Research Article

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Chaos synchronization of fractional–order discrete–time systems with different dimensions using two scaling matrices

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Abstract: In this paper, we study the synchronization of fractional–order discrete–time chaotic systems by means of two scaling matrices Θ and Φ . The considered synchronization scheme can be tailored to encompass several types of classical synchronization types. We proposed two nonlinear control strategies for the Θ – Φ synchronization of an m –dimensional drive system and an n –dimensional response system, whereby the synchronization dimension $d = m$ and $d = n$, respectively. Numerical examples are presented to test the findings of the study.

Keywords: Chaotic maps, discrete-time fractional calculus, synchronization, different dimensions, linear systems, stability

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1 Introduction

Discrete–time chaotic systems have attracted a low of attention in recent years due to their many applications especially in the fields of control [1, 2] and secure communi-

cations [3–6]. Several discrete–time chaotic systems have been proposed throughout the years including the well known 2–dimensional Hénon map [7], the Lozi system [8], the Zeraoulia–Sprott system [9], the generalized Hénon map [10] and the Baier–Klein system [11], and the discrete Rössler system [12]. These systems are all integer–order in the sense that the differences involved in the dynamical system’s equations has an integer order. In recent years, researchers have picked up an interest in the fractional systems corresponding to the above mentioned integer–order systems. The basic idea is that fractional difference equations have an infinite memory, which both improves the flexibility of the dynamic model in representing physical phenomena as well as having the interesting property of infinite memory.

Continuous–time fractional calculus has been around for centuries. However its discrete counterpart is relatively new. Recent studies have attempted to formulate a framework for discrete–time fractional calculus whereby its stability and transformation theory is established [13–21]. Consensus has yet to be achieved as to the general notation and terminology. For that reason, we choose to adopt those of [22]. Over the last couple of years, substantial progress has been made in the subject of fractional discrete calculus. For instance, in [23], the authors considered a class of Caputo derivative–based generalized differential optimization problems. Interesting results were presented regarding the stability and discrete approximation of these problems. The stability of linear fractional difference systems with delay and impulse effects was treated in [24] using a novel Mittag-Leffler discrete function. Furthermore, a generalization of the Gronwall inequality was achieved in [25] and employed to establish the stability of discrete fractional delay systems.

The main aspect of dynamical chaotic systems that makes them so appealing is the seemingly unpredictable (random–like) nature of their trajectories in phase–space. Mathematically speaking, these are systems that have at least one positive Lyapunov exponent, which is defined

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as the exponential growth in the difference between two trajectories starting from infinitesimally close initial states. At the end of the 1980s, chaos synchronization emerged whereby one chaotic or hyperchaotic (more than one positive Lyapunov exponent) can be controlled to follow the trajectory of another. This idea was revolutionary as it gave birth to chaos-based secure communications. In the simplest form of synchronization, the drive and response system states are controlled to become identical in infinite time. This is referred to as complete synchronization. In many applications, this is not required. Instead, what is required is that a certain function of the slave states synchronizes with another function of the master states. This led to numerous synchronization types.

Numerous types of synchronization and control strategies have been proposed in the literature in relation to integer-order discrete-time systems including, for instance, [26–34]. As for fractional order discrete systems, the available literature is scarce and includes [35–39]. In our study, we are concerned with scalar matrix $\Theta - \Phi$ synchronization, which was first proposed in [40, 41] as a generalization of the matrix projective synchronization (MPS) scheme. The scheme was developed for continuous-time chaotic systems and numerical applications were shown. In [42], the authors extended the work to fractional-order continuous systems. The importance of the $\Theta - \Phi$ lies in the fact that depending on the choice of the two scaling matrices, we may obtain different synchronization types including complete synchronization, anti-synchronization, matrix projective synchronization, and inverse matrix projective synchronization. The main contribution of this paper is the application of $\Theta - \Phi$ synchronization to fractional-order chaotic maps with different dimensions. To the best of the authors' knowledge, most of the majority of results reported in the literature apply only to systems with identical dimensions, which makes the results presented herein both novel and interesting.

In the following section of this paper, we will define the notation and terminology to be used throughout the study. Section 3 details the proposed $\Theta - \Phi$ synchronization control laws with dimension $d = n$ and $d = m$. The convergence of the synchronization error is established by means of the stability theory of linear fractional-order discrete-time systems. In Section 4, we give two numerical examples that confirm the findings of our study. Finally, Section 5 summarizes the results of this paper.

