Being(x) # label("Definition VI: God"). [clausify(6)].
57 -God(x) | AbsolutelyInfinite(x) # label("Definition VI: God"). [clausify(6)].
58 God(a). [assumption].
63 -AbsolutelyInfinite(x) | InItself(x). [resolve(35,b,30,a)].
70 AbsolutelyInfinite(a). [resolve(58,a,57,a)].
85 Being(a). [resolve(58,a,56,a)].
86 -Being(x) | HasEssence(x) # label("Auxiliary assumption 7: If x has being, then x
has essence"). [clausify(19)].
87 -InItself(x) | EssenceInvExistence(x). [resolve(25,b,23,a)].
88 -EssenceInvExistence(x) | -HasEssence(x) | Exists(x) # label("Auxiliary assumption
8: if the essence of x involves the existence of x and x has essence, then x exists").
[clausify(20)].
100 InItself(a). [resolve(70,a,63,a)].
101 -InItself(x) | -HasEssence(x) | Exists(x). [resolve(87,b,88,a)].
110 -Exists(a) # label("Prop. XI: God exists"). [deny(21)].
135 HasEssence(a). [resolve(85,a,86,a)].
150 -HasEssence(a) | Exists(a). [resolve(101,a,100,a)].
151 $F. [copy(150),unit_del(a,135),unit_del(b,110)].

===== end of proof =====

. . .

===== end of search =====

```

THEOREM PROVED

Exiting with 1 proof.

. . .

Figure 10: An extract from the proof of (GE) that is generated by *prover9*, using the file shown in Figure 9 as input. The full output of *prover9* for this case is contained in Appendix 8. “. . .” signifies text in the original that I deleted.

Here's how to read Figure 10. *prover9* first translates some of the input file (in this case, the DAPI, together with a proposition to be derived) to a set of sentences in a specific clausal form that is logically equivalent to sentences input to *prover9*.

Locate the PROOF section in Figure 10. Each of the clausal sentences in the PROOF section of Figure 10 is part of a translation of *prover9*'s inputs to *prover9*'s clausal form. Each line in the proof begins with a line number. The line numbers of the first few lines in the proof shown in Figure 10, for example, are “1”, “3”, and “6”. (It is not significant that the line numbers have some “gaps”.)

By default, *prover9* proofs are proofs by contradiction. Here's a narrative of the details of the PROOF section in Figure 10. x and y are variables and are universally quantified. *prover9* universally converts “ $P \rightarrow Q$ ” to “ $\neg P \mid Q$ ”. The “resolution inference rule” mentioned below is a generalization of modus ponens.⁴⁶

Line 1. x is self-caused if and only if x 's essence involves existence, and that of which the nature is only conceivable as existent. (from Df. I of “self-caused”)

Line 3. x is substance if and only if x is in itself and conceived through itself. (from Df. III of “substance”)

Line 6. x is God if and only if x is a being and x is absolutely infinite. (from Df. VI of “God”)

Line 7. x is absolutely infinite if and only if x is substance and x consists in infinite attributes, and each attribute expresses eternal and infinite essentiality. (from Df. VI of “absolutely infinite”)

Line 18. If x is in itself, x is self-caused. (from Auxiliary Assumption 4)

Line 19. If x has being, x has essence. (from Auxiliary Assumption 7)

Line 20. If the essence of x involves existence and x has essence, x exists. (from Auxiliary Assumption 8)

Line 21. a Exists. (This is a statement of the proposition that will be proven, i.e., it is a “goal”, not an assumption, of the derivation. a is asserted to be God at Line 58)

Line 23. x is not self-caused or x 's essence involves x 's existence. (This follows from Line 1.)

Line 25. x is not in itself or x is self-caused. (This follows from Line 18.)

Line 30. x is not substance or x is in itself. (This follows from Line 3.)

Line 35. x is not absolutely infinite or x is substance (This follows from Line 7.)

Line 56. x is not God or x has being. (This follows from Line 6.)

Line 57. x is not God or x is absolutely infinite. (This follows from Line 6.)

Line 58. Let a be God.

Line 63. x is not absolutely infinite or x is in itself. (This follows from Lines 35 and 30 by the resolution inference rule.)

Line 70. a is absolutely infinite. (This follows from Lines 58 and 57 by the resolution inference rule.)

Line 85. a has being. (This follows from Lines 58 and 56 by the resolution inference rule.)

Line 86. x does not have being or x has essence. (This follows from Line 19.)

Line 87. x is not in itself or x 's essence involves its existence. (This follows from Lines 25 and 23 by the resolution inference rule.)

Line 88. x 's essence does not involve its existence or x does not have essence or x exists. (This follows from Line 20.)

Line 100. a is in itself. (This follows from Lines 70 and 63 by the resolution inference rule.)

Line 101. x is not in itself or x does not have essence or x exists. (This follows from Lines 87 and 88 by the resolution inference rule.)

Line 110. a does not exist. (This is the negation of the goal of the derivation (see Line 21).)

⁴⁶ See Leitsch, *Resolution Calculus*, for further details.

Line 135. **a** has essence. (This follows from Lines 85 and 86 by the resolution inference rule.)

Line 150. **a** does not have essence or **a** exists. (This follows from Lines 101 and 100 by the resolution inference rule.)

Line 151. From Lines 135 and 150 it follows that that **a** exists. That result contradicts Line 110. Thus, by proof by contradiction, the negation of Line 21 is not the case. Therefore, **a**, i.e, God (see Line 58), exists. QED.

4 Discussion and conclusions

The results of Sections 2.0 and 3.0 yield several conclusions. An ADS can facilitate showing

1. that SFOA is not derivable from the DAPI.
2. that (SE) (and therefore Prop. 7) is not derivable from the DAPI.
3. the model generated by the script shown in Figure 8 is shown in Appendix 10. Thus, (SE) is independent of the DAPI conjoined with Auxiliary Assumptions 4, 7, and 8. This result, together with Appendix 3 (which shows that Auxiliary Assumption 1 is independent of the DAPI conjoined with Auxiliary Assumptions 4, 7, and 8) and Appendix 8, collectively show that (SE), contrary to Spinoza's suggestion, is not required to derive (GE).
4. that by conjoining some modest auxiliary assumptions (4, 7, and 8) with the DAPI, we can, a la Figure 10/ Appendix 8, overcome the deficiencies of SFOA identified in (1) and (2), without assuming/implying (SE).
5. the relative implicational strength of theories (such as extensions or modifications of the DAPI). From this perspective, Auxiliary Assumptions 1, 4, 7, and 8 can be regarded as elements of a moment of the ongoing dialectical search for Spinoza's "true" intentions.

To my knowledge,⁴⁷ the automated derivation of (GE) shown in Figure 10/Appendix 8 is novel.

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⁴⁷ More precisely, the argument in Appendix 8 is not mentioned in: Oppy, "Ontological Arguments"; Nadler, *Spinoza's Ethics*; or Oppy, *Ontological Arguments and Belief in God*.

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