Research Article

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Bounds for partition dimension of M-wheels

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Abstract: Resolving partition and partition dimension have multipurpose applications in computer, networking, optimization, mastermind games and modelling of chemical substances. The problem of finding exact values of partition dimension is hard so one can find bound for the partition dimension of a general family of graph. In the present article, we give the sharp upper bounds and lower bounds for the partition dimension of m -wheel, $W_{n,m}$ for all $n \geq 4$ and $m \geq 1$. Presented data generalise some already available results.

Keywords: Metric dimension, partition dimension, generalized Wheel

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1 Introduction

Resolving set and metric basis appeared on the scene way back in 1953 for an arbitrary metric space by Blumenthal [1]. But it did not attract much attention at that probably

because of linear continuum nature of $\mathbb{R}^n$. Almost twenty years later, Slater pointed out a potential application of detection problem in graph theory, which brought these ideas again in the spotlight but now in the context of discrete cases like graphs and networks.

In graph and network theory, computer networks are treated as graphs with vertices as nodes and edges as communication media. One is interested to assign a unique address to each node to easily communicate and identify the failure of any device or node in this network of computers. The concept of resolving sets and metric basis are the derived to handle these situations. Slater and independently in Harary and Melter in [7] laid down the basis of these ideas in the realm of graph theory. Resolving sets play an important part in image proceeding and digital geometry [9], robot navigation and pattern recognition [11, 12], mastermind games [20], and pharmacy chemistry and drug design [6].

Let us denote by G , a simple connected graph, V be the set of vertices of graph, a metric $d : V \times V \rightarrow W$, where W is the set of non-negative integers and $d(u, v)$ is the minimum number of edges in any path between u and v . Let $W = w_1, w_2, \dots, w_k$ be an ordered set of vertices of G and let v be a vertex of G . The representation $r(v|W)$ of v with respect to W is the k -tuple $(dG(v, w_1), dG(v, w_2), \dots, dG(v, w_k))$. If distinct vertices of G have distinct representations with respect to W , then W is called a resolving set of G , see [8–12]. A resolving set having minimum cardinality is called basis of G and its cardinality is the metric dimension of G , represented by $\beta(G)$. This version of dimension of graph is perhaps the most famous and relatively easy to compute. It is an established fact that the problem of computing metric dimension is NP -Complete [5]. Buczkowski *et al.* proved that $\beta(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n = 7$ [18], for wheel graph W_n . Caceres *et al.* proved that metric dimension of fan to be $\lfloor \frac{2n+2}{5} \rfloor$ for $n = 7$ and Tomescu *et al.* proved that the dimension of Jahangir graphs J_{2n} to be $\lfloor \frac{2n}{3} \rfloor$ for all $n = 4$. It is desirable to obtain a family of graph in which metric dimension remains independent of the particular elements of the family. Such families are called graphs of constant metric dimension. Authors in [2] proved that a graph has constant metric dimension 1 if and only if it is a path. Author computed

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the closed forms of metric dimension of cartesian products of graphs in [3] and metric dimension of finite connected graphs in [4]. Javaid *et al.* discussed some families of regular graphs having constant metric dimension, [8]. Murtaza *et al.* established some incomplete results about $\beta(M_{2,n})$ where $M_{2,n}$ is a Mobius ladder with $2n$ number of vertices, in [14] whereas Munir *et al.* established complete and exact for $M_{2,n}$ in [16]. Recently Zaffar *et al.* established bounds for the metric dimension of a kind of generalization classical Mobius Ladders in [38] which on specializing give already established results for this ladders. For other closed results about metric dimensions of families of graph, please see [12–16].

Slater introduced a new concept, for a connected graph [10, 20], closely related to metric dimension. It is a partition dimension which is even harder to compute. Let us take a subset $S \subseteq V(G)$, a vertex set v , and distance $d(v, S) = \min\{d(v, x) : x \in S\}$. If $P = \{S_1, \dots, S_t\}$ is considered as an ordered t -partition of $V(G)$, then $r(v|P) = \{d(v, S_1), \dots, d(v, S_t)\}$ is the t -tuple representation of v with respect to P . If this t -tuple representation of v , $r(v|P)$ for all $v \in V(G)$, are all distinct, then this P is called resolving partition. A resolving partition having minimum cardinality is often required and this minimum cardinality is partition dimension, represented as $pd(G)$. We illustrate this with a simple example. Consider the graph G in the Figure 1 below.