2 System Model

Let us consider the master system given by

$${}^C\Delta^\nu X(t) = AX(t + \nu - 1) + f(X(t + \nu - 1)), \quad (1)$$

with $X(t) = (x_1(t), \dots, x_n(t))^T$ denoting its state vector, A being an $n \times n$ real matrix, and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ being a nonlinear function. Let us also consider as slave the m -component system

$${}^C\Delta^\nu Y(t) = BY(t + \nu - 1) + g(Y(t + \nu - 1)) + U, \quad (2)$$

where, again, $Y(t) = (y_1(t), \dots, y_m(t))^T$ are the states, $B \in \mathbb{R}^{m \times m}$, and $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a nonlinear function. The vector $U = (u_i)_{1 \leq i \leq m}$ is a controller used to drive the states of the slave system towards a specific trajectory to synchronize with the master. Finding a suitable closed form for this controller is the aim of our paper.

Throughout this paper, we adopt the notation ${}^C\Delta_a^\nu X(t)$ denoting the Caputo type delta difference of a function $X(t) : \mathbb{N}_a \rightarrow \mathbb{R}$ with $\mathbb{N}_a = \{a, a + 1, a + 2, \dots\}$ [16], which is of the form

$$\begin{aligned} {}^C\Delta_a^\nu X(t) &= \Delta_a^{-(n-\nu)} \Delta^n X(t) \\ &= \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} (t-s-1)^{(n-\nu-1)} \Delta^n X(s), \end{aligned} \quad (3)$$

for $\nu \in \mathbb{N}$ is the fractional order, $t \in \mathbb{N}_{a+n-\nu}$, and $n = \lceil \nu \rceil + 1$. In (3), the ν -th fractional sum of $\Delta_s^n X(t)$ is defined similar to [14] as

$$\Delta_a^{-\nu} X(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} X(s), \quad (4)$$

with $\nu > 0$. The term $t^{(\nu)}$ denotes the falling function defined in terms of the Gamma function Γ as

$$t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}. \quad (5)$$

Normally, when we talk about synchronization, what comes to mind is trying to force the states of a slave dynamical system to coincide (synchronize) with those of a master. Traditionally, the slave and master were considered as the same system but with different initial conditions. For chaotic systems, this meant that the chaotic behavior of the slave is controlled through some parameter to synchronize to the master. Several forms of synchronization have been proposed throughout the years with different applications.

In our paper, we are concerned with what we call scaling matrix synchronization, which aims to show that

through appropriate choice of U , there exist matrices $\Theta \in \mathbb{R}^{d \times m}$ and $\Phi \in \mathbb{R}^{d \times n}$, with d a certain dimension, such that the error

$$e(t) = \Theta Y(t) - \Phi X(t), \quad (6)$$

decays to zero as $t \rightarrow +\infty$, i.e.

$$\lim_{t \rightarrow +\infty} \|e(t)\| = 0. \quad (7)$$

If this is the case, then systems (1) and (2) are said to be $\Theta - \Phi$ synchronized in dimension d . It is easy to see that depending on our choice of the matrices Θ and Φ , we may have several synchronization types:

- The pair $(\Theta, \Phi) = (I, I)$ yields complete synchronization as

$$\lim_{t \rightarrow +\infty} \|Y(t) - X(t)\| = 0.$$

- The pair $(\Theta, \Phi) = (I, -I)$ yields anti-synchronization as

$$\lim_{t \rightarrow +\infty} \|Y(t) + X(t)\| = 0.$$

- The pair $(\Theta, \Phi) = (I, \Phi)$ yields matrix projective synchronization as

$$\lim_{t \rightarrow +\infty} \|Y(t) - \Phi X(t)\| = 0.$$

- The pair $(\Theta, \Phi) = (\Theta, I)$ leads to inverse matrix projective synchronization as

$$\lim_{t \rightarrow +\infty} \|\Theta Y(t) - X(t)\| = 0.$$

3 Scaling Matrix Synchronization

In this section, we present control law for the proposed $\Theta - \Phi$ synchronization scheme corresponding to two different cases of the synchronization dimension d , namely $d = m$ and $d = n$. In order to establish the convergence of the synchronization error in the two scenarios, we make use of the stability theory of linear fractional discrete systems, which can be summarized in the following theorem.

