Research Article

Muhammad Rafiq, Baljeet Singh, Samreen Arifa, Muhammad Nazeer, Muhammad Usman, Shoaib Arif, Mairaj Bibi, and Adnan Jahangir*

Harmonic waves solution in dual-phase-lag magneto-thermoelasticity

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Abstract: The current work analyzes the transmission behavior of plane harmonic waves in an isotropic medium. The observation is made for homogeneous type solid in the context of generalized dual phase lag model of thermoelasticity. Concept micro-temperature, where the microelements have different temperatures has also been considered. The basic focus of the work is to predict the influence of initially applied magnetic field on plane waves through the elastic solid. We have made an attempt to find exact solution of the problem using an analytical technique of a normal mode analysis method. The theoretical results are obtained for a generalized solid in order to test the numerical calculation of a magnesium crystal. It is found that the magnetic field reduces the strength of the attenuation factor.

Keywords: Time harmonic waves; thermo-elasticity; magnetic field; micro-temperature; dual phase lag; normal mode analysis

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*Corresponding Author: **Adnan Jahangir:** Dept. of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt, Pakistan; Email: adnan_jahangir@yahoo.com

Muhammad Rafiq, Samreen Arifa: Dept. of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt, Pakistan

Baljeet Singh: Dept. of Mathematics, Post Graduate Government College, Sector 11, Chandigarh, India

Muhammad Nazeer: Dept. of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt, Pakistan; Dept. of applied Mathematics, ERICA Hanyang University Ansan, South Korea

Muhammad Usman: Dept. of Eng., Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, Sawabi, Pakistan

Shoaib Arif: Department of Mathematics, Air University, PAF Complex, E-9, Islamabad, Pakistan

Mairaj Bibi: Dept. of Mathematics, COMSATS University Islamabad, Islamabad Pakistan

1 Introduction

The field of microstructure in which microelements also contain micro-temperatures, was initiated by Grot [1]. Riha [2] extended the work and analyzed heat conduction along with micro-temperature in elastic solids. Later, Iesan and Quintanilla [3] extended the concept-presented by Grot and introduced the theory of thermo-elastic bodies with micro-temperatures, which allows the transmission of thermal waves at finite speed. Several experts worked on the extensions of the theory, some notable authors are [4–9].

Fourier law was modified by considering phase-lags of heat flux vector and temperature gradient by [10] to bring in the theory of dual-phase-lag (DPL). The updated form of Fourier's law becomes $q(P, t + \tau_q) = -[k \nabla T(P, t + \tau_T)]$. The τ_q is the relaxation time of thermal inertia, and τ_T is the time lag due to micro-structural interactions. Quintanilla [11] studied the stability of DPL model. Parasad *et al.* [12] analyze the propagation of plane waves under thermoelasticity with DPL model. Some interesting problems related with DPL model are being encountered by Othman [13], Baljeet [14], Model *et al.* [15]. Reflection phenomena and some special effects on waves propagating through thermoelastic medium in context of DPL model, is investigate in detail by Zenkour and his colleagues [16–22].

The relationship of the Maxwell electromagnetic field with the motion of elastic solids is very important due to its appliance to geophysical problems and certain topics in optics and acoustics. The magneto-elastic nature of the Earth's material may influence the wave propagating through it. The magneto-thermoelastic disturbance in an elastic half-space having a finite conductivity, has been investigated by Baljeet *et al.* [23]. Puri [24] studied the propagation of plane waves in a solid under the influence of an electromagnetic field. Authors like [25–31] have studied the related problems.

The idea of the paper is to study the propagation of plane harmonic waves in an isotropic half space medium in the framework of a generalized DPL thermoelastic

model. We have also considered the influence of an initially applied magnetic field on waves generated in an elastic medium because of linear deformation. Firstly, we obtained the theoretical solution of the problem by using a Normal mode analysis method. Secondly, results obtained theoretically are computed by using the packages of Matlab and shown graphically.