Let $P_1 = \{A_1, A_2, A_3\}$, where $A_1 = \{a_0\}$, $A_2 = \{a_1, a_2, a_3, a_4, a_5\}$ and $A_3 = \{a_6, a_7, a_8, a_9, a_{10}\}$. The partition P_1 does not resolve G because $r(a_1|P_1) = r(a_5|P_1) = (1, 0, 2)$. Now let $P = \{B_1, B_2, B_3\}$, where $B_1 = \{a_0, a_5\}$, $B_2 = \{a_1, a_2, a_3, a_4\}$ and $B_3 = \{a_6, a_7, a_8, a_9, a_{10}\}$.

Next we use a_i instead of $r(a_i|P)$, $i = 0, 1, 2, \dots, 10$. The distance vectors belonging to various vertices of G relative to partition P are

$a_0 = (0, 1, 1)$, $a_1 = (1, 0, 2)$, $a_2 = (2, 0, 3)$,
 $a_3 = (2, 0, 4)$, $a_4 = (1, 0, 3)$, $a_5 = (0, 1, 2)$,
 $a_6 = (1, 2, 0)$, $a_7 = (2, 3, 0)$, $a_8 = (2, 4, 0)$,
 $a_9 = (1, 3, 0)$, $a_{10} = (0, 2, 1)$

Since the distance vectors belonging to various vertices of G relative to P are distinct so the partition P resolves G . Moreover any partition of G having cardinality less than 3 does not resolve G . So $pd(G) = 3$.

This type of dimension has been utilized in chemistry and drug design [6]. Chartrand *et al.* discussed partition dimension of a general graph in [25, 26]. Chappel *et al.* computed bounds on the metric and partition dimension of some graphs in [27]. Fernau computed partition dimension of uni-cyclic graphs in [29]. It is natural to discuss the rela-

tion of these two dimensions. In [25, 26], Chartrand *et al.* established that

$$pd(G) = \beta(G) + 1$$

for a connected graph G . Tomescu *et al.* in [30] established that it can be even more smaller than the metric dimension. Authors, in fact, discussed all 23 families of connected graphs having order n with partition dimensions n , $n-1$ or 2. Recently Hernando *et al.* in [28] identified the repetition in this list and proved that there are only 15 such families. In [27], authors computed some bounds for metric and partition dimension of a connected graph.

Chartrand *et al.* proved in [26] that if G is a connected graph of order $n \geq 2$ then $pd(G) = 2$ if and only if G is a path, $pd(G) = n$ if and only if $G = K_n$ and for $n = 5$, $pd(G) = n-1$ if and only if G is one of the graphs $K_{1,n-1}$, K_{n-2} , $K_1 + (K_1 \cup K_{n-2})$. In [38] Zaffar *et al.* computed bounds for the partition dimension of generalization of Mobius ladders and computed that these bounds depend upon the parity of m and n . Authors proved that, for $m \geq 3$ and $n \geq 2$, $3 \leq pd(M_{m,n}) \leq 4$ when both m and n are opposite parity and $3 \leq pd(M_{m,n}) \leq 5$ when both parameters are of same parity.

Tomescu *et al.* in [31] computed partition dimension and connected partition dimension of wheel graphs and

Figure 1: G

showed that for $n \geq 4$, $\lceil (2n)^{\frac{1}{3}} \rceil = pd(G) = 2\lceil n^{\frac{1}{2}} \rceil + 1$. We shall generalize some results about wheel graph to generalized wheel graph. Following lemma relates bounds for the partition dimension of a graph of size n and its diameter.