Theorem 1 [45] *The zero equilibrium of the linear fractional-order discrete-time system*

$${}^C\Delta_a^\nu e(t) = \mathbf{M}e(t + \nu - 1), \quad (8)$$

where $e(t) = (e_1(t), \dots, e_n(t))^T$, $0 < \nu \leq 1$, $\mathbf{M} \in \mathbb{R}^{n \times n}$ and $\forall t \in \mathbb{N}_{a+1-\nu}$, is asymptotically stable if

$$\lambda \in \left\{ z \in \mathbb{C} : |z| < \left(2 \cos \frac{|\arg z| - \pi}{2 - \nu} \right)^\nu \right\} \quad (9)$$

$$\text{and } |\arg z| > \frac{\nu\pi}{2},$$

for all the eigenvalues λ of $\mathbf{M}$.

3.1 Case 1: $d = m$

Since in this case $d = m$, the scaling matrices are of the form $\Theta = (\Theta_{ij})_{m \times m}$ and $\Phi = (\Phi_{ij})_{m \times n}$. The ν -Caputo fractional difference of the error system (6) can be derived as

$$\begin{aligned} {}^C\Delta^\nu e(t) &= \Theta {}^C\Delta^\nu Y(t) - \Phi {}^C\Delta^\nu X(t) \\ &= \Theta BY(t + \nu - 1) + \Theta g(Y(t + \nu - 1)) + \Theta U \\ &\quad - \Phi AX(t + \nu - 1) + \Theta f(X(t + \nu - 1)). \end{aligned} \quad (10)$$

By defining a control matrix $C \in \mathbb{R}^{m \times m}$ and

$$\begin{aligned} R &= (C - B)e(t) + \Theta BY(t) + \Theta g(Y(t)) - \Phi AX(t) \\ &\quad - \Theta f(X(t)), \end{aligned} \quad (11)$$

we can rewrite (10) in the form

$${}^C\Delta^\nu e(t) = (B - C)e(t + \nu - 1) + R + \Theta U, \quad (12)$$

To achieve synchronization between systems (1) and (2), we assume that Θ is an invertible matrix and denote its inverse by Θ^{-1} . The following theorem presents the control laws.

Theorem 2 *The master-slave pair (1)–(2) is globally m -dimensional $\Theta - \Phi$ synchronized by means of the control law*

$$U = -\Theta^{-1} \times R, \quad (13)$$

subject to the control matrix C being selected such that all the eigenvalues λ of matrix $B - C$ satisfy

$$-2^\nu < \lambda \leq 0. \quad (14)$$

Proof. Substituting (13) into (12), the fractional $\Theta - \Phi$ error system can be described as

$${}^C\Delta^\nu e(t) = (B - C)e(t + \nu - 1). \quad (15)$$

It is easy to see that subject to (13), all eigenvalues λ of matrix $B - C$ satisfy $|\arg \lambda| = \pi > \frac{\nu\pi}{2}$ and $|\lambda| < 2^\nu = \left(2 \cos \frac{|\arg \lambda| - \pi}{2 - \nu} \right)^\nu$. It, then, follows directly from Theorem 1 that the zero solution of the error system (15) is globally asymptotically stable and thus the master (1) and slave (2) are globally m -dimensional $\Theta - \Phi$ synchronized. $\square$

3.2 Case 2: $d = n$

The second case we are going to consider is the one where the synchronization dimension $d = n$, where $n < m$, leading to the scaling matrices $\Theta = (\theta_{ij})_{n \times m}$ and $\Phi = (\phi_{ij})_{n \times n}$. Assuming a controllable matrix $L \in \mathbb{R}^{n \times n}$ and using the notation

$$T = (L - A)e(t) + \Theta BY(t) + \Theta g(Y(t)) - \Phi AX(t) - \Theta f(X(t)), \quad (16)$$

we may describe the error system in the form

$${}^C\Delta^\nu e(t) = (A - L)e(t) + \Theta \times U + T. \quad (17)$$

We assume that the matrix $\hat{\Theta} = (\theta_{n \times n})$ is invertible and denote its inverse by $\hat{\Theta}^{-1}$. The following theorem describes the proposed n -dimensional $\Theta - \Phi$ synchronization control laws.