2 Field equations of the problem

Rectangular coordinate system (x, y, z) is selected to represent the linear form of field equations for a homogeneous isotropic medium in the presence of a micro-temperature and magnetic field $\mathbf{H}_0$. The problem is designed as a half space medium $z \geq 0$ with origin on the surface $y = 0$ and z -axis pointing vertically into the medium. The equations of motion for linear generalized thermo-elasticity and without body force and heat source is given by [6],

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) - \beta \nabla T + \mathbf{F}_i = \rho \ddot{\mathbf{u}}_i, \quad (1)$$

where, $\mathbf{F}_i = (\mathbf{J} \times \mathbf{B})_i$ is the Lorentz force that appeared because of initially applied magnetic field. The medium is supposed to be perfectly electrically conducting and is half space $(x, 0, z)$ such that all the variables are independent of the dimension y . For simplicity we have fixed the direction and intensity of the initially applied magnetic field along y -axis, i.e., $\mathbf{H}_0 = (0, H_0, 0)$. Maxwell's equations with slowly moving media is represented by [32]

$$\begin{aligned} \vec{J} &= \text{Curl} \vec{h} - \epsilon_0 \dot{\vec{E}}, \quad \text{Curl} \vec{E} = -\mu_0 \dot{\vec{K}}, \\ \vec{E} &= -\mu_0 (\dot{\vec{u}} \times \vec{h}), \quad \vec{\nabla} \cdot \vec{h} = 0, \end{aligned} \quad (2)$$

where $\vec{J}$, $\vec{E}$, ϵ_0 , and μ_0 are the electric current density, induced electric field, permittivity, and permeability, respectively. The total magnetic field is given by $\vec{H} = \vec{H}_0 + \vec{h}$, $\vec{h}$ is representing the induced magnetic field. Equation (2) can be represented as,

$$\mathbf{E} = -\mu_0 H_0 (-\dot{u}_3, 0, \dot{u}_1), \quad (3)$$

$$\mathbf{J} = \left(-\frac{\partial H y}{\partial z} - \epsilon_0 \dot{E}_x, 0, -\frac{\partial H y}{\partial x} - \epsilon_0 \dot{E}_z \right). \quad (4)$$

By applying all the values, the component of Lorentz force becomes,

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} = \mu_0 (\mathbf{J} \times \mathbf{H}), \quad (5)$$

$$F \quad (6)$$

$$= \left(\mu_0 H_0^2 \left(\frac{\partial e}{\partial x} - \epsilon_0 \mu_0 \ddot{u}_1 \right), 0, \mu_0 H_0^2 \left(\frac{\partial e}{\partial z} - \epsilon_0 \mu_0 \ddot{u}_3 \right) \right).$$

This can be represented in tensor form as,

$$F_i = \mu_0 H_0^2 (e_{,i} - \epsilon_0 \mu_0 \ddot{u}_i), \quad (7)$$

where, $h = -H_0(0, e, 0)$. The dual phase heat conduction equation, along with influence of micro-temperature, is [10],

$$\begin{aligned} & \left(1 + \tau_T \frac{\partial}{\partial t} \right) K T_{,ij} + k_1 (\nabla \cdot \vec{w}) \\ & = \left(1 + \tau_q \frac{\partial}{\partial t} \right) (\rho c_e \dot{T} + \beta T_0 \dot{e}_{kk}), \end{aligned} \quad (8)$$

The equation of micro-temperature [1]

$$k_6 \nabla^2 w + (k_4 + k_5) \nabla (\nabla \cdot \mathbf{w}) - k_3 \nabla T - k_2 w - b w' = 0 \quad (9)$$

Where $\lambda, \mu, \alpha, \beta, k, k_i (i = 1, 2, \dots, 6)$ are constitutive coefficients, and $\rho, T = T_t - T_0, T_0$ is the density, temperature, and reference temperature of the medium. The quantity $\vec{u}(x, z, t)$ is a displacement vector and $\vec{w}$ is the micro-temperature vector. The constitutive relations for the said problem are given by [6].