Lemma: If $|G| \geq 3$, then $pd(G) \leq n - \text{diam}(G) + 1$. In the present article we are interested to find sharp bounds for $pd(W_{n,m})$. The graph $(W_{n,m})$, also called m -step or level wheel graph consisting of m number of C_n cycles and a vertex v which are connected in such a way that every vertex of each C_n is adjacent to v , as shown in Figure 1, [29]. The vertices of C_n are termed as rim vertices. The order of graph $W_{n,m}$ is $nm + 1$ and its diameter is 2. This graph can be taken as generalization of simple wheel graph W_n . The following Figure 2 contains an m -step wheel with 8 rim vertices in a cycle.

Figure 2: Wheel $W_{8,m}$

Wireless sensor networks and the vulnerability of networks are some of the uses of this graph [32, 34]. The wheel graph has many useful properties. From the standpoint of the hub vertex, all elements, including vertices and edges, are in its one-hop neighborhood, which indicates that the wheel structure is fully included in the neighborhood graph of the hub vertex. Wheel and related graphs are extensively studied recently. In [37] authors computed metric dimension of some wheel related graphs. In [32], authors gave an algorithm to compute average lower 2-domination number and also computed this number for some wheel related graphs. Authors computed different kinds of energies associated to matrices of $W_{n,m}$ in [36], and bounds for Stanley depth for $W_{n,m}$ in [35].

2 Main results

The main result of this article is given in the following theorem

Theorem 2.1. For all $m \geq 1$ and for $n \geq 4$ $\lceil (2nm)^{1/3} \rceil \leq pd(W_{n,m}) \leq 2\lceil (nm)^{1/2} \rceil + 1$

We need following lemmas to prove our main results.

Lemma 2.2. Let $\Pi = \{S_1, S_2, \dots, S_q\}$ be a resolving q -partition of $V(W_{n,m})$. If $c \in S_1$, then $|S_1| \leq 1 + \binom{q-1}{2} + \binom{q-1}{1} + \binom{q-1}{0}$ where c is the central vertex.

Proof. We analyze that $r(c|\Pi) = (0, 1, 1, \dots, 1)$ and $r(v|\Pi) = (0, \dots)$ for $v \in S_1 \setminus \{c\}$. We also know that the diameter of $W_{n,m}$ is 2, the elements of the q -vector representation $r(v|\Pi)$ of each rim vertex $v \in S_1 \setminus \{c\}$ except the first element can be 1 or 2. But in vector representation there can be at most two elements which will be equal to 1 apart from the first position of the vector. For the rim vertices in the vector representation there are $q - 1$ positions which can be filled by at most two 1's and the other can be filled by 2's. Thus, for all vertices $v \in S_1 \setminus \{c\}$ there are at most $\binom{q-1}{2} + \binom{q-1}{1} + \binom{q-1}{0}$ different vector representations. Together with these vector representations of the center, we arrive at most $1 + \binom{q-1}{2} + \binom{q-1}{1} + \binom{q-1}{0}$ different representations. Therefore $|S_1| \leq 1 + \binom{q-1}{2} + \binom{q-1}{1} + \binom{q-1}{0}$. $\square$

Lemma 2.3. Let $\Pi = \{S_1, S_2, \dots, S_q\}$ be a resolving q -partition of $V(W_{n,m})$. If $c \in S_1$, then $|S_i| \leq \binom{q-2}{2} + \binom{q-2}{1} + \binom{q-2}{0}$ for each $2 \leq i \leq q$.

Proof. We consider another class not S_1 , say S_2 , where c does not belong to S_2 . Then for $w \in S_2$, the vector representation becomes $r(w|\Pi) = (1, 0, \dots)$. For rim vertices in the vector representation there are $q - 2$ which can be filled by at most two 1's and the other can be filled by 2's. Thus for all vertices $w \in S_2$, there are at most $\binom{q-2}{2} + \binom{q-2}{1} + \binom{q-2}{0}$ different representations. Therefore $|S_i| \leq \binom{q-2}{2} + \binom{q-2}{1} + \binom{q-2}{0}$ for each $2 \leq i \leq q$. $\square$

With the help of above two lemmas, we can now find the lower bound for the partition dimension of $W_{n,m}$.

Proposition 2.4. We have $\lceil (2nm)^{1/3} \rceil \leq pd(W_{n,m})$, for every $n \geq 4$ and $m \geq 1$.