Theorem 3 *The master-slave pair (1)–(2) are globally n -dimensional $\Theta - \Phi$ synchronized if*

$$\hat{U} = (u_1, u_2, \dots, u_n)^T = -\hat{\Theta}^{-1} \times T, \quad (18)$$

and $u_i = 0$ for $i = n + 1, \dots, m$, subject to the control matrix L being selected such that all the eigenvalues of $A - L$ are situated between -2^ν and 0.

Proof. Assuming that the components u_i are equal to zero for $i = n + 1, \dots, m$, the product $\Theta \times U$ reduces to

$$\Theta \times U = \hat{\Theta} \times \hat{U},$$

where $\hat{U} = (u_1, u_2, \dots, u_n)^T$ and $\hat{\Theta} = (\theta_{n \times n})$. Hence, the error system (17) becomes

$${}^C\Delta^\nu e(t) = (A - L)e(t + \nu - 1) + \hat{\Theta} \times \hat{U} + T. \quad (19)$$

Substituting the proposed law (18) into (19) yields

$${}^C\Delta^\nu e(t) = (A - L)e(t + \nu - 1). \quad (20)$$

In much the same way as the proof of Theorem 2 and keeping in mind that the eigenvalues of $A - L$ satisfy the condition (14), the zero solution of error system (20) is globally asymptotically stable and, consequently, systems (1) and (2) are globally n -dimensional $\Theta - \Phi$ synchronized. $\square$

4 Applications

Let us now apply the findings of our study to a particular pair of Hénon-type discrete-time fractional-order maps.

We use the 2-dimensional system proposed in [43] as a fractional extension of the original integer-order system [7] to drive our synchronization. This master system is of the form

$${}^C\Delta^\nu X(t) = AX(t + \nu - 1) + f(X(t + \nu - 1)), \quad (21)$$

where

$$A = \begin{pmatrix} -1 & 1 \\ b_1 & -1 \end{pmatrix}, \quad (22)$$

and

$$f(X(t)) = \begin{pmatrix} -a_1 x_1^2(t) + 1 \\ 0 \end{pmatrix}. \quad (23)$$

As for the slave system to be controlled, we use the fractional-order generalized chaotic 3-dimensional Hénon-like map [44], which is given by

$${}^C\Delta^\nu Y(t) = BY(t + \nu - 1) + g(Y(t + \nu - 1)) + U, \quad (24)$$

where

$$B = \begin{pmatrix} -1 & 0 & -b_2 \\ 1 & -1 & b_2 \\ 0 & 1 & -1 \end{pmatrix}, \quad (25)$$

and

$$g(Y(t)) = \begin{pmatrix} 0 \\ 0 \\ 1 - a_2 y_3^2(t) \end{pmatrix} \quad (26)$$

It is easy to show that these systems exhibit a chaotic behavior. Consider, for instance, the case where $(a_1, b_1) =$

Figure 1: Phase space plot for the fractional Henon map with $(a_1, b_1) = (1.4, 0.3)$, $\nu = 0.984$, and $x_1(0) = x_2(0) = 0$.

Figure 2: Phase portraits for the hyperchaotic Henon map with $(a_2, b_2) = (0.99, 0.2)$, $\nu = 0.984$ and $(y_1, y_2, y_3) = (0.1, 0.2, 0.5)$.

$(1.4, 0.3)$, $(a_2, b_2) = (0.99, 0.2)$, and $\nu = 0.984$. For the uncontrolled slave, i.e. $U = (0, 0, 0)^T$, Figures 1 and 2 depict the chaotic trajectories of the master and slave, respectively.

We would like to put the results of Theorems 2 and 3 to the test and for that we consider two examples.