$$\begin{cases} \sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} - \beta T \delta_{ij}, \\ q_{ij} - \kappa_4 w_{r,r} \sigma_{ij} - \kappa_5 w_{i,j} - \kappa_6 w_{j,i}, \\ Q_i = (K - \kappa_3) T_{,i} + (\kappa_1 - \kappa_2) w_i, \\ q_i = K T_{,i} + \kappa_1 w_i, \end{cases} \quad (10)$$

where $\sigma_{ij} = (i, j = 1, 2, 3)$ is the stress tensor q_{ij} , Q_i , and q_i are components of first heat flux moment vector; components of mean heat flux vector; and heat flux vector, σ_{ij} is the Kronecker delta, and e_{ij} is the strain tensor given by $e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$,

To non dimensionalize the system, the variables selected are,

$$\begin{aligned} x'_i &= \frac{x_i}{l_0}, \quad t' = \frac{c_1 t}{l_0}, \quad u'_i = \frac{u_i}{l_0}, \quad T' = \frac{T}{T_0}, \\ w'_i &= w_i l_0, \quad \tau'_T = \frac{c_1}{l_0} \tau_T, \quad \tau'_q = \frac{c_1}{l_0} \tau_q, \quad \nabla'^2 = \frac{1}{l_0^2} \nabla'^2, \\ h' &= \frac{h}{H_0}, \quad z'_i = \frac{z_i}{l_0}, \quad e = u_{i,i}. \end{aligned} \quad (11)$$

Where, l_0 is standard length and c_1 is the standard velocity given by $c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$. By non-dimensionalizing the governing equations we get, (after dropping primes for convenience)

$$\alpha_0 u_{i,jj} + (\alpha_1 + R_H) u_{j,ji} - \Gamma_1 T_{,i} - \Gamma_2 \ddot{u}_i = 0, \quad (12)$$

$$\left(1 + \tau_T \frac{\partial}{\partial t} \right) \alpha_2 T_{,jj} + \alpha_3 w_{j,j} \quad (13)$$

$$= \left(1 + \tau_q \frac{\partial}{\partial t}\right) (\dot{T} + \alpha_4 \dot{u}_{j,j}),$$

$$\beta_0 w_{i,jj} + \beta_1 w_{j,ji} - \beta_2 T_{,i} - \beta_2 w_i - \beta_4 \dot{w}_i = 0, \quad (14)$$

Where,

$$\begin{aligned} \beta_0 &= \frac{\kappa_6}{\rho l_0^3 c_1^3}; \quad \beta_1 = \frac{(\kappa_4 + \kappa_5)}{\rho l_0^3 c_1^3}; \quad \beta_2 = \frac{\kappa_3 T_0}{\rho l_0 c_1^3}; \quad \beta_3 = \frac{\kappa_2}{\rho l_0 c_1^3}; \\ \beta_4 &= \frac{b}{\rho l_0^2 c_1^2}; \quad \alpha_2 = \frac{\kappa}{l_0 \rho c_e c_1}; \quad \alpha_3 = \frac{\kappa_1}{l_0 \rho T_0 c_e c_1}; \\ \alpha_4 &= \frac{\beta}{\rho c_e}; \quad \alpha_0 = \frac{\mu}{\rho c_1^2}; \quad \alpha_1 = \frac{(\lambda + \mu)}{\rho c_1^2}; \quad R_H = \frac{\Phi^2}{c^2}; \\ c^2 &= \frac{1}{\varepsilon_0 \mu_0}; \quad \Phi^2 = \frac{\mu_0 H_0^2}{\rho}; \quad \Gamma_1 = \frac{\beta T_0}{\rho c_1^2}; \quad \Gamma_2 = \frac{\Phi^2}{c^2} + 1. \end{aligned}$$