Proof. Let us assume $pd(W_{n,m}) = q$ and $\Pi = \{S_1, S_2, \dots, S_q\}$ be a resolving q -partition of $V(W_{n,m})$. Let $c \in S_1$ and by lemma (2.2) we have $|S_1| \leq 1 + \binom{q-1}{2} + \binom{q-1}{1} + \binom{q-1}{0}$ and by lemma (2.3) we have $|S_i| \leq \binom{q-2}{2} + \binom{q-2}{1} + \binom{q-2}{0}$

for $2 \leq i \leq q$. Using these two lemmas we get $|V(W_{n,m})| = nm + 1 = \sum_{i=1}^q |S_i| \leq 1 + \sum_{i=0}^{q-1} \binom{q-1}{i} + (q-1) \binom{q-2}{i}$.
 $nm < (q^3 - 3q^2 + 6q - 2)/2 \leq \frac{q^3}{2}$ for every $q \geq 2$. It follows that $q \geq \lceil (2nm)^{1/3} \rceil$. $\square$

We now move towards the upper bounds. Following two results give upper bounds for the partition dimension of $W_{n,m}$ but these bounds are not sharp enough.

Proposition 2.5. For every $n \geq 2$ we have $pd(W_{n,m}) \leq \lfloor \frac{2n+2}{5} \rfloor + (m-1) \lfloor \frac{2n+4}{5} \rfloor + 1$.

Proof. We easily obtain $dim(W_{n,m}) = \lfloor \frac{2n+2}{5} \rfloor + (m-1) \lfloor \frac{2n+4}{5} \rfloor$, by ([29], theorem 2.2). By using elementary inequality relating metric and partition dimension we arrive at the required result. $\square$

Proposition 2.6. For every $n \geq 2$ we have $pd(W_{n,m}) \leq \frac{2}{5}(mn + 2m - 1) + 1$.

Proof. From above, we obtain $dim(W_{n,m}) = \lfloor \frac{2n+2}{5} \rfloor + (m-1) \lfloor \frac{2n+4}{5} \rfloor$. But using definition of floor function we have $\lfloor \frac{2n+2}{5} \rfloor + (m-1) \lfloor \frac{2n+4}{5} \rfloor < \frac{2n+2}{5} + \frac{2}{5}(m-1)(2n+2) = \frac{2}{5}(mn + 2m - 1)$. Using relation between pd and dim we obtain the required result. $\square$

We now move forward to find upper bound which is sharper than the above bounds.

Proposition 2.7. We have $pd(W_{n,m}) \leq p+1$, For every $n \geq 2$ and p is the least prime number, such that $p(p-1) \geq nm$

Proof. Let p be a the smallest prime number in the sense that $p(p-1) \geq nm$. Since p is the prime number, so the sequence $\{0, i, 2i, 3i, \dots, (p-1)i\}$, where $1 \leq i \leq p-1$ and all the above numbers $0, i, 2i, 3i, \dots, (p-1)i$ are reduced modulo p . It is obvious that these numbers are the permutation of the set $\{0, 1, \dots, p-1\}$. Now assume that the sequence

$$(x_k)_k = 1, 2, \dots, p(p-1) = X_1, X_2, \dots, X_{(p-1)/2}$$

where for every $1 \leq i \leq (p-1)/2$ there is a subsequence

$$X_i = 0, 0, i, i, 2i, 2i, \dots, (p-1)i, (p-1)i$$

which contains $2p$ terms and every pair of equal elements in above sequence different from $0, 0$. And we can obtain all these numbers from the previous one by adding i module p to every component. Thus, for $V(W_{n,m})$ the resolving partition $\Pi = \{S_1, \dots, S_{p+1}\}$ is defined as :

- a) if $nm = p(p-1)$ then $S_{p+1} = \{c\}$ and also in this case every element $V_i (0 \leq i \leq nm-1)$ is assigned to the class $S_{x_{i+1}+1}$;

- b) if $nm < p(p-1)$ then $S_p = \{c, v_{nm-1}\}$ and also every element $v_i (0 \leq i \leq nm-2)$ is assigned to the class $S_{x_{i+1}+1}$.