Example 1 Based on our approach described in Section 3.1, the m -dimensional Θ - Φ error system is given by

$$(e_1(t), e_2(t), e_3(t))^T = \Theta(y_1(t), y_2(t), y_3(t))^T - \Phi(x_1(t), x_2(t))^T \quad (27)$$

where

$$\Theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \text{ and } \Phi = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 3 \end{pmatrix}. \quad (28)$$

Obviously, Θ is invertible and its inverse is given by

$$\Theta^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}. \quad (29)$$

According to Theorem 2, there exists a matrix C such that $(B - C)$ is negative-definite, which achieves Θ - Φ synchronization. We choose, for instance,

$$C = \begin{pmatrix} 0 & 0 & b_2 \\ 0 & 0 & -b_2 \\ 0 & 0 & 0 \end{pmatrix}. \quad (30)$$

It is clear that the eigenvalues of

$$B - C = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \quad (31)$$

are equal to -1 and satisfy the condition of Theorem 1. Control law (13) can be achieved by means of matrix R as defined in (11) and Θ^{-1} as in (29). According to Theorem 2, once U is defined as in (13) the master-slave pair becomes m -dimensional Θ - Φ synchronized. This may be easily verified as the resulting error system is given by

$$\begin{cases} {}^C\Delta^\nu e_1(t) = -e_1(t + \nu - 1), \\ {}^C\Delta^\nu e_2(t) = e_1(t + \nu - 1) - e_2(t + \nu - 1), \\ {}^C\Delta^\nu e_3(t) = e_2(t + \nu - 1) - e_3(t + \nu - 1). \end{cases} \quad (32)$$

Figure 3: The evolution of errors over time for Example 1.

Figure 4: The evolution of errors over time for Example 2.

Given the initial values

$$\begin{pmatrix} e_1(0) \\ e_2(0) \\ e_3(0) \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.4 \\ 1.4 \end{pmatrix},$$

Figure 3 shows the convergence of the errors towards zero in sufficient time.

Example 2 Let us now move to the case considered in Section 3.2, the n -dimensional Θ - Φ error system is of the form

$$\begin{aligned} (e_1(t), e_2(t))^T &= \Theta(y_1(t), y_2(t), y_3(t))^T \\ &\quad - \Phi(x_1(t), x_2(t))^T, \end{aligned} \quad (33)$$

with

$$\Theta = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \end{pmatrix} \text{ and } \Phi = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}. \quad (34)$$

Using the notation of Section 3.2, we have

$$\hat{\Theta} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } \hat{\Theta}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \quad (35)$$

According to Theorem 3, there exists a control matrix L , which can be selected as

$$L = \begin{pmatrix} 0 & 0 \\ b_1 + \frac{9}{16} & -\frac{3}{2} \end{pmatrix}, \quad (36)$$

leading to

$$A - L = \begin{pmatrix} -1 & 1 \\ -\frac{9}{16} & \frac{1}{2} \end{pmatrix},$$

which clearly satisfies the condition of Theorem 3. Next, T can be easily formed as in (16). Next, we may define the control law stated in (18) and append it with a zero at the end. Theorem 3 established that the master-slave pair (21)–(24) becomes n -dimensional inverse Θ - Φ synchronized. To verify this, we use the error system

$$\begin{cases} {}^C\Delta^\nu e_1(t) = -e_1(t+\nu-1) + e_2(t+\nu-1), \\ {}^C\Delta^\nu e_2(t) = -\frac{9}{16}e_1(t+\nu-1) + \frac{1}{2}e_2(t+\nu-1). \end{cases} \quad (37)$$

Figure 4 shows the time evolution of the errors given the initial values

$$\begin{pmatrix} e_1(0) \\ e_2(0) \end{pmatrix} = \begin{pmatrix} -0.4 \\ 2.4 \end{pmatrix}.$$

5 Summary

In this paper, we dealt with the synchronization of fractional-order discrete-time chaotic systems. We considered the case of Θ - Φ synchronization, which forces a linear function of the n slave states represented as a matrix $\Phi \in \mathbb{R}^{d \times n}$ to converge towards a linear function of the m master states represented by matrix $\Theta \in \mathbb{R}^{d \times m}$. This type of synchronization is powerful in that it encompasses numerous different synchronization schemes. We developed nonlinear control laws for two distinct cases of the dimension d , namely $d = m$ and $d = n$. The asymptotic stability and convergence of the synchronization errors were established by means of the stability theory of linear fractional discrete-time systems. Two numerical examples were presented using as master the fractional Hénon map and as slave the fractional 3-dimensional generalized Hénon-type discrete map. Numerical results confirmed the existence of appropriate control matrices and the convergence of the synchronization error towards zero.

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