Displacement and micro-temperature functions could be converted in terms of potential function, by the following expressions,

$$\begin{aligned} u_1 &= \left(\frac{\partial R}{\partial x} + \frac{\partial \psi}{\partial z}\right), \quad u_3 = \left(\frac{\partial R}{\partial z} - \frac{\partial \psi}{\partial x}\right), \quad (15) \\ w &= \nabla v, \end{aligned}$$

where R and v are scalar potential functions, while ψ is vector potential function. With the use of equation (15), the equations (12) to (14) become

$$\left(\pi_1 \nabla^2 - \Gamma_2 \frac{\partial^2}{\partial t^2}\right) R - \Gamma_1 T = 0, \quad (16)$$

$$\left(\alpha_1 \nabla^2 - \Gamma_2 \frac{\partial^2}{\partial t^2}\right) \psi = 0, \quad (17)$$

$$\begin{aligned} \alpha_2 \left(\nabla^2 T + \tau_T \nabla^2 \dot{T}\right) + \alpha_2 \nabla^2 v \\ - \dot{T} \left(1 + \tau_1 \frac{\partial}{\partial t}\right) - \left(1 + \tau_q \frac{\partial}{\partial t}\right) (\alpha_4 \nabla^2 \dot{R}) = 0, \end{aligned} \quad (18)$$

$$\sigma \nabla^2 v - \beta_2 T - \left(\beta_3 + \beta_4 \frac{\partial}{\partial t}\right) v = 0. \quad (19)$$

3 Normal mode analysis

Now let us consider that each field variable is propagating through the medium in terms of harmonic waves as,

$$\begin{aligned} \{R, T, v, \psi\}(x, z, t) \\ = \{R^*, T^*, v^*, \psi^*\}(z) \exp(\omega t + i a x), \end{aligned} \quad (20)$$

where ω is the angular frequency, $i = \sqrt{-1}$; a is the positive wave number in the z direction; and R^*, T^*, v^*, ψ^* are the

amplitudes depending on the variable z . By using equation (20), the governing equations (16) to (19) are represented as,

$$(\pi_1 D^2 - A_1) R^* - \Gamma_1 T^* = 0, \quad (21)$$

$$(\alpha_1 D^2 - A_2) \psi^* = 0, \quad (22)$$

$$\begin{aligned} (\alpha_2 D^2 \pi_2 + A_3) T^* \\ + (\alpha_3 D^2 - \alpha_2 a^2) v^* + (-\alpha_4 D^2 \pi_3 + A_4) R^* = 0, \end{aligned} \quad (23)$$

$$(\sigma D^2 - A_5) v^* - \beta_2 T^* = 0, \quad (24)$$

where,

$$\begin{aligned} A_1 &= -(\pi_1 a^2 + \Gamma_2 \omega^2), \quad A_2 = \alpha_1 a^2 + \Gamma_2 \omega^2, \\ A_3 &= -\alpha_2 a^2 (1 + \tau_T \omega) - \omega (1 + \tau_q \omega), \\ A_4 &= \alpha_4 a^2 (\omega + \tau_q \omega^2), \quad \sigma = \beta_0 + \beta_1, \\ A_5 &= \sigma a^2 - \beta_3 - \beta_4 \omega, \quad \pi_1 = \alpha_0 + \alpha_2 + R_H. \end{aligned}$$

Condition of non trivial solution leads us to the equation.

$$[AD^6 + BD^4 - CD^2 + E] \{v^*, R^*, T^*\}(z) = 0. \quad (25)$$

Where

$$\begin{aligned} A &= \sigma \alpha_2 \pi_1 \pi_2 \\ B &= -A_1 \pi_2 \alpha_2 \sigma + \pi_1 A_3 - A_5 \alpha_2 \pi_1 \pi_2 - \beta_2 \alpha_3 \pi_1 + \Gamma_1 \alpha_4 \pi_3 \sigma \\ C &= -A_1 A_3 \sigma + A_1 A_5 \pi_2 \alpha_2 - \pi_1 A_3 A_5 + \beta_2 \alpha_3 a^2 \pi_1 + A_4 \Gamma_1 \sigma \\ &\quad - A_5 \Gamma_1 \alpha_4 \pi_3 \\ E &= A_1 A_3 A_5 - \beta_2 \alpha_3 a^2 A_1 - A_5 A_4 \Gamma_1. \end{aligned}$$