so, Π is a resolving connected partition of $V(W_{n,m})$ having $p+1$ classes, which implies $pd(W_{n,m}) \leq p+1$. $\square$

Proposition 2.8. For every $n \geq 4$ we have $\lceil (2nm)^{1/3} \rceil \leq pd(W_{n,m}) \leq 2 \lceil (nm)^{1/2} \rceil + 1$

Proof. Since p is prime then it should satisfy that $p(p-1) \geq nm$ so select $p \geq \lceil (nm)^{1/2} \rceil + 1$. Bertrand's postulate can be used here which states that for every $n \geq 1$, there always exists a prime with

$$n < p \leq 2n.$$

It was proved by Chebyshev for the first time in the history, see [30]. So we can easily infer that

$$\lceil (nm)^{1/2} \rceil < p \leq 2 \lceil (nm)^{1/2} \rceil.$$

So it can be derived that

$$pd(W_{n,m}) \leq p+1 \leq 2 \lceil (nm)^{1/2} \rceil + 1.$$

which is the required result. $\square$

It is clear that by putting $m = 1$ we obtain simple wheel W_n with the result, for $4 \leq n \leq 7$ the $pd(W_n) = 3$, and for $8 \leq n \leq 19$ the $pd(W_n) = 4$ and $pd(W_3) = 4$ see [25]. So our results is an extension of this result.

Example. Let $n = 3$ and $m = 10$, we have $\lceil (2(3)(10))^{1/3} \rceil \leq pd(W_{3,10}) \leq 2 \lceil ((3)(10))^{1/2} \rceil + 1$
 $\lceil (60)_{1/3} \rceil \leq pd(W_{3,10}) \leq 2 \lceil (30)^{1/2} \rceil + 1$
 $3 \leq pd(W_{3,10}) \leq 11$

Example. Take $W_{2,10}$ with resolving partition $\Pi = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ and $S_1 = \{v_2, v_7, v_8, v_{13}\}$, $S_2 = \{v_3, v_{11}, v_{12}, v_{19}\}$, $S_3 = \{v_5, v_9, v_{10}, v_{16}\}$, $S_4 = \{v_6, v_{14}, v_{15}, v_{17}\}$, $S_5 = \{v_0, v_1, v_4, v_{18}\}$ and $S_6 = \{c\}$
 $(c|\Pi) = (1, 1, 1, 1, 1, 0)$, $(v_0|\Pi) = (2, 2, 1, 2, 0, 1)$,
 $(v_1|\Pi) = (1, 2, 2, 2, 0, 1)$,
 $(v_2|\Pi) = (0, 1, 2, 2, 1, 1)$, $(v_3|\Pi) = (1, 0, 2, 2, 1, 1)$,
 $(v_4|\Pi) = (2, 1, 1, 2, 0, 1)$,
 $(v_5|\Pi) = (2, 2, 0, 2, 1, 1)$, $(v_6|\Pi) = (1, 2, 1, 0, 2, 1)$,
 $(v_7|\Pi) = (0, 2, 2, 1, 2, 1)$,
 $(v_8|\Pi) = (0, 2, 1, 2, 2, 1)$, $(v_9|\Pi) = (1, 2, 0, 2, 1, 1)$,
 $(v_{10}|\Pi) = (2, 1, 0, 2, 2, 1)$,
 $(v_{11}|\Pi) = (2, 0, 1, 2, 2, 1)$, $(v_{12}|\Pi) = (1, 0, 2, 2, 2, 1)$,
 $(v_{13}|\Pi) = (0, 1, 2, 1, 2, 1)$,
 $(v_{14}|\Pi) = (1, 2, 2, 0, 2, 1)$, $(v_{15}|\Pi) = (2, 2, 1, 0, 2, 1)$,
 $(v_{16}|\Pi) = (2, 2, 0, 1, 2, 1)$,

$(v_{17}|\prod) = (2, 2, 1, 0, 1, 1)$, $(v_{18}|\prod) = (2, 1, 2, 1, 0, 1)$,
 $(v_{19}|\prod) = (2, 0, 1, 2, 1, 1)$.

Also in the case of $W_{3,10}$ the partition dimension is 8 which is also $\leq p + 1$, where p is 7.

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