Equation (25) is written in factorized form as

$$(D^2 - k_1^2) (D^2 - k_2^2) (D^2 - k_3^2) \{v^*, R^*, T^*\}(z) = 0. \quad (26)$$

where k_i , $i = (1, 2, 3)$ are the roots of equation (25). The solution of equation (26) has the form

$$R^* = \sum_{n=1}^3 M_n e^{-k_n z}, \quad (27)$$

$$T^* = \sum_{n=1}^3 H_{1n} M_n e^{-k_n z}, \quad (28)$$

$$v^* = \sum_{n=1}^3 H_{2n} M_n e^{-k_n z}, \quad (29)$$

Where M_n are some parameters to be determined and,

$$H_{2n} = -\beta_2 \left(\pi_1 k_n^2 - A_1 \right) / \left(\Gamma_1 \left(\sigma k_n^2 - A_5 \right) \right),$$

$$H_{1n} = \left[\pi_1 k_n^2 - A_1 \right] / \Gamma_1,$$

Solution of vector potential function ψ can be obtained by using equation (22)

$$\psi = M_4 e^{-k_4 z} \quad (30)$$

where, $k_4 = \sqrt{\frac{A_2}{\alpha_1}}$ and solution exists only if

$$\left(\frac{\mu_0 H_0^2}{\rho c^2} + 1 \right) \omega^2 \geq 0.$$

It may be noted that $\left(\frac{\mu_0 H_0^2}{\rho c^2} + 1 \right)$ is positive depending on initially applied magnetic field and magnetic permeability of the material,

4 The boundary conditions

The boundary conditions on the plane $z = 0$ are:

1. Stress condition for surface of the medium

$$\sigma_{zz}(x, 0, t) = f \exp(\omega t + iax), \quad (31)$$

$$\sigma_{zx}(x, 0, t) = 0,$$

2. Thermal condition,

$$\frac{\partial T}{\partial z}(x, 0, t) = 0 \quad (32)$$

3. Normal component of heat flux moment tensor is

$$q_{zz}(x, 0, t) = 0. \quad (33)$$

Using above equations, the boundary conditions (31) to (33) result in the following set of equations,

$$f_1^* = \sum_{n=1}^3 Y_{1n} M_n e^{-k_n z} + 2\mu k_4 i a M_4, \quad (34)$$

$$0 = \sum_{n=1}^3 Y_{2n} M_n e^{-k_n z} + \mu \left(a^2 + k_4^2 \right) M_4, \quad (35)$$

$$0 = \sum_{n=1}^3 -k_n H_{1n} M_n e^{-k_n z}, \quad (36)$$

$$0 = \sum_{n=1}^3 Y_{3n} M_n e^{-k_n z}, \quad (37)$$

Where

$$Y_{1n} = \left[-a^2 \lambda + (\lambda + 2\mu) k_n^2 - \beta T_0 H_{1n} \right],$$

$$Y_{2n} = -2ia\mu k_n,$$

$$Y_{3n} = \left[-K_4 \left(k_n^2 - a^2 \right) - k_n^2 (K_5 + K_6) \right] H_{2n},$$

Applying the boundary conditions (29) – (32) and using Eqs. (22) – (25), we obtained the following matrix transform,

$$\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & 2\mu i a k_4 \\ Y_{21} & Y_{22} & Y_{23} & \mu (a^2 + k_4^2) \\ -k_1 H_{11} & -k_2 H_{12} & -k_3 H_{13} & 0 \\ Y_{31} & Y_{32} & Y_{33} & 0 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \begin{pmatrix} f_1^* \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (38)$$

By solving the above matrix we can obtain the values of five constants M_n , $n = 1, 2, \dots, 5$. Therefore, we obtain the solution for each field variable.

5 Numerical discussion

The evaluated theoretical results in equations (38) are computed numerically by using the relevant parameters for the case of a magnesium crystal. The relevant physical values of elastic constants and micro-temperatures are [14] $\rho = 1.74 \times 10^3 \text{ kg m}^{-3}$, $\lambda = 9.4 \times 10^{10} \text{ Nm}^{-2}$, $\mu = 4.0 \times 10^{10} \text{ Nm}^{-2}$, $\gamma = 7.779 \times 10^{-8} \text{ N}$, $C_E = 1.04 \times 10^3 \text{ NmKg}^{-1} \text{K}^{-1}$, $T_0 = 0.298$, $K = 1.7 \times 10^2 \text{ N sec}^{-1} \text{K}^{-1}$, $b = 0.15 \times 10^{-9} \text{ N}$

The micro-temperature parameters are

$$k_1 = 0.0035 \text{ N s}^{-1}, k_2 = 0.045 \text{ N s}^{-1}, k_3 = 0.055 \text{ N s}^{-1}, k_4 = 0.064 \text{ N s}^{-1}, k_5 = 0.075 \text{ N s}^{-1}, k_6 = 0.096 \text{ N s}^{-1}$$

Figures 1-6 give effects of initially applied magnetic field on waves propagating through the medium. To represent the magnetic field we have selected the variable Φ as defined in equation (11). In the figures, the solid line represents the curve without a magnetic field, dash with dots are curves for $\Phi = 25$ and dotted lines are for $\Phi = 50$.

Figure 1 predicts the response of a horizontal component of a displacement distribution function against vertical distance, for different values of Φ . It is found that the magnetic field plays an important role in reduction of the peak value of amplitude *i.e.*, greater the value of magnetic field, lower will be the amplitude of horizontal component of displacement against depth of medium. It is also found that the curve with $\Phi = 0$ will converge to zero earlier than other indicating that Φ reduces the attenuation and hence results in slow convergence toward zero amplitude.

Figure 1: Horizontal component of displacement against vertical distance for different magnetic field

Figure 4: Temperature distribution against vertical distance for different magnetic field

Figure 2: Vertical components of displacement against vertical distance for different magnetic field

Figure 5: Normal stress against vertical distance for different magnetic field

Figure 3: Micro-temperature vector distribution against vertical distance for different magnetic field

Figure 6: Tangential stress distribution against vertical distance for different magnetic field

Figure 2 gives the graphical analysis of vertical displacement component for different intensity of initially applied magnetic field. Same as that of displacement, the higher the intensity of magnetic field results in a lower peak value

(a)

(b)

Figure 7: (a,b) 3D propagation of displacement distribution function

(a)

(b)

Figure 8: (a,b) 3D propagation of temperature distribution function

of amplitude for curve propagating through the medium. For the case of micro-temperature w , temperature T and normal component of stress σ_{zz} , the initially applied magnetic field is having decreasing effects on waves propagating through the medium, i.e., greater the intensity of initially applied magnetic field, greater the amplitude of wave propagating through the medium. During the case of normal component of stress distribution function, the magnetic field is having decreasing effect for $0 < z < 3.5$ and $19 < z < 39$, increasing for all other values of z . The curves of all field variables will converge to zero as distance from surface $z = 0 = y$. The curves during zero magnetic fields will converge to zero with a high rate of convergence.

In the graphical presentation of solution, we have also shown 3D behavior of harmonic waves propagating through the medium along the x -direction, while the amplitude of each curve reduces along the depth of the medium.

6 Conclusions

Following are some main points which could be concluded after considering the solutions of the problem:

Figure 9: (a,b) 3D propagation of stress distribution function

1. All curves obtained converge to zero as depth from the surface of the medium increases. Satisfying the condition of decaying waves.
2. For the case of u_1 and u_3 , the magnetic field is having a reducing effect on absolute amplitude values i.e., Maximum amplitude is obtained for the case of $\Phi = 0$.
3. Higher the intensity of initially applied magnetic field, greater will be the absolute value of the thermodynamics temperature, micro-temperature, and normal stress.
4. Magnetic field plays important role in reducing the strength of attenuation factor, i.e., the curve with low magnetic field will converge to zero with faster rate of convergence.
5. 3D curves indicate that whatever the value of horizontal distance is considered, the curves moving on the plane of medium will converge to zero by increasing the distance from the surface.